Internat. J. Math. & Math. Sci.
Vol. 7 No. 4 (1984) 625-661

INVARIANT MANIFOLDS AND ALTERNATIVE APPROACHES TO REDUCTIVE PERTURBATION THEORY: DISSIPATIVE SYSTEMS

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(Received June 19, 1984)

ABSTRACT: This paper is concerned with the limiting process known as Reductive Perturbation Theory. A new view of both conventional reductive perturbation theory and the higher order theory of Kodama is presented for a general class of dissipative systems. An alternative iterative approach to the determination of higher order corrections is outlined.

KEY WORDS AND PHRASES: *Reductive Perturbation Theory, Dissipative Systems, Invariant Manifolds, Higher Order Corrections, Contraction Mapping.*
1980 MATHEMATICS SUBJECT CLASSIFICATION CODE. 35Q99(76B15)

1. INTRODUCTION

The past twenty years has seen a explosion of interest in the special equations such as the Korteweg-deVries and sine-Gordon equations that can be solved by techniques associated with INVERSE SPECTRAL TRANSFORM method [1]. One reason for that continued interest has been the regular appearance of those special equations in the analysis of a wide range of different physical phenomena. Very frequently complicated systems of equations modelling such complex behaviour as wave propagation in a plasma or stratified fluids have been reduced, by a special limiting process, to one of the basic equations mentioned previously. The limiting process used is known as REDUCTIVE PERTURBATION THEORY.

Until recently most of the work on this technique was carried out by Japanese physicists and mathematicians and consequently has appeared in Japanese journals. A good recent review of the earlier work can be found in the paper by Taniuti [2] and the book [3]. This later reference contains a very complete bibliography on the existing theory and application of this method.

In spite of its fundamental importance in soliton physics the procedures and

techniques of reductive perturbation theory are badly defined. It is the purpose of this paper to address some of the outstanding problems that are associated with the method as currently formulated.

We propose, in the following sections, to expound a particular view of what is being constructed when the reductive perturbation method is applied. We start in section two with some alternative approaches to quasi-linear hyperbolic systems of a fairly general form. The ideas of far-field solutions and simple waves are developed and interrelated. In the following section the results of section two are extended to a dissipative generalisation of systems previously considered. The form of reductive perturbation theory that we develop is equivalent to the higher order renormalised reductive perturbation theory of Kodama [4]. In our final section four we propose a more general formulation of the method as a whole in which the standard method is looked upon as an 'outer expansion' solution to a singular perturbation problem. As an alternative to power series solutions to this problem we propose an iterative scheme. In developing this approach we have drawn upon ideas in the papers by Foy [5] , Koppel and Howard [6] and Sattinger [7] . This alternative is still in an exploratory stage and so we present only the germ of the ideas involved and refer the interested reader to a forthcoming paper by the authors [8] .

2. THE REDUCTIVE PERTURBATION METHOD

In this section we present the motivational concepts that underlie the method of reductive perturbation theory. As the name suggests there are two essential features involved with this technique. Most standard perturbation techniques involve a basic linearisation about some known solution followed by the solution of the resulting linear equations to calculate correction terms. Such asymptotic methods are usually based upon the existence, within the system equations, of a small parameter. This can occur either in the dynamical equations or possibly in the boundary conditions which define the full system. The exact way in which the parameter occurs, whether in a regular or singular way, determines the appropriate form of asymptotic expansion used to represent the solution function. The method of reductive perturbation theory is not a perturbation method in quite the same way. Rather, it involves a limiting process based on the existence of classes of one parameter solutions. Whilst that parameter may be thought of as small, and the expansions in terms of it as perturbation expansions, it does not appear in the dynamical equations. It can be thought of as intrinsically involved in the initial conditions but in the reductive perturbation method the initial conditions are usually given less prominance. In general the conditions imposed are of a spatially asymptotic character. One requires the solutions to be bounded or to tend to some given asymptotic form. The method is not concerned with the initial value problem but with the construction of special solutions which represent the solutions to the equations concerned in some asymptotic, 'far field', regions. The method seeks to extract the effect of dominant nonlinearities. In that sense it is an improvement over normal perturbation procedures as the simplified 'reduced' equations are nonlinear. The area of principal application is hyperbolic systems

of nonlinear partial differential equations. For such systems the reductive perturbation technique reduces very complicated systems to simple canonical forms. For dissipative systems we have canonical forms such as the Burgers equation. For dispersive systems it is the celebrated Korteweg-de Vries (K.dV) equation which most frequently appears as the canonical form. When such 'reductions' are made there is clearly a considerable loss of information about the system. What is actually shown is that certain projections of the solution function to a system of nonlinear hyperbolic partial differential equations satisfy the canonical equations mentioned above. In fact it is possible to determine different canonical equations for different projections and to obtain information about the whole solution function. However, it is the structure of the nonlinear terms which determines the specific limiting process used for each projection. This means that we cannot assemble these results together to obtain an asymptotic picture of the complete solution. We have here a further reflection of the fact that this method is not concerned with the general initial value problem. Rather, it is concerned with the construction of some type of invariant manifold for the systems involved. Can we determine a one parameter family of functions which are invariant under the flow induced on the function space of initial functions by such a system? We shall make some further comments upon this view of reductive perturbation theory as we develop the method. We are then, concerned with the idea that there exist regions of space in which special types of solution to systems of hyperbolic partial differential equations behave in a simpler, canonical, fashion. Some elementary examples will demonstrate the physical origins of these ideas.

Consider the wave equation

$$W_{tt} - W_{xx} = 0 \quad . \tag{2.1}$$

If we introduce the new variables u and v defined by

$$u = \frac{1}{2}(W_x + W_t) \qquad v = \frac{1}{2}(W_x - W_t) \tag{2.2}$$

equation (2.1) can be recast into the form of a hyperbolic system of linear partial differential equations. The vector $U = (u,v)^T$, where 'T' is the operation of matrix transposition, satisfies the equation

$$U_t + A_o U_x = 0 \tag{2.3}$$

with $A_o = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$. This real symmetric matrix has two real eigenvalues 1 and -1 with eigenvectors $R_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $R_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ respectively. If we consider the initial value problem defined by (2.3) together with the initial condition

$$U(x,t=0) = U_o(x) \tag{2.4}$$

we can obtain the general solution in the following way. Expand both the initial vector valued function $U_o(x)$ and the solution vector $U(x,t)$ in terms of the basis defined by the eigenvectors R_1 and R_2,

$$U_o(x) = w_{1o}(x)R_1 + w_{2o}(x)R_2 \tag{2.5}$$

$$U(x,t) = w_1(x,t)R_1 + w_2(x,t)R_2 \quad . \tag{2.6}$$

When (2.6) is substituted into (2.3) we find that the coordinate functions w_i satisfy the decoupled initial value problems

$$w_{it} + e_i w_{ix} = 0 \qquad w_i(x,t=0) = w_{io}(x) \tag{2.7}$$

where $e_i = \begin{cases} 1 & i=1 \\ -1 & i=2 \end{cases}$. These linear partial differential equations can be solved in the following classical way. Define the family of curves $C^{(i)}$ to be the trajectories of the system of ordinary differential equations

$$\frac{dx}{dt} = e_i \quad . \tag{2.8}$$

From (2.7) we see that w_i is constant along the family of curves $C^{(i)}$. The curves $C^{(i)}$ are called <u>characteristic curves</u> and the characteristic which originates from the point $(\xi,0)$ on the initial data curve $t=0$ is the straight line

$$x - e_i t = \xi \quad . \tag{2.9}$$

As the function w_i is constant along this curve we have the result

$$w_i(x,t) = w_{io}(\xi) = w_{io}(x-e_i t) \tag{2.10}$$

and the complete solution is given by

$$U(x,t) = w_{1o}(x-t)R_1 + w_{2o}(x+t)R_2 \quad . \tag{2.11}$$

Let us first of all observe the mathematical structure of the result. The coupled system of equations (2.3) was decoupled by introducing two new variables w_i which were the projections of the solution vector onto the eigenvectors R_i. Each of these new projections satisfied a simple, canonical, equation (2.7). The solutions to these equations are travelling waves of the form (2.10) in which the initial wave profile w_{io} propagates along the x-axis at a speed e_i. For general initial data we will have both leftward and rightward travelling waves. This is most easily envisaged if we consider initial data on compact support,

$$U_o(x) = \begin{cases} F(x) & |x| \leq d \\ 0 & |x| \geq d \end{cases} \quad . \tag{2.12}$$

The projected initial data w_{io} will also have compact support and for $t > d$ the solution consists of two nonoverlapping progressive waves. To generate a single wave it is necessary to choose an initial function which is an eigenvector of the matrix A_o. If the function U_o was required to satisfy the subsidiary condition

$$(A_o - e_i I)U_o = 0 \tag{2.13}$$

it could only generate uni-directional waves. This can also be stated by saying

that the space of functions $U(x,t)$ with the property

$$(A_o - e_i I)U(x,t) = 0 \tag{2.14}$$

is invariant under the flow induced by the partial differential equation (2.3). This follows from the simple fact that the operator $(A_o - e_i I)$ commutes with the time evolution operator $-A_o \partial_x$.

The extension of the simple 2 dimensional case above to the general $n \times n$ system

$$U_t + AU_x = 0 \tag{2.15}$$

where A is an $n \times n$ matrix with n distinct real eigenvalues $e_i (i=1,\ldots,n)$ is straightforward and automatic. Expanding the solution function in terms of the eigenvectors R_i $(i=1,\ldots,n)$ of A we can reduce system (2.15) to the system of decoupled 1-dimensional equations

$$w_{it} + e_i w_x = 0 \tag{2.16}$$

where w_i is the projection of U onto the ith eigenvector R_i. A convenient way to express w_i is by means of the left eigenvectors L_i of A defined by

$$L_i^T(A - e_i I) = 0 \quad . \tag{2.17}$$

It is easily shown that the bases $\{R_i\}$ and $\{L_i\}$ are dual to each other with the orthogonality relations

$$L_i^T R_j = 0 \qquad \text{for} \qquad i \neq j \quad . \tag{2.18}$$

Consequently we can write

$$w_i = (L_i^T U)/(L_i^T R_i) \tag{2.19}$$

for the ith projection.

The complete solution is now given by

$$U(x,t) = \sum_{i=1}^{n} w_{io}(x - e_i t)R_i \quad . \tag{2.20}$$

As the problem is linear we are able to obtain the complete solution from the reduced equations (2.16). This will cease to be possible when we transfer our attention to nonlinear systems. After a certain time $t=t^*$ the initial data with compact support in the interval $|x| \leq d$ will have been transported along the straight line characteristics to n distinct regions $S_1,\ldots,S_n$. The figure

below illustrates this situation for the case n=4.

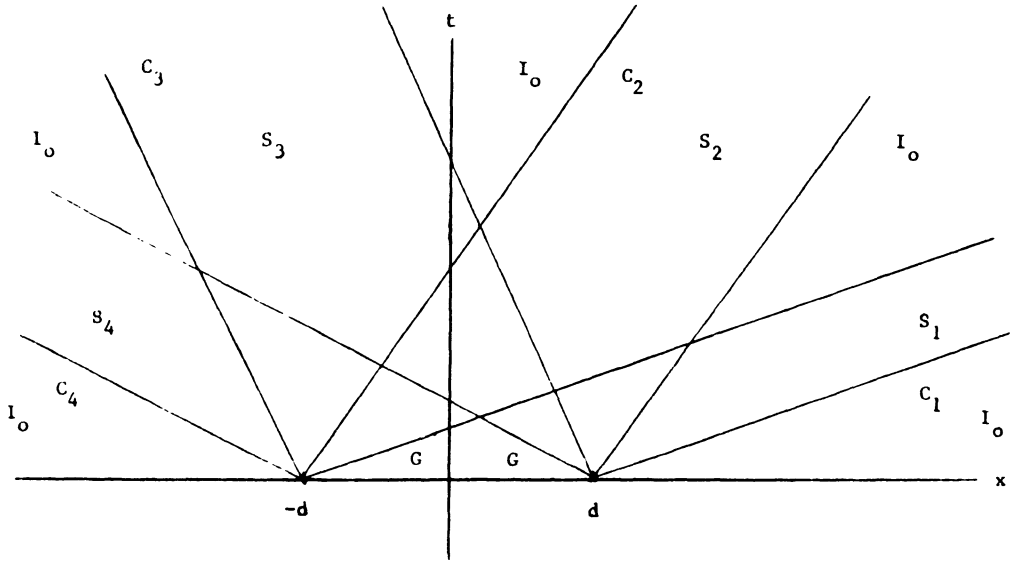

Fig 2.1: An illustration of far field regions for the case n=4. The regions S_i are the far field regions. The region G is the region of complete determinacy and the regions I_o are regions of constant state.

Within any single region only a single wave $w_k(x,t)$ will be observed. Thus, after the critical time $t=t^*$ the solution (2.20) represents n progressive waves which do not overlap in space or time. These wave solutions are sometimes called the far field solutions of the hyperbolic system (2.15) and the regions of space-time S_k, where they are to be found, are called the far field regions.

Many of the systems of nonlinear partial differential equations that occur in physics can be recast, like the basic wave equation (2.1), as systems of partial differential equations of the form

$$U_t + A(U)U_x = 0 \tag{2.21}$$

where $A(U)$ is an $n \times n$ matrix depending upon the system state vector U. A typical example is provided by the equations for an isentropic gas. In that case the state variable is the vector $U = (\rho,u)^T$ where ρ is the gas density and u is its velocity. The matrix $A(U)$ takes, in that case, the form

$$A(U) = \begin{bmatrix} u & \rho \\ a^2\rho^{-1} & u \end{bmatrix} \tag{2.22}$$

and for simplicity we consider the case of constant a. For a system of the type (2.21) we can ask if the notions of far field solution and region retain any validity. The analysis of the case of constant A depended largely upon the construction of the families of characteristic curves $C^{(i)}$. Even for the quasi-linear case (2.21) we can still introduce the idea of characteristic curves given certain assumptions about $A(U)$. Before we consider the general

case let us investigate a simple one dimensional example. The equation

$$u_t + uu_x = 0 \tag{2.23}$$

will prove adequate to show the problems involved. The characteristic curves in this case are defined to be the trajectories of the single ordinary differential equation

$$\frac{dx}{dt} = u \quad . \tag{2.24}$$

Along such curves the solution vector $u(x,t)$ is a constant and so, as in the simple constant case, the trajectories of (2.24) are a family of straight lines. The problem that exists in this case is that each characteristic originating from a point $(\xi,0)$ has a different slope determined by the initial function $u_o(\xi)$. The solution to the equation (2.23) is given implicitly by the following nonlinear analogue of (2.10),

$$u(x,t) = u_o(\xi(x,t)) \tag{2.25}$$

where $\xi(x,t)$ is determined implicitly by the algebraic equation

$$x = \xi + tu_o(\xi) \quad . \tag{2.26}$$

The existence of the function $\xi(x,t)$ is guaranteed by the inverse function theorem provided

$$1 + tu_o'(\xi) \neq 0 \quad . \tag{2.27}$$

The critical time on the trajectory originating from $x=\xi$ we denote by

$$t_1(\xi) = -(u_o'(\xi))^{-1} \quad . \tag{2.28}$$

When $t=t_c(\xi)$ the equation (2.26) is noninvertible because two characteristics cross at the point $(u_o(\xi)t_c(\xi)+\xi,t_c(\xi))$ and so a unique point $(\xi,0)$ on the initial data axis $t=0$ cannot be ascribed to it. The locus of such points determines a critical curve Γ_c beyond which the solution is ill defined. In fact it becomes necessary to generalise the whole notion of what is meant by a solution to proceed outside of the region bounded by Γ_c. This is the important subject of shock solutions but as we shall bypass the problem by introducing higher order derivative terms into the basic equations we shall not pursue it at this juncture.

Already it is clear that there are going to be problems. The solution may not exist in the classical sense long enough for any far-field solutions to separate and that means no far field regions. However, let us place these thoughts at the back of our mind confident that we will think of some way of circumventing the problem. How can we handle the general system (2.21)? Start with the $n=2$ case and assume that the matrix $A(U)$ has two real distinct eigenvalues $E_i(u)(i=1,2)$, with right and left eigenvectors $R_i(U)$ and $L_i(U)$ both of which define a basis for R^2. If we premultiply (2.21) by the left eigenvector $L_i(U)$

we obtain the equations

$$L_i^T(U)(U_t+E_i(U)U_x) = 0 \quad . \tag{2.29}$$

This suggests that the appropriate analogue of characteristic curves in this non-linear situation is provided by the trajectories of the ordinary differential equations

$$C^{(i)}: \qquad \frac{dx}{dt} = E_i(U) \tag{2.30}$$

as in (2.24). Equations (2.29) then tells us that if s_i is a path parameter along a member of the family $C^{(i)}$ we have

$$L_i \frac{dU}{ds_i} = 0 \quad . \tag{2.31}$$

By means of an appropriate integrating factor $\mu_i(U)$ this equation can be converted into a complete differential $\frac{df_i}{ds_i}$ which defines the quantity $f_i(U)$. Along the characteristic curve $C^{(i)}$ we have the result

$$f_i(U) = r_i(s_j) \qquad j \neq i \tag{2.32}$$

where $r_i(s_j)$ is a function of the other characteristic variable s_j $(j \neq i)$ determined by the initial data $U_o(x)$. By means of illustration consider the isentropic gas equations defined by (2.22). The eigenvalues and right and left eigenvectors are easily calculated to be

$$E_1(U) = (u+a) \qquad R_1(U) = \begin{bmatrix} 1 \\ a\rho^{-1} \end{bmatrix} \qquad L_1(U) = \begin{bmatrix} a\rho^{-1} \\ 1 \end{bmatrix} \tag{2.33}$$

$$E_2(U) = (u-a) \qquad R_2(U) = \begin{bmatrix} 1 \\ -a\rho^{-1} \end{bmatrix} \qquad L_2(U) = \begin{bmatrix} a\rho^{-1} \\ -1 \end{bmatrix} \tag{2.34}$$

and we observe that the orthogonality properties (2.18) are fulfilled. If we suppose initial data of the form

$$U(x,t=0) = \begin{bmatrix} g(x) \text{ for } |x| \leq d \text{ and } \rho_o \text{ for } |x| \geq d \\ 0 \text{ for all } x \end{bmatrix} . \tag{2.35}$$

then the constants r_i, which are known as the Riemann invariants, are given by

$$f_1 = u + a\log(\rho) = r_1(s_2) \quad \text{along} \quad C^{(1)}: \frac{dx}{dt} = (u+a) \tag{2.36}$$

$$f_2 = u - a\log(\rho) = r_2(s_1) \quad \text{along} \quad C^{(2)}: \frac{dx}{dt} = (u-a) \quad . \tag{2.37}$$

In the x-t plane the characteristic curves determine special regions. These are indicated schematically in the figure below.

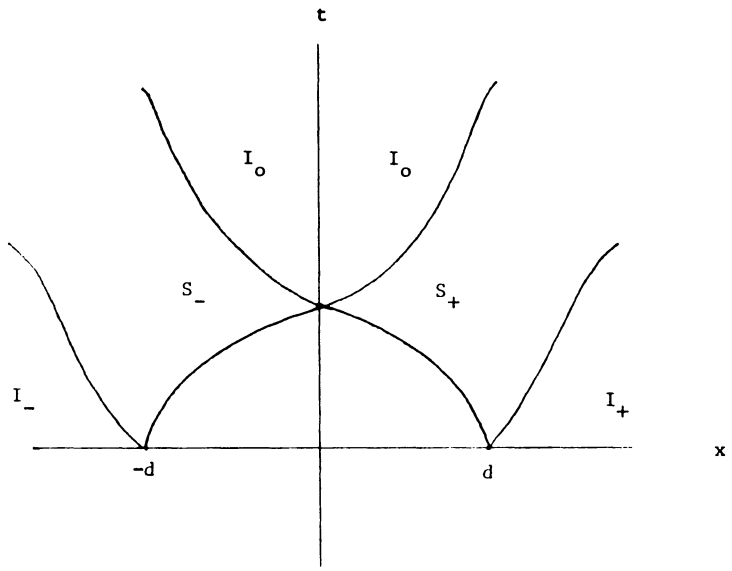

Fig 2.2 : The regions I_a ($a=\pm,o$) are constant state regions. The regions S_a ($a=\pm$) are the simple wave regions.

In the region S_1 equations (2.1.36-37) become

$$u + \text{alog}(\rho) = \text{alog}(g(\xi)) \quad \text{on} \quad C^{(1)} \tag{2.38}$$

$$u - \text{alog}(\rho) = -\text{alog}(\rho_o) \quad \text{on} \quad C^{(2)} \tag{2.39}$$

which can be solved to give

$$u = \tfrac{1}{2}\,\text{alog}(g(\xi)/\rho_o) \tag{2.40}$$

$$\rho = (\rho_o g(\xi))^{\frac{1}{2}} \tag{2.41}$$

Now that we know the state vector in S_1 we can integrate the equations for the characteristic curves in that region. The result is the family of straight lines

$$C^{(1)}: \quad x = \xi + (u(\xi)+a)t \tag{2.42}$$

where $u(\xi)$ is the function given in (2.40). The time t^* indicated on figure (2.2) is the time after which the solution has divided into two distinct components existing in the regions S_1. The family of characteristics (2.42) has a breaking time t_b, defined as the minimum positive value of $t_c(\xi)$, given by

$$t_b = \min_{|\xi|<d} \left(-\frac{2g(\xi)}{ag'(\xi)} \right) \tag{2.43}$$

The distinguishing feature of the regions S_1 is that within them one of the Riemann invariants $r_j (j \neq i)$ is a constant. Such regions are known as <u>simple</u>

wave regions and the solutions within them as simple waves. The equations (2.40-2) define a simple wave solution to the gas dynamics equations for $t^* < t < t_b$ provided that ordering of times is valid. The subregions of S_i corresponding to that time slice clearly represent a generalisation of the 'far fields' of the constant matrix system (2.15). We also see that the simple wave solutions are the most natural generalisation of the 'far field solutions' of the linear case.

Unfortunately, the theory of Riemann invariant does not generally extend to the $n \times n$ situation. This means that if we wish to use the idea of simple wave solutions to generalise the notion of 'far field' solutions to the non-linear case we must find some alternative definition, equivalently defining them in the $n=2$ case, which can be extended to the general system. One way in which it can be done is to seek special solutions to the system (2.21) of the form

$$U = (u_1, u_2(u_1), \ldots, u_n(u_1))^T \tag{2.44}$$

with each of the component functions u_i $(i=2,\ldots,n)$ a function of one single independent component taken, without loss of generality, to be u_1. For example, in the case $n=2$ the substitution of the ansatz (2.44) into the 2-dimensional version of (2.21) yields the equation

$$(u_{1t} + A(U)u_{1x}) \begin{bmatrix} 1 \\ u_2' \end{bmatrix} = 0 \quad . \tag{2.45}$$

From this we see that $-u_{1t}/u_{1x}$ is an eigenvalue of $A(U)$ with eigenvector $R(U) = \begin{bmatrix} 1 \\ u_2' \end{bmatrix}$. This gives us two types of simple wave corresponding to the two distinct eigenvalues $E_i(U)$,

$$u^i_{1t} + E_i(U)u^i_{1x} = 0 \qquad (i=1,2) \tag{2.46}$$

$$\begin{bmatrix} 1 \\ u_2' \end{bmatrix} = CR_i(U) \qquad \text{with } C \text{ a constant} \tag{2.47}$$

For the gas dynamics example this gives for $i=1$

$$u_{1t} + (u_1+a)u_{1x} = 0 \tag{2.48}$$

$$u_2' = a(u_1)^{-1} \tag{2.49}$$

The second of these can be immediately integrated to give

$$u_2 = a\log(u_1) + K \quad \text{where } K \text{ is a constant} \tag{2.50}$$

Eliminating the function $g(\xi)$ between (2.1.40) and (2.1.41) we see that

$$u = u_2 = a\log(\rho) - a\log(\rho_o) = a\log(u_1) - a\log(\rho_o) \tag{2.51}$$

and we have agreement. The solution for (2.48) is determined by the same family of characteristic curves (2.42) as before. Naturally, when we deter-

mine solutions by means of an ansatz such as (2.44) we are no longer solving an initial value problem such as that determined by the initial data (2.35). Just as in that case we were able to show that there were simple wave regions but were not able to exactly locate them so, in the ansatz case, we are able to find the solutions but know not where they exist nor the initial data from which they derive.

We are now in a position to introduce the reductive perturbation approach to the determination of simple wave type solutions for nonlinear systems such as (2.21). The systems (2.21) have one very simple class of solutions. These are the solutions given by constant state vectors $U(x,t) = U_o$. Of course this will not satisfy the initial conditions unless $U_o(x) = U_o$ a constant. Let us consider an initial value problem in which the initial condition is some deformation of the constant state U_o. Let us take

$$U_o(x) = U_o + \epsilon \hat{U}(x,\epsilon) \tag{2.52}$$

where $\hat{U}(x,\epsilon) = O(1)$ as $\epsilon \to 0$. In fact we will assume that $\hat{U}(x,\epsilon)$ is analytic in ϵ with a Taylor series of the form

$$\hat{U}(x,\epsilon) = U_{1o}(x) + \epsilon U_{2o}(x) + \cdots + \epsilon^{n-1} U_{no}(x) + \cdots \tag{2.53}$$

The physical idea is that the system is started off in a 'near equilibrium' configuration. The parameter ϵ is thought of as 'small'. However, in practice it acts as a formal expansion parameter and need not be small.

We concentrate first upon the simple wave solutions of the general system (2.21). These are determined by the 'reduced equations'

$$u_t + E(u)u_x = 0 \tag{2.54}$$

where u is the independent coordinate of U and $E(u)$ is one of the eigenvalues of $A(U)$ expressed as a function of the single function u as demanded by the simple wave hypothesis. The initial data that will give rise to a simple wave is not known to us. Let us suppose that it is possible to find a one parameter family of initial functions of the form

$$u_o(x,\epsilon) = u_{o1} + \epsilon \hat{u}(x,\epsilon) \tag{2.55}$$

corresponding in the simple wave case to the general initial data (2.52) of the full equations. The function $\hat{U}(x,\epsilon)$ is assumed to be analytic in ϵ. The initial value problem defined by (2.54-55) can be solved by the method of characteristics and is given in the implicit form by the equations

$$u(x,t,\epsilon) = u_o(\xi(x,t,\epsilon),\epsilon) \tag{2.56}$$

where the new coordinate $\xi(x,t,\epsilon)$ is given implicitly by the algebraic equation

$$x = \xi + E(u_o(\xi,\epsilon))t \tag{2.57}$$

As $u_o(x,\epsilon)$ is analytic in ϵ, equation (2.57) has a solution $\xi(x,t,\epsilon)$ which is also analytic in ϵ. We easily find that the structure of this analytic

function is given by

$$\xi(x,t,\epsilon) = \xi_o + a_1(\xi_o)(\epsilon t) + a_2(\xi_o)(\epsilon t)^2 + a_3(\xi_o)(\epsilon^2 t) + 0(\epsilon^3) \tag{2.58}$$

where the variable ξ_o is defined by

$$\xi_o = (x-E(u_{o1})t) \tag{2.59}$$

The constant $E(u_{o1})$ is readily seen to be any one of the eigenvalues of the constant matrix A_o defined by

$$A_o = A(U_o) \tag{2.60}$$

The functions $a_1(\xi_o)$ are determined by the initial data function $u_o(x,\epsilon)$ and, of course, the matrix $A(U)$. The structure of the simple wave solution (2.56-57) in the parameter ϵ suggests an alternative way in which simple wave solutions, and possibly generalisations, might be extracted from the general equations (2.21) with initial conditions (2.52). The function $\xi(x,t,\epsilon)$ can be regarded as a function of the infinite set of 'scaled variables'

$$\xi_{oj} = (x-E_j(U_o)t) \qquad (j=1,\ldots,n) \tag{2.61}$$

$$T_i = \epsilon^i t \qquad (i=1,\ldots,\infty) \tag{2.62}$$

where $E_j(U_o)$ is an eigenvalue of $A(U_o)$. The variables ξ_{oj} and T are known as the Gardner-Morikawa variables (Gardner,Morikawa, [9]). Simple wave solutions are now seen as part of the class of solutions to (2.21&52) which have formal asymptotic expansions of the form

$$U(x,t,\epsilon) = \tilde{U}(\xi_o,T_1,\ldots,T_n,\ldots,\epsilon) \tag{2.63}$$

where $\tilde{U}(\cdot;\epsilon)$ is an analytic function of ϵ. Each of the variables (2.61-62) is regarded as independent. We see from (2.58) that in order to obtain the simple wave solution we need to retain all of the higher order 'slow times' T_n. We refer to them as 'slow times' because if the parameter ϵ is small the variable T_n will be small also for periods of time t for which $t < \epsilon^{-n}$. For such periods of time it may be possible to neglect the effect of times T_i for which $i > n$.

If we assume the existence of a solution to the $n \times n$ initial value problem of the form (2.63) we obtain the following set of variational equations

$$\left(\sum_{i=1}^{\infty} \epsilon^i a_{T_i} - e\partial_\xi\right)\left(\sum_{j=1}^{\infty} U_j\epsilon^j + U_o + A\Big(U_o + \sum_{j=1}^{\infty} U_j\epsilon^j\Big)\right)\left(\sum_{j=1}^{\infty} U_{j\xi}\epsilon^j\right) = 0 \tag{2.64}$$

where ξ is the Gardner-Morikawa variable $(x-et)$ corresponding to the eigenvalue e of A_o. We have expanded $\tilde{U}(\cdot;\epsilon)$ in its Taylor series

$$\tilde{U}(\cdot,\epsilon) = U_o + \sum_{j=1}^{\infty} A_j(U_1,\ldots,U_j)\epsilon^j \tag{2.65}$$

and we can write

$$A(U_o + \sum_{j=1}^{\infty} U_j \epsilon^j) = A_o + \sum_{j=1}^{\infty} A_j(U_1,\ldots,U_j)\epsilon^j \qquad (2.66)$$

where the coefficient functions $A_j(U_1,\ldots,U_j)$ depend upon the coefficient functions U_i for $i \leq j$. They also depend on U_o, but as this is fixed we suppress its appearance in the notation. Equating the coefficient of ϵ^n in (2.64) to zero we obtain the equation

$$\sum_{i=1}^{n} \partial_{T_i} U_{n-i} - eU_{n\xi} + \sum_{i=0}^{n} A_i(U_1,..,U_1)U_{n-i\xi} = 0 \qquad (2.67)$$

The lowest order equation corresponding to n=1 is given by

$$(A_o - eI)U_{1\xi} = 0 \qquad (2.68)$$

which is reminiscent of the subsidiary condition (2.14) required in the constant linear case to ensure the generation of asymptotic far field solutions. This constraint implies that U_1 must be representable in the form

$$U_1 = \omega(\xi,T_1,\ldots,T_\infty)\, R + C(T_1,..,T_\infty) \qquad (2.69)$$

where $\omega(.)$ is a scalar function and R is the right eigenvector of A_o corresponding to the eigenvalue e.

As in the simple wave anstatz we have brought about a reduction in the nature of the state variable. The vector quantity U_1 has been reduced to a single function ω. Whether ω will prove sufficient to parameterise the rest of the state vector remains to be seen. Whilst it would be possible to continue the analysis and retain the integration constant C (.) it does lead to the introduction of a large number of additional terms that obscure the underlying process. For that reason it is convenient to introduce an extra boundary condition at spatial infinity which implies that they are all zero. If we impose the additional requirement upon our basic initial value problem that we are only concerned with initial functions and solutions with the property

$$U(x,t,\epsilon) \to 0 \quad |x| \to \infty \qquad (2.70)$$

then all such integration constants, depending upon time variables alone, will be zero. In order to determine the dependence of $\omega(.)$ upon ξ and the 'slow times' T_i we must look at higher order terms $n \geq 2$.

The equation (2.1.67) becomes, for n=2, the equation,

$$\partial_{T_1} U_1 - eU_{2\xi} + A_1(U_1)U_{1\xi} + 0 \quad . \qquad (2.71)$$

The content of this equation is best analysed by recasting it into the form

$$(A_0 - eI)U_{2\xi} = -(\partial_{T_1} U_1 + A_1(U_1)U_{1\xi}) \tag{2.72}$$

$$= -(\partial_{T_1} \omega + A_1(\omega R)\omega_\xi)R \ . \tag{2.73}$$

The functional $A_1(U)$ is linear. Let us write

$$A_1(U) = \sum_{i=1}^{n} A_1^i U_{(i)} \tag{2.74}$$

where $U_{(i)}$ is the i-th component of the vector U. If we define the matrix B_1 by

$$B_1 = \sum_{i=1}^{n} A_1^i R_{(i)} \tag{2.75}$$

we can finally express (2.1.73) in the form

$$(A_0 - eI)U_{2\xi} = -(\omega_{T_1} + \omega\omega_\xi B_1)R \tag{2.76}$$

Premultiplying each side of this equation by the left eigenvector L corresponding to the eigenvalue e we obtain the following equation for the projection ω

$$\omega_{T_1} + a_1\omega\omega_\xi = 0 \tag{2.77}$$

where a_1 is the constant defined by

$$a_1 = (L^T B_1 R)/(L^T R) \tag{2.78}$$

Equation (2.76) yields, upon the use of (2.77), the result

$$(A_0 - eI)U_2 = \tfrac{1}{2}(a_1 - B_1)R\omega^2 \ . \tag{2.79}$$

In (2.79) we have set the integration constant to zero by virtue of our asymptotic hypothesis (2.70). The solution to (2.79) is ambiguous to the extent that we can add to any solution U_2 an element of the kernel of $(A_0 - eI)$. If E_R is the eigenspace of R^n spanned by the eigenvector R then A_0 defines a vector space decomposition of R^n,

$$R^n = E_R \oplus N_R \tag{2.80}$$

If we denote the projection operators onto these complementary subspaces by P_R and Q_R

$$P_R \oplus Q_R = I \tag{2.81}$$

the P_R and Q_R can be expressed in terms of the left and right eigenvectors L and R. The appropriate formulae, which define N_R, are

$$(P_R)_{ij} = \frac{L_{(j)}R_{(i)}}{(L^T R)}, \quad (Q_R)_{ij} = \delta_{ij} - \frac{L_{(j)}R_{(i)}}{(L^T R)} \quad . \tag{2.82}$$

Let ϕ_2 be the unique solution to the equations

$$(A_o - eI)\phi_2 = \tfrac{1}{2}(a_1 - B_1)R, \quad P_R\phi_2 = 0 \quad . \tag{2.83}$$

We can then express the solution (2.76) in the form

$$U_2 = \omega^2\phi_2 + s_2 R \tag{2.84}$$

where s_2 is a function of all of the system variables. The essential idea behind the construction of simple wave solutions would suggest that all of the components of the vector should be expressed in terms of a single function. The natural choice for that function is ω and that, in turn, suggests that we should choose the only unique solution (2.76) corresponding to the choice $s_2 = 0$ in (2.84). However, let us for the present retain the term $s_2 R$ in (2.1.84) to see what complications its retention would involve.

The equations for the n=3 level in (2.67) are given by

$$(A_o - eI)U_{3\xi} = -(A_1(U_1)U_{2\xi} + A_2(U_1, U_2)U_{1\xi} + \partial_{T_2}U_1 + \partial_{T_1}U_2) \quad . \tag{2.85}$$

Once again, premultiplication by the left eigenvector L leads to an evolution equation for the function ω with respect to the 'slow time' T_2

$$(\omega_{T_2} + s_{2\,T_1} + \omega a_1 s_{2\xi})(L^T R) + (L^T B_1 \phi_2)\omega(\omega^2)_\xi +$$
$$+ L^T A_2(\omega R, \omega^2\phi_2 + s_2 R)R\,\omega_\xi = 0 \quad . \tag{2.86}$$

If we write out the Taylor expansion of A(U) in the form

$$A(U) = A_o + \sum_i^n A_1^i U_{(i)} + \sum_{ij}^n A_2^{ij} U_{(i)}U_{(j)} + O(|U|^3) \tag{2.87}$$

it is easily shown that

$$A_2(U_1, U_2) = \sum_i^n A_1^i U_{2(i)} + \sum_{ij}^n A_2^{ij}\,(U_{1(i)}U_{1(j)}) \quad . \tag{2.88}$$

The final simplified form of (2.86) can be expressed as

$$(\omega_{T_2} + a_2\omega^2\omega_\xi) + (s_{2T_1} + a_1 s_{2\xi}\omega + a_1 s_2\omega_\xi) = 0 \tag{2.89}$$

where a_2 is defined to be the constant

$$a_2 = (2L^T B_1\phi_2 + L^T B_2 R)/(L^T R) \tag{2.90}$$

in which B_2 is the matrix

$$B_2 = \sum_{ij}^n A_2^{ij} R_{(i)}R_{(j)} + \sum_i^n A_1^i \phi_{2(i)} \quad . \tag{2.91}$$

If we do make the choice $s_2 = 0$ which is both special and arbitrary we obtain the following evolution equation for the projected variable ω

$$\omega_{T_2} + a_2\omega^2\omega_\xi = 0 \quad . \tag{2.92}$$

Note that had the dependence of ω on T_2 been omitted equation (2.92) would be a nonsensical equation in the sense that it would, in conjunction with (2.77), allow only the solution $\omega = 0$.

Let us summarise our results to date. We have, by making various and arbitrary assumptions, produced an asymptotic representation for the solutions of the general initial value problem (2.21 & 52) supplemented with the asymptotic constraint (2.70). The solution we have constructed is given in terms of a single unknown scalar function ω and is given by

$$U(x,t,\varepsilon) = U_o + \varepsilon\omega(\xi,T_1,T_2)R + \varepsilon^2\omega(\xi,T_1,T_2)^2\vartheta_2 + O(\varepsilon^3) \tag{2.93}$$

where, as we have only collated terms up to $O(\varepsilon^3)$, we have only indicated the dependence of the projected solution ω on the principal higher order 'slow times' T_1 and T_2. The function $\omega(\xi,T_1,T_2)$ is required to satisfy the 'reduced evolution equations'

$$\omega_{T_1} + a_1\omega\omega_\xi = 0 \tag{2.94}$$

$$\omega_{T_2} + a_2\omega^2\omega_\xi = 0 \tag{2.95}$$

which must be solved subject to the projected initial data

$$\omega(\xi,0,0)R = P_R U_{1O}(\xi) \tag{2.96}$$

for U_1 and, in the case of U_2,

$$\omega^2(\xi,0,0)\vartheta_2 = Q_U U_{2o}(\xi) \quad . \tag{2.97}$$

We see that the initial data for different components of the solution vector must be related. The situation we have is exactly that which pertains for the simple wave solution. The choice $s_2 = 0$ would seem to correspond exactly with the simple wave case. The analysis we have just made requires the dimension of the state space to be at least two. Making the minor modifications required we find that the related results for the one dimensional situation are given by

$$u(x,t,\varepsilon) = u_o + \varepsilon\omega(\xi_o,T_1,T_2) + \varepsilon^2 s(\xi_o,T_1,T_2) + O(\varepsilon^3) \tag{2.98}$$

where the functions ω and s satisfy the equations

$$\omega_{T_1} + A_1\omega\omega_{\xi_o} = 0 \tag{2.99}$$

$$\omega_{T_2} + s_{T_1} + A_1 s_{\xi_o}\omega + (A_2\omega^2 + A_1 s)\omega_{\xi_o} = 0 \quad . \tag{2.100}$$

In this case it is possible to have solutions which depend only upon ξ_o and T_1 as that possibility only requires s to satisfy the equation

$$s_{T_1} + A_1 s_{\xi_o}\omega + (A_2\omega + (A_2\omega^2 + A_1 s)\omega_{\xi_o} = 0 \quad . \tag{2.101}$$

If s is zero we obtain the equation determing the T_2-evolution of ω to be

$$\omega_{T_2} + A_2\omega^2\,\omega_{\xi_o} = 0 \quad . \tag{2.102}$$

We note that (2.99 & 102) are exactly the reduced equations (2.94 & 95) of the general system. Our process has reduced the nXn system to an equivalent one dimensional problem

$$\omega_t + (a_o + a_1\omega + a_2\omega^2)\omega_x + 0 \tag{2.103}$$

for times $t<\varepsilon^{-3}$.

To express the equations (2.67) for $n\geq 4$ it is convenient to use a multi-index notation. If $I=(i_1,\ldots,i_n)$ is an n-tuple of integers belonging to the alphabet $\{1,2,3,\ldots,n\}$ define the symbol $U_{(I)}$ to be the monomial

$$U_{(I)} = U_{(i_1)}U_{(i_2)}\cdots U_{(i_n)} \quad . \tag{2.104}$$

Also, given a sequence U_j $(j=1,2,..,)$ of n-tuples, and a multi-index $J=(j_1,\ldots,j_n)$ drawn from the alphabet of positive, nonzero integers, we define the symbol $U_{J(I)}$ by

$$U_{J(I)} = U_{j_1(i_1)}\cdots U_{j_n(i_n)} \quad . \tag{2.105}$$

In terms of these multi-index symbols the Taylor expansion of A(U) can be written in the concise form

$$A(U) = \sum_{|I|\geq 1} A^I U_{(I)} + A_o \tag{2.106}$$

where for multi-indices drawn from either alphabet

$$|I| = i_1+i_2+\ldots+i_n \quad . \tag{2.107}$$

The summation in (2.106) is over all such multi-indices obeying the constraint shown. In this case $|I|\geq 1$. The Taylor coefficients $A_j(U_1,\ldots,U_j)$ are expressible in terms of the A^I by

$$A_j(U_1,..U_j) = \sum_{\substack{|I.J|=j \\ 1\leq|I|\leq j}} A^I U_{J(I)} \tag{2.108}$$

and this allows us to express the general equation (2.67) in the form

$$(A_o-eI)U_{n\xi} = -\Big(\sum_{i=1}^{n-1} \partial_{T_i} U_{n-i} + \sum_{j=1}^{n}\Big(\sum_{\substack{|J.I|=j \\ 1\leq|I|\leq j}} A^I U_{J(I)}\Big)U_{n-j\xi}\Big) \quad . \tag{2.109}$$

It is not clear that it is possible to solve these equations in terms of polynomials in ω and its derivatives with respect to ξ as we can for n=1 and

2. The equation to be solved for U_3 can be written in the form

$$-(A_0-eI)U_{3\xi} = (2B_1\Phi_2 + B_2R)\omega^2\omega_\xi + (\omega_{T_2})R + (\omega^2)_{T_1}\hat{v}_2 \qquad (2.110)$$

where we have the requirements upon the higher 'slow time' evolutions of ω given by equations (2.94-95). Using those equations, (2.110) can be re-expressed as

$$-(A_o-eI)U_3 = (2B_1\Phi_2 + B_2R - a_2R - 2a_1\Phi_2)\omega^2\omega_\xi \ . \qquad (2.111)$$

Therefore if we denote by Φ_3 the unique solution to the equations

$$(A_o-eI)\Phi_3 = \frac{1}{3}[(a_2-B_2)R + 2(a_1-B_1)\Phi_2], P_R\Phi_3 = 0 \qquad (2.112)$$

we can write the general solution to (2.1.110) as

$$U_3 = \omega^3\Phi_3 + s_3R \qquad (2.113)$$

where s_3 is arbitrary. One way of ensuring the absence of such terms is to impose the constraint

$$P_RU_j = 0 \quad \text{for all } j>1 \ . \qquad (2.114)$$

This requirement can be expressed as a condition upon the full state vector U by means of the condition

$$P_R\, U = P_R(U_o + \epsilon\omega R) \qquad (2.115)$$

defining ω to be the unique projection of the state vector U onto the eigenspace E_R as defined by the projection operator P_R.
The reductive perturbation technique that we are developing for the general system is a projection technique similar to that used in bifurcation theory Joseph & Ioos [10]. If we do impose condition (2.115) we see that the parameter ϵ is defined as an 'amplitude parameter' for the E_R projection of the initial data function $U_o(x)$

$$P_R(U_o(x)-U_o) = \epsilon\omega(x) \qquad (2.116)$$

with $\omega(x) = O(1)$ as $\epsilon \to 0$.

Let us suppose that for $1<i\leq n$ the coefficient U_i has the form

$$U_i = v_i(\omega) \quad \text{with } P_Rv_i = 0 \qquad (2.117)$$

and that dependence of ω upon the scaled time T_i is determined by an evolution equation of the form

$$\omega_{T_i} + P_i(\omega)\omega_\xi = 0 \quad (i=1,\ldots,n-1) \ . \qquad (2.118)$$

We have already shown that this is true for i=2 and hope that that result will serve as the foundation of an induction proof. The equation which determines the dependence of ω on T_n requires the function $A_j(U_1,..,U_j)$ for

$1 \leq j < n$. Substituting (2.117) into (2.108) we obtain

$$A_j(U_1,..,U_j) = \sum_{\substack{|I.J|=j \\ 1\leq|I|\leq j}} A^I v_{J(I)}(\omega) \equiv F_j(\omega) \qquad (2.119)$$

where this equation defines the function $F_j(\omega)$ and we have extended the notation expressed in equations (2.117-8) by defining

$$v_1 = R\omega \quad . \qquad (2.120)$$

From equation (2.109) we are now able to determine the T_n-evolution equation as

$$L^T(\omega_{T_n} R + \sum_{i=1}^{n-1} \partial_{T_i}(v_{n+1-i}(\omega)) + \sum_{j=1}^{n} F_j(\omega)(v_{n+1-j}(\omega))_\xi) = 0 \qquad (2.121)$$

which has the form

$$\omega_{T_n} + P_n(\omega)\omega = 0 \qquad (2.122)$$

with $P_n(\omega)$ defined recursively by

$$P_n(\omega) = (\sum_{j=1}^{n} L^T F_j(\omega) v'_{n+1-j}(\omega))/(L^T R) \quad . \qquad (2.123)$$

In order to determine $F_j(\omega)$ we need $v_j(\omega)$ for $j=1,..,n$. That same information is sufficient to determine $P_n(\omega)$ also. Our basic assumption (2.117) was that these were known. To complete the induction proof we have only to show that U_{n+1} can be represented in the form (2.117). The equation that must be solved to obtain U_{n+1} is given by

$$-(A_o - \sigma I)U_{n+1\xi} = (\sum_{j=1}^{n} (F_j(\omega) - P_j(\omega))v'_{n+1-j}(\omega))\omega_\xi \quad . \qquad (2.124)$$

The operator $Q_R(A_o - \sigma I)Q_R : N_R \to N_R$ is, by the assumed hyperbolic nature of A_o, a nonsingular invertible matrix. Denote the inverse by K_o. As, by construction, the right hand side of (2.124) belongs to N_R it follows that the unique solution of (2.124) in N_R is simply the inverse image of the righthand side of that equation under K_o. This produces the result

$$v_{n+1}(\omega) = \sum_{j=1}^{n} (-\int_o^\omega K_o((F_j(\omega) - P_j(\omega))v'_{n+1-j}(\omega))d\omega) \qquad (2.125)$$

and completes our induction proof. Each of the vectors $v_j(\omega)$ is a vector valued polynomial in ω of degree j and $P_j(\omega)$ is a polynomial function of ω of degree j. The final complete asymptotic expansion for the solution vector U is given by

$$U(x,t,\epsilon) = U_o + \epsilon\omega(\xi,T_1,..,T_\infty)R + \sum_{j=2}^{\infty} v_j(\omega)\epsilon^j \qquad (2.126)$$

where the function ω, which is proportional to the projection of $(U-U_o)$ onto the eigenvector R, must satisfy the heierachy of evolution equations

$$\omega_{T_n} + P_n(\omega)\omega_\xi = 0 \quad . \qquad (2.127)$$

Our next important observation, that can be proved by induction, is that the basic functions $F_j(\omega), P_j(\omega)$ and $v_j(\omega)$ are simple monomials.
We are able to write

$$F_j(\omega) = \omega^j \hat{F}_j \quad P_j(\omega) = a_j \omega^j \quad v_j(\omega) = \omega^j \hat{v}_j \tag{2.128}$$

and the T_n-evolution equations are given by

$$\omega_{T_j} + a_j \omega^j \omega_\xi = 0 \quad . \tag{2.129}$$

Note that all of these flows commute and so there is a solution for ω.
The final representation for $U(x,t,\epsilon)$ is a Taylor series in $(\epsilon\omega)$

$$U(x,t,\epsilon) = U_o + (\epsilon\omega))R + \sum_{i=2}^{\infty} \Phi_j (\epsilon\omega)^j \quad . \tag{2.130}$$

From (2.128-29) we see that we can forget about the expansion methods we have used. Equations (2.129) are the 'reductive perturbation' equations for the scalar partial differential equation

$$\omega_t + a(\omega)\omega_\xi = 0 \tag{2.131}$$

together with the initial condition

$$\omega(x,t=0) = u_o + \epsilon\omega_o(x) = P_R U_o(x,\epsilon) \cdot \tag{2.132}$$

The function $a(\omega)$ is defined by

$$a(\omega) = (e + \sum_{j=1}^{\infty} a_j \omega^j) \quad . \tag{2.133}$$

What we have shown is that any solution of the scalar equation (2.131) generates a solution of the full system (2.21) given by,

$$U(x,t) = U_o + \sum_{j=1}^{\infty} \Phi_j \omega^j \tag{2.134}$$

and consequently we refer to (2.131) as a CANONICAL FORM FOR THE SYSTEM (2.21). We would like to adopt the view that the reductive perturbation method is simply a way of determining that canonical form.

From the structure of (2.131 & 134) it is clear that the solution we have obtained is nothing more than the simple wave solution (2.44) based upon the use of a coordinate system determined by A_o. The function $a(\omega)$ is nothing more than the eigenvalue of the matrix $A(U_o+\omega)$ previously denoted by $E(U)$ in (2.54). Similarly the Φ_j are the Taylor coefficients of the eigenvectors of that same matrix expanded as a power series in ω.

3. DISPERSIVE AND DISSIPATIVE NONLINEAR SYSTEMS

We have considered in the last section systems modelled by equation (2.21). A significant property of the solutions of such equations was the existence of a breaking time t_b after which solutions do not exist. This property is associated with the existence of discontinuous solutions corresponding to shock waves.

In physical systems genuinely discontinous phenomena do not usually occur. This is because such transitions are associated with rapidly changing functions and high field gradients. In such regimes nonlinear effects, involving higher partial derivatives, play an important role. There is a basic division of evolution equations into those which are dissipative and those which are dispersive. In a dissipative linear system individual Fourier modes decay exponentially in time and energy is lost from the system. Physically, it is frictional types of force that are involved and they are modelled by derivative terms of even order. A generalization of the basic system (2.21) which incorporates dissipative effects is given by the (nxn) nonlinear partial differential equation

$$U_t + A(U)U_x + K_1(U)(K_2(U)U_x)_x = 0 \tag{3.1}$$

where $A(U)$ and $K_i(U)(i=1,2)$ are (nxn) matrix functions.

In a dispersive linear system individual Fourier modes have phases which vary with time without affecting their amplitude. This leads to interference effects and a gradual spreading out of wave packets. Physically, dispersion is associated with cooperative effects, with a sharing of the available energy between different Fourier modes rather than a loss of energy from the system. Such mechanisms are associated with derivative terms of odd order. A generalization which incorporates dispersive effects is given by the (nxn) nonlinear partial differential equation

$$U_t + A(U)U_x + K_1(U)[K_2(U)(K_3(U)U_x)_x]_x = 0 \tag{3.2}$$

where $A(U)$ and $K_i(U)$ $(i=1,2,3)$ are (nxn) matrix functions.

Let us consider first equation (3.1) where for convenience we take $K_2(U)=I$ so that we will consider

$$U_t + A(U)U_x + K(U)U_{xx} = 0 \quad . \tag{3.3}$$

What we are trying to do is to determine the analogue of simple waves for systems that can be expressed in such a form. The substitution of an ansatz such as (2.44) produces an inhomogeneous form of that equation and it is completely unclear how that equation should be analysed. In the previous section we showed that by the introduction of the formal parameter ϵ it was possible to construct the simple wave solutions in a manner independent of the ansatz (2.44). This suggests that our search for an appropriate generalization of those solutions to the system (3.2) or (3.1) might be based on a generalization of that technique. As a first guess we might think that the second derivative terms should be small and that we could develop our theory as a singular perturbation of the system (2.21). However, the outer expansion of such a solution would be determined by the system (2.22) with its inherent shock solution problems. To obtain a balance between the three terms in (3.3) we introduce a new set of variables.

$$\xi = \epsilon^a(x-\lambda t) \quad , \quad t_i = \epsilon^{b+i}t \tag{3.4}$$

and assume that the solution vector can be expressed in the asymptotic form

$$U(x,t;\epsilon) = U_o + \sum_{j=1} U_j(\xi,t_1,t_2,\ldots)\epsilon^j \tag{3.5}$$

with the associated initial function

$$U(x,t=0,\epsilon) = U_0 + \sum_{j=1} U_{j0}(\xi)\epsilon^j \quad . \tag{3.6}$$

The real numbers a and b are to be determined so that the leading t_1-evolution equation involves a balance between all three types of term in (3.3). The substitution of the variables (3.4) into (3.3) gives

$$\epsilon^b \sum_{i=1} U_{t_i} \epsilon^i + (A(U) - \lambda I)U_\xi \epsilon^a + K_1(U)U_{\xi\xi}\epsilon^{2a} = 0 \quad . \tag{3.7}$$

The matrix function A(U) is still assumed to have the Taylor expansion (2.106). We still wish to have U defined as the same projection ω as in the first order case considered in section 2. This means that we must choose $b > a$. If both $K(U_o)$ and $A_1(U_2)$ are non zero, we will obtain a t_1-evolution equation incorporating shock eliminating higher order derivatives if we make the choice

$$1 + b = 1 + a = 2a \tag{3.8}$$

The scaled variables are therefore

$$\xi = \epsilon(x-\lambda t) \quad t_i = \epsilon^{i+1}t \quad . \tag{3.9}$$

The representation (3.5), together with an assumed regularity of the matrix function K(U), allows us to write

$$K(U) = \sum_{i=0}^{\infty} K_i(U_1,..,U_i)\epsilon^i \quad . \tag{3.10}$$

and the equation obtained by equating the coeffecient of ϵ^n to zero in equation (3.7) has the appearance

$$\sum_{i=1}^{n} U_{it_{n-i+1}} + (A_o-\lambda I)U_{n+1,\xi} + \sum_{i=1}^{n} A_i(U_1,,U_i)U_{n-i+1,\xi}$$

$$+ \sum_{\substack{i,j=0 \\ i+j=n}}^{\infty} K_i(U_1,,U_i)U_{j\xi\xi} = 0 \tag{3.11}$$

for $n > o$. There are also negative powers of ϵ once the equation has been divided through by ϵ^2, these lead to

$$(A_o - \lambda I)U_{1\xi} = 0 \tag{3.12}$$

which gives

$$U_1 = u_1(\xi,t_1,t_2,..)R \tag{3.13}$$

where R is the eigenvector of A_o with eigenvalue λ.
For n=1 we obtain

$$U_{1t_1} + (A_o-\lambda I)U_{2\xi} + A_1(U_1)U_{1\xi} + K_oU_{1\xi\xi} = 0 \quad . \tag{3.14}$$

Premultiplying by the left eigenvector of A_o, L^T, we obtain the compatibility condition for (3.14)

$$u_{1t_1} + a_1u_1u_{1\xi} + \mu u_{1\xi\xi} = 0 \tag{3.15}$$

which is the familiar Burgers equation, with the coefficients defined by

$$a_1 = \frac{L^T B_1 R}{L^T R} \tag{3.16}$$

$$\mu = \frac{L^T K_o R}{L^T R} \quad . \tag{3.17}$$

On the other hand, solving (3.14) gives

$$U_2 = u_2 R + u_1^2 V_2 + u_{1\xi} W_2 \tag{3.18}$$

where V_o and W_o are the unique solutions to

$$(A_o - \lambda I) V_2 = \tfrac{1}{2}(Q_1 - B_1)R \ , \ P_R V_2 = 0 \tag{3.19}$$

$$(A_o - \lambda I) W_2 = (U - K_o)R \ , \ P_R W_2 = 0 \quad .$$

The function u_2 is dependent on the variables $\{\xi, t_1, \ldots\}$. For n=2, equation (3.11)

$$U_{1t_2} + U_{2t_1} + (A_o - \lambda I)U_{3\xi} + A_1(U_1)U_{2\xi} + A_2(U_1, U_2)U_{1\xi} +$$
$$+ K_o U_{2\xi\xi} + K_1(U_1)U_{1\xi\xi} = 0 \quad . \tag{3.20}$$

The compatibility condition of (3.20) is

$$u_{2t_1} + a_1(u_1 u_2)_\xi + \mu u_{2\xi\xi} = -u_{1t_2} - a_2 u_1^2 u_{1\xi} + \sigma_1 u_1 u_{1\xi\xi}$$
$$+ \sigma_2 u_{1\xi}^2 + \sigma_3 u_{1\xi\xi\xi} \tag{3.21}$$

where a_2 is as in equation (2.90) and the constants $\sigma_1, \sigma_2, \sigma_3$ are

$$\sigma_1 = -(L^T B_1 W_2 + 2L^T K_o V_2 + L^T \textstyle\sum K_1^i R_{(i)} R)/L^T R$$
$$\sigma_2 = -(L^T \textstyle\sum A_1^i W_{2(i)} R + 2L^T K_o V_2)/L^T R$$
$$\sigma_3 = -L^T K_o W_2 / L^T R \quad . \tag{3.22}$$

Defining the operator $L(u_1)$ by

$$L(u_1)V = V_{t_1} + a_1(u_1 V)_\xi + \mu V_{\xi\xi}$$

allows us to write (3.21)as

$$L(u_1)u_2 = u_{1t_2} - a_2 u_1^2 u_{1\xi} + \sigma_1 u_1 u_{1\xi\xi} + \sigma_2 u_{1\xi}^2 + \sigma_3 u_{1\xi\xi\xi} \tag{3.23}$$

Indeed, to $O(\epsilon^n)$ equation (3.11) may be solved to give

$$U_{n+1} = u_{n+1} R + \hat{V}_{n+1} \quad . \tag{3.24}$$

where U_{n+1} is arbitrary and $\hat{V}_{n+1}$ is a vector valued function belonging to N_R the compatibility condition to the same order is

$$L(u_1)u_n = -u_{it_n} - \sum_{i=2}^{n-1} u_{it_{n-i+1}} + S^{(n)}(u_1,,u_{n-1}) \quad . \tag{3.25}$$

From (3.5) we see that the asymptotic form of U is now

$$U = U_o + \epsilon u_1 R + \sum_{j=2} \epsilon^j u_j R + \sum_{j=2} \hat{v}_j \epsilon^j \tag{3.26}$$

and thus

$$P_R(U-U_o) = \epsilon u_1 + \sum_{j=2} \epsilon^j u_j \tag{3.27}$$

which means that ϵ cannot be interpreted as an amplitude parameter. It might be thought possible to set all of the u_i's to zero as was done in section 2. Unfortunately, if this is allowed, the resulting flows do not commute.

We require to solve equations (3.15) and (3.25). Consider first Burgers equation (3.15). As it stands this is a non linear equation but by the Cole-Hopf transformation

$$\frac{a_1 u_1}{2\mu} = \frac{\partial}{\partial \xi} (\ell n \phi) \tag{3.28}$$

it may be transformed into the heat equation

$$\phi_{t_1} + \mu \phi_{\xi\xi} = 0 \quad . \tag{3.29}$$

The solution to (3.29) is then easily found to be

$$\phi(\xi,t_1) = \int_{-\infty}^{\infty} \Gamma(\xi-y,t_1)\phi(y,o)dy \tag{3.30}$$

where the function $\Gamma(\xi-y,t_1)$ is given by

$$\Gamma(\xi-y,t_1) = \begin{cases} \dfrac{e^{-(\xi-y)^2/4ht_1}}{\sqrt{4ht_1}} & , \ -\infty<\xi<\infty, \ t_1>0 \\ 0 & , \ -\infty<\xi<\infty, \ t_1\leq 0 \end{cases} \tag{3.31}$$

where $h=-\mu$ ($\mu<o, h>o$).

Equation (3.30) may be further simplified by making a change of variable

$$\eta = \frac{y-\xi}{\sqrt{4ht_1}} \tag{3.32}$$

and with $\phi(y,o) = f(y)$ we get

$$\phi(\xi,t_1) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} f(\xi + y\sqrt{4ht_1}) \, e^{-y^2} dy \ . \tag{3.33}$$

An interesting initial function is

$$\phi(\xi,o) = f(\xi) = 1 + \xi + \xi^2 + \ldots + \xi^n \tag{3.34}$$

From (3.33) it can be seen that

$$\phi(\xi,t_1) = \sum_{i=o}^{n} \theta_i(\xi,t_1) \tag{3.35}$$

where the $\theta_i(\xi,t_1)$ (i=1,..n),called heat polynomials, may be generated by

$$e^{xz + htz^2} = \sum_{n=o}^{\infty} \frac{\theta_n(x,t)z^n}{n!} \tag{3.36}$$

or more conveniently, from

$$\theta_n(x,t) = n! \sum_{m=o}^{[n/2]} \frac{(ht)^m}{m!} \frac{x^{n-2m}}{(n-2m)!} \tag{3.37}$$

where [n/2] ≡ largest integer $\leq n/2$.

For example

$$\theta_o = 1,\ \theta_1 = x\ ,\ \theta_2 = x^2 + 2ht\ ,\ \theta_3 = x^3 + 6nth$$
$$\theta_4 = x^4 + 12x^2ht + 12h^2t^2 \quad \text{etc.} \tag{3.38}$$

It must, of course, be remembered that we are not solving an initial solve problem for the Burgers equation. Rather we are hoping to isolate solutions having the simple wave property of the preceding chapter. The introduction of dissipative terms into (2.21) was necessary in order to prevent discontinuous shock type solutions appearing at some finite time t_b. It is then possible for travelling wave solutions to exist. These have the functional form

$$u(x,t) = u(x-\lambda t) = u(\zeta) \tag{3.39}$$

the explicit form of which may be found on substituting (3.39) into the Burgers equation. This leads to the equation

$$-\lambda u_\zeta + a u u_\zeta + \mu u_{\zeta\zeta} = 0\ . \tag{3.40}$$

Integration gives

$$-\lambda u + \frac{1}{2} a u^2 + \mu u_\xi = A \tag{3.41}$$

which may be rewritten as

$$\frac{-2\mu u_\xi}{a} = \left[u^2 - \frac{2\lambda}{a} u - \frac{2A}{a}\right] = (u-\alpha)(u-\beta) \tag{3.42}$$

where

$$\alpha = \frac{\lambda}{a} + \sqrt{\frac{\lambda^2}{a^2} + \frac{2A}{a}}$$

$$\beta = \frac{\lambda}{a} - \sqrt{\frac{\lambda^2}{a^2} + \frac{2A}{a}} \quad . \tag{3.43}$$

We require the roots to be real which implies that

$$\frac{\lambda^2}{a^2} + \frac{2A}{a} > 0$$

$$\alpha > \beta \quad . \tag{3.44}$$

Integrating (3.42) gives

$$u(x-\lambda t) = u(\zeta) = \frac{\lambda}{a} - \sqrt{\left(\frac{\lambda^2}{a^2} + \frac{2A}{a}\right)} \tanh\left[\frac{a}{2\mu}\sqrt{\left(\frac{\lambda^2}{a^2} + \frac{2A}{a}\right)}\,\zeta\right] \tag{3.45}$$

where the constant of integration has been chosen subject to the conditions

$$u(\zeta) \to \beta \text{ as } \zeta \to +\infty$$

$$u(\zeta) \to \alpha \text{ as } \zeta \to -\infty \tag{3.46}$$

The function in (3.45) describes a wave that varies continuously from a value of α at $\zeta = -\infty$ to a value of β at $\zeta = +\infty$. As such it represents a smoothed out shock wave and is often called the Burgers shock wave. The velocity λ is given by the Rankine-Hugoniot relation

$$\lambda = \frac{a}{2}(\alpha + \beta) \quad . \tag{3.47}$$

The effect of removing the dissipation is determined by letting $\mu \to 0$ in equation (3.45). We obtain the discontinuous shock solution

$$u_d = \begin{cases} \alpha \ , x < \lambda t \\ \\ \beta \ , x > \lambda t \end{cases} \quad . \tag{3.48}$$

Thus, as expected, when the dissipation is non zero ($\mu \neq 0$) the discontinuous shock solution (3.48) does not appear. This marks an improvement upon the results of the previous section since in that case discontinuous shock solutions were shown to appear after some finite time t_b. From (3.45) it can be seen the $u(\zeta)$ is a rightward travelling wave which undergoes no change in shape as it travels. It seems natural to consider this to be the far-field solution to the Burgers equation. These considerations motivate the introduction of the shock wave boundary conditions which we will take to be

$$u_1 \to K \text{ as } \xi \to -\infty$$

$$\to 0 \text{ as } \xi \to +\infty \tag{3.49}$$

and will be the conditions that solutions to (3.15) must satisfy. Corresponding to (3.49), the function ϕ obeys the corresponding asymptotic boundary conditions

$$\phi \sim O(e^{-k\xi}) \quad \text{as} \quad \xi \to -\infty \quad \left(\kappa = \tfrac{1}{2}\frac{a_1 K}{n} > 0\right)$$

$$\sim O(1) \quad \text{as} \quad \xi \to +\infty \tag{3.50}$$

Note that under such boundary conditions it is not permissible to choose an initial function of the form (3.34). Such solutions, because they are unbounded in time, are called secular solutions.

Now that we have imposed appropriate boundary conditions on the solution function $u_1(\xi,t_1)$ let us turn now to the question of solving equations (3.25). In order to gain a clearer appreciation of the significance of the u_{1t_n} terms in (3.25) suppose for the moment that there is no t_n-dependence ($n\geq 2$) in (3.25). The compatibility conditions then become

$$L(u_1)u_n = S^{(n)}(u_1,..,u_{n-1}) \tag{3.51}$$

where for notational convenience we shall not distinguish between u_n in (3.25) and (3.51). If $S^{(n)}(u_1,..,u_{n-1})$ contains a term $\bar{S}^{(n)}$, say, that satisfies the linearized Burgers equation, that is, the homogeneous equation associated with (3.51), then the solution for u_n will contain a part that increases without bound in t_1. As we have already seen this is called a secular term. In short, if $S^{(n)} = \bar{S}^{(n)} + g^{(n)}$ and $L(u_1)\bar{S}^{(n)} = 0$ then obviously $L(u_1)(t\bar{S}^{(n)} + f^{(n)}) = \bar{S}^{(n)} + L(u_1)f^{(n)}$. If $f^{(n)}$ is such that $L(u_1)f^{(n)} = g^{(n)}$ there the solution will be $u_n = t_1\bar{S}^{(n)} + f^{(n)}$ and is secular.

Such a result will be recognized as reminiscent of a forced oscillation type problem. For example a mass attached to a spring, of spring constant k, undergoing oscillation due to a forcing term F obeys the equation

$$m\ddot{x} + kx = F \tag{3.52}$$

If the forcing term F contains a part F_R that oscillates with the natural frequency of the system then it is easily verified that the amplitude of the oscillations will increase without bound. This is called resonance. By direct analogy we may refer to $\bar{S}^{(n)}$ as a resonant term.

Most techniques for eliminating secular behaviour are based on multiple scaling (Nayfeh [11]). This involves the introduction of extra space and time scales into the solution function resulting in a freedom to eliminate the secular causing resonant terms. In our formulation of the reductive perturbation method the time scales $(t_2,..)$ have been introduced a priori. Differentiating the Burgers equation with respect to t_n shows that u_{1t_n} is a solution to the linearized Burgers equation and therefore on the right hand side of (3.25) is a secular producing term. However we anticipate the occurence of resonant terms in $S^{(n)}(u_1,.., u_{(n-1)})$ and by choosing an appropriate t_n-evolution of u_1 we may eliminate such resonances.

Recalling equation (2.127) of the previous section, we inquire whether it is possible to find a corresponding set of commuting flows for the dependent variable u_1 in this dissipative case. The Burgers equation, invariant under the scaling symmetry

$$u_1 \to \lambda u, \quad ,\xi \to \lambda\,\xi \quad , \; t\to \lambda^2\, t \tag{3.53}$$

leads us to define the weighting function W, assigning to a monomial in the variables W_i, where $W_{i+1}\equiv(\frac{\partial}{\partial\zeta})^i u_1$, an integer as follows. Each of the W_i is assigned the following weight

$$W(W_i) = i \quad . \tag{3.54}$$

Also if I and J are n-tuples we have

$$W\left(W_I^J\right) = \left|I.J\right| \quad . \tag{3.55}$$

We also define

$$W(u_j) = j \quad . \tag{3.56}$$

We will now show that equations (2.127) may be extended to the dissipative case to give the infinite set of commuting flows

$$W_{1t_n} + P_{n+2}(W_1,\ldots,W_{n+2}) = 0 \tag{3.57}$$

where $P_{n+2}(W_1,..,W_{n+2})$ is a differential polynemial in W_I of weight n + 2. It is convenient to rescale u as

$$u_1 = \frac{-2\mu\nu_1}{a_1} \quad . \tag{3.58}$$

Burgers equation then takes the form

$$\nu_{1t_1} - 2\mu\nu_1\nu_\xi + \mu\nu_{1\xi\xi} = 0 \quad . \tag{3.59}$$

Defining the operators M and N by

$$Mf = f_\xi - \nu_1 f\,, \quad Nf = f_{t1} - 2\mu\nu_1 f_\xi + \mu f_{\xi\xi} \tag{3.60}$$

allows equation (3.59) to be written in terms of the commutator of M and N

$$[M,N] = 0 \quad . \tag{3.61}$$

The C-densities C_n are defined as polynomials in $\{\nu_I\}$ of weight n lying in the kernel of N

$$NC_n = 0 \quad . \tag{3.62}$$

It is easily seen that

$$L(\nu_1)\,\nu_{1t_n} = 0 \quad . \tag{3.63}$$

Writing

$$\nu_{1t_n} = F_{n\xi}$$

we find from (3.63) that F_n satisfies

$$F_{nt_1} - 2\mu\nu_1 F_{n\xi} + \mu F_{n\xi\xi} = 0 \quad . \tag{3.64}$$

This is the adjoint equation to the linearized Burgers equation. Equation (3.64) is just (3.62) with C_n replaced by F_n. Then the natural choice for the t_n evolution of ν_1 is obtained by choosing

$$P_{n+2}(W_1,\ldots,W_{n+2}) = \alpha_{n+2}\, C_{n+1,\xi} \tag{3.65}$$

where α_{n+2} is a normalization constant to be determined by our problem.

Using equation (3.61) we may determine a recurrence relation for the C_n-densities. We have

$$[M,N]\, C_n = 0 = MNC_n - NMC_n = -N(MC_n) \Rightarrow N(MC_n) = 0 \tag{3.66}$$

Since the operator M increases the weight of a differential polynomial by one we see that a polynomial of weight (n+1) is given by

$$C_{n+1} = MC_n \quad . \tag{3.67}$$

Observing that the function ν_1 of weight one satisfies (3.62) and is $C_1 = \nu_1$ allows us to generate, using (3.67), the hierarchy of C-densities. We find, for C_2 and C_3

$$C_2 = \nu_2 - \nu_1^2 \quad , \quad C_3 = \nu_3 - 3\nu_1\nu_2 + \nu_1^3 , \nu_{n+1} = (\nu_1)_{(n\xi)} \, . \tag{3.68}$$

We have thus determined the t_n-evolution of W to be

$$W_{t_n} + \alpha_{n+2}\, C_{n+1,\xi} = 0 \tag{3.69}$$

and it may be shown that the flows generated by (3.69) do indeed commute.

Equation (3.25) becomes on substituting for W_{1t} from (3.69)

$$L(W_1)u_n = \alpha_{n+2}\, C_{n+1,\xi} - \sum_{i=2}^{n-1} u_{it_{n-i+1}} + S^{(n)}(W_1, \ldots, u_{n-1}) \, . \tag{3.70}$$

The elimination of resonant terms in $S^{(n)}(W_1, \ldots, u_{n-1})$ is then seen to be accomplished by an appropriate choice of α_{n+2}. The first two flows from (3.69) are

$$W_{1t_1} + \frac{a_1}{2\mu}\alpha_3 \left(W_{1\xi} + \frac{a_1}{2\mu} W_1^2\right)_\xi = 0 \tag{3.71}$$

$$W_{1t_2} \frac{a_1\alpha_4}{2\mu} \left(W_{1\xi\xi} + \left(\frac{3a_1}{2\mu}\right) W_1 W_{1\xi} + \left(\frac{a_1}{2\mu}\right)^2 W_1^3\right)_\xi = 0 \tag{3.72}$$

the constant α_3 in (3.71) is given by $\alpha_3 = -(2\mu^2/a_1)$ in order to convert (3.71) into Burgers equation (3.15). The determination of α_4 is not quite so obvious. One possibility is to choose α_4 by requiring that as $\mu \to 0$ equation (3.72) reduces to $W_{1t_2} + a_2 W_1^2 W_{1\xi} = 0$. At the moment it is not clear that such a condition is compatible with the requirement that α_4 must be chosen in order to eliminate the resonant part of $S^{(n)}$. That this is possible we now show. The operator $L(u_1)$ terms of ν_1 is simply

$$L(\nu_1)u_n \equiv u_{t_1} - 2\mu\,(\nu_1 u_n)_\xi + \mu u_{n,\xi\xi} = G^{(n)} \tag{3.73}$$

where $G^{(n)}$ is the right hand side of (3.25). The introduction of $R^{(i)}$ through the change of variable

$$u_i = (R^{(i)}/\phi)_\zeta \tag{3.74}$$

transforms equation (3.72) into

$$R^{(n)}_{t_1} - hR^{(n)}_{\xi\xi} = \phi \int^{\xi} G^{(n)}\, d\xi \tag{3.75}$$

where $G^{(n)} = -W_{1t_n} - \sum_{i=2}^{n-1} u_{i_1} t_{n-i+1} + S^{(n)}$, and $h = -\mu$. This is just the inhomogeneous heat equation with source term $\int^{\xi} G^{(n)}\, d\xi$. For n=2 equation (3.74) takes the form

$$R^{(2)}_{t_1} - hR^{(2)}_{\xi\xi} = \phi \left[\alpha_4 C_3 - \frac{a_2}{3} u_1^3 + \sigma_1 \int^{\xi} u_1 u_{1\xi\xi} d\xi + \sigma_2 \int^{\xi} u_{1\xi}^2 \, d\xi + \sigma_3 u_{1\xi\xi} \right] \quad (3.76$$

where the σ_i are constants which determine $S^{(2)}$ in (3.22).
The C-densities can be expressed in terms of the associated heat potential ϕ by

$$C_n = -(\phi_{(n\xi)}/\phi) \quad . \tag{3.77}$$

This allows us to immediately identify the secular nature of the first term on the right hand side of (3.75). The problem is to identify the secular component of the remainder of the right hand side that must cancel with the first term. To determine that secular component we restrict ourselves to a special class of solutions. Consider solutions that are related to heat potentials with the following asymptotic form,

$$\phi = \begin{cases} 0\,(e^{-\kappa\theta}) & \text{for } \theta \to -\infty \\ 0\,(1) & \text{for } \theta \to +\infty \end{cases} \quad . \tag{3.78}$$

Without loss of generality we can use the special choice

$$\Phi = (1 + e^{-\kappa\theta}) \text{ where } \theta = (\xi - \kappa h t_1) \tag{3.79}$$

in order to determine the special choice of the coefficients α_n that will give us the best possible perturbation expansion for the full solution function.
From (3.78) we obtain the following asymptotic limiting forms,

$$u_1 = \begin{cases} 0\,(1) \\ 0\,(e^{-\kappa\theta}) \end{cases} \qquad u_{1n} = \begin{cases} 0\,(e^{\kappa\theta}) \\ 0\,(e^{-\kappa\theta}) \end{cases} \text{ as } \theta \to \begin{cases} -\infty \\ +\infty \end{cases} \quad . \tag{3.80}$$

The asymptotic form of the right hand side of (3.75) is then easily calculated to be

$$-\kappa^3(\alpha_4 - \frac{a}{3}\, 2\, (\frac{2\mu}{a_1})^3) e^{-\kappa\theta} \quad \text{as } \theta \to -\infty \quad . \tag{3.81}$$

Thus the function $R^{(2)}$ will be asymptotically secular unless we choose α_4 to make this term zero. A related procedure can be applied to (3.70) to arrive at the result

$$\alpha_{n+2} = \frac{a_n}{(n+1)} (2\mu/a_1)^{n+1} \quad . \tag{3.82}$$

Thus our commuting flows are given by

$$W_{t_n} + \alpha_{n+2} C_{n+1,\xi} = 0 \tag{3.83}$$

with the α_{n+2} determined as in (3.81). In terms of the associated heat potential ϕ

corresponding to W_1 we have the flows

$$\phi_{t_n} - (2\mu/a_1)\,\alpha_{n+2}\,\phi_{(n+1)\xi} = 0 \quad . \tag{3.84}$$

These are the perturbation equations that result from applying the reductive perturbation method to the linear equation

$$B_t + \lambda B_\xi - \left(\frac{2\mu}{a_1}\right)\left(\sum_{i=2}^{\infty} \alpha_{i+1}\, B_{(ix)}\right) = 0 \quad . \tag{3.85}$$

Consequently we can regard this equation as the canonical form for equation (3.3). If we define

$$w = (2\mu/a_1)\ln(B)_\xi \tag{3.86}$$

the function w satisfies the equation

$$w_t + \lambda w_\xi - \sum_{i=1}^{\infty} (2\mu/a_1)^{i+1} C_{i+1}(-2\mu w/a_1)\left(\frac{a_1}{i+1}\right) = 0 \tag{3.87}$$

where the coefficients a_i are those in (2.133) and (3.86) is the generalisation of that canonical equation to the dissipative case. It is most natural to regard this equation as THE CANONICAL FORM FOR THE SYSTEM (3.3).

There still remains a considerable problem. We have obtained the best behaviour possible for the perturbation series as $\xi \to -\infty$ but $\xi \to +\infty$ yields only

$$u_{n+1}/u_n = 0\ (\xi) \text{ as } \xi \to \infty \quad . \tag{3.89}$$

This means that these expansions are only useful near to the shock front. In the following section we will propose an alternative approach to the determination of higher order corrections based, not on series, but on a process of iteration.

4. AN ITERATIVE APPROACH TO HIGHER ORDER CORRECTIONS:

In this section we attempt to place the reductive perturbation technique on a more rigorous foundation. To do so we will prove the existence of invariant manifolds by means of fixed point methods. Not only will this establish the existence of such manifolds but it will provide an iterative procedure for constructing a convergent sequence of approximants. We regard these higher approximants as the most appropriate higher order corrections.

Consider the model set of equations of section three.

$$U_t + A(U)U_x + K(U)U_{xx} = 0 \tag{4.1}$$

where A and K are matrix functions of an n-dimensional vector U. The simplest solutions of (4.1) are the constant solutions

$$U = U_o \text{ a constant} \quad . \tag{4.2}$$

Let us consider solutions which are in some sense 'close' to U_o by writing U in the form

$$U = U_o + W \quad . \tag{4.3}$$

Intuitively we are thinking of W as being 'small' but we make no approximations. We assume, as in earlier sections, that the matrix $A(U_o)$ has n distinct eigenvalues $a_o,..,a_{n-1}$ with associated eigenvectors $R,R_1,..,R_{n-1}$,

$$A(U_o)R = eR \qquad A(U_o)R_i = e_iR_i \ . \tag{4.4}$$

A single eigenvalue e is selected from those available and a new variable ξ defined by

$$\xi = (x-et) \tag{4.5}$$

is introduced in place of x. If the function W defined in (4.3) is regarded as a function of ξ and t it must satisfy the equation

$$W_t + (A(U_o + W) - eI)\ W_\xi + K(U_o + W)\ W_{\xi\xi} = 0 \ . \tag{4.6}$$

We now suppose that the functions A (U) and K(U) are differentiable at U_o and that they have representations of the form

$$A(U) = A(U_o) = A_1(W) + B(W) \tag{4.7}$$

$$K(U) = K(U_o) + K_1(W) = C(W) \tag{4.8}$$

where the remainder terms B (W) and C(W) are both $o\ (|W|)$ for $|W|\to 0$. Our next step is to project equation (4.6) down onto the subspaces E_R and N_R defined in section one. We write for W the decomposition

$$W = w(\xi,t)R + q(\xi,t) \tag{4.9}$$

where $P_Rq = 0$. Equation (4.6) can then be recast into the form

$$Rw_t + q_t + (A(U_o) - eI)\ q_\xi + A_1(R)Rww_\xi + F[w,q] + K(U_o)Rw_{\xi\xi} = 0 \tag{4.10}$$

where the functional F[w,q] is defined by

$$F[w,q] = (A(U_o + wR + q) - eI)\ (Rw_\xi + q_\xi) - (A(U_o) - eI)q_\xi - A_1(R)Rww_\xi +$$

$$K(U_o + Rw + q)\ (Rw_{\xi\xi} + q_{\xi\xi}) - K(U_o)Rw_{\xi\xi} \ . \tag{4.11}$$

If we now apply the projection operators P_R and Q_R to equation (4.10) we obtain

$$w_t + aww_\xi + kw_{\xi\xi} = -f_P[w,q] \tag{4.12}$$

$$q_t + K_o^{-1}q_\xi = -f_Q[w,q] - bww_\xi - cw_{\xi\xi} \tag{4.13}$$

where the constants a, b, c and k are defined by

$$a = \frac{L^TA_1(R)R}{L^TR} \qquad b = Q_RA_1(R)R \qquad c = Q_RK(U_o)R \qquad k = \frac{L^TK(U_o)R}{L^TR} \tag{4.14}$$

and the functionals f_P and f_Q are defined by the decomposition of F[w, q] onto the subspaces E_R and N_R given by

$$F[w,q] = R(f_P[w,q]) + f_Q\ [w,q] \tag{4.15}$$

We want to think of the contribution of F to the equations (4.12-13) as 'small' without explicity neglecting any particular terms. To help us maintain that picture clearly in our mind we introduce the scaled variables η, τ, Θ, and Ψ defined by

$$\eta = \varepsilon\xi, \quad \tau = \varepsilon^2 t, \quad w = \varepsilon\Theta, \quad q = \varepsilon^2\psi \tag{4.16}$$

where ε is an arbitrary real parameter introduced into the problem for the first time at this juncture. In terms of these new scaled variables the equations (4.12-13) take the form,

$$\Theta_\tau + a\Theta\Theta_\eta + k\Theta_{\eta\eta} = -\varepsilon h_p[\Theta\Psi,\varepsilon] \tag{4.17}$$

$$\varepsilon\Psi + K_o^{-1}\Psi_\eta = -b\Theta\Theta_\eta - c\Theta_{\eta\eta} - \varepsilon h_Q[\Theta, \Psi, \varepsilon] \quad . \tag{4.18}$$

If we think of ε as a small parameter, equations (4.17-18) define a singular perturbation problem. The leading term in an outer expansion must satisfy the equations

$$\Theta_\tau + a\Theta\Theta_\eta + k\Theta_{\eta\eta} = 0 \tag{4.19}$$

$$K_o^{-1}\Psi_\eta = -b\Theta\Theta_\eta - c\Theta_{\eta\eta} \quad . \tag{4.20}$$

These equations are simply the lowest order reductive perturbation solution. If Θ_o is any solution of the Burgers equation (4.19) the second equation has the solution Ψ_o given by

$$\Psi_o = -K_o(\tfrac{1}{2}b\Theta_o^2 + c\Theta_{o_\eta}) \tag{4.21}$$

where we have set a possible constant of integration to zero. We do not attempt to solve equations (4.17-18) by series methods because that is essentially the path of the regular reductive perturbation scheme. Instead we recast these general equations as a fixed point problem and define an iterative sequence of approximate solutions. The function pair (Θ_o, Ψ_o) defines the initial point of a, hopefully convergent, sequence of iterates. Our first step in establishing our iterative scheme is to recast our basic equations (4.17-18) in an integral equation form. We begin by introducing the new variable θ defined by

$$\Theta = \Theta_o + \varepsilon\theta \quad . \tag{4.22}$$

Introducing this representation into (4.17) yields the equation

$$L(\theta_o)\theta = -h_p[\Theta_o, \Psi_o, 0] - \varepsilon(G[\theta, \Psi, \varepsilon] + a\theta\theta_\eta) \tag{4.23}$$

where the functional G is defined by

$$h_p[\Theta_o + \varepsilon\theta, \Psi\varepsilon] = h_p[\Theta_o, \Psi_o, 0] + \varepsilon G[\theta, \Psi, \varepsilon] \quad . \tag{4.24}$$

The operator $L(\Theta_o)$ is the linearisation of the Burgers operator introduced in the previous section and has the form

$$L(\Theta_o)\theta = \theta_\tau + a(\theta\Theta_o)_\eta + k\theta_{\eta\eta} \quad . \tag{4.25}$$

The second basic equation (4.18) can be expressed in terms of the new variable θ as

$$\varepsilon\Psi_\tau + K_o^{-1}\Psi_\eta = K_o^{-1}\Psi_{o_\eta} - \varepsilon(h_Q[\Theta_c + \varepsilon\theta,\Psi,\varepsilon] + b(\theta\Theta_o)_\eta + b\varepsilon\theta\theta_\eta) \quad . \tag{4.26}$$

In order to represent these equations in an integral equation format we must be able to solve the two basic equations

$$L(\Theta_o)\theta = m(\eta,\tau) \tag{4.27}$$

and

$$\varepsilon\Psi_\tau + K_o^{-1}\Psi_\eta = n(\eta,\tau) \quad . \tag{4.28}$$

The second of these has a unique solution if we specify the asymptotic condition

$$\Psi(\eta,\tau) \to 0 \qquad \text{as} \quad |\tau| \to \infty \quad . \tag{4.29}$$

Without loss of generality let us suppose that we have chosen our basis in κ so that $A(U_o)$ is diagonal. This means that the subspace N_R is spanned by the vectors $R_1,\ldots,R_{m-1}$. In that basis we have

$$(K_o^{-1})_{ij} = (e_i - e)\delta_{ij} = \kappa_i\,\delta_{ij} \tag{4.30}$$

and the solution to the initial value problem defined by equations (4.28 & 29) is

$$\Psi(\eta,\tau|n) = -\sum_{i=1}^{n-1} \int_{\tau\varepsilon^{-1}}^{\infty} n_i(\kappa_i(u-\tau\varepsilon^{-1})+\eta,\ \varepsilon u)\,du\,R_i \quad . \tag{4.31}$$

The functions $n_i(\eta,\tau)$ are the components of n relative to the basis R_i (i=1,., (n-1). We denote by T_Q the mapping $n \to \Psi(*,*|n)$. That is we write

$$T_Q[n](\eta,\tau) = \Psi(\eta,\tau|n) \quad . \tag{4.32}$$

In the previous section we showed that the operator $L(\Theta_o)$ can be linearised by means of an extension of the Cole-Hopf transformation. Each solution Θ_o of the Burgers equation (4.19) can be expressed in the form

$$\Theta_o = 2(k/a)\,\ln(S_o)_\eta \tag{4.33}$$

where S_o is a solution of the heat equation

$$S_{o\tau} + kS_{o\eta\eta} = 0 \quad . \tag{4.34}$$

If we introduce a new variable H by means of the formula

$$\theta = (H/S_o)_\eta \tag{4.35}$$

we find that the function H is a solution of the heat equation

$$H_\tau + kH_{\eta\eta} = M(\eta,\tau)S_o(\eta,\tau) \tag{4.36}$$

where $M(\eta,\tau)$ is a primative of $m(\eta,\tau)$ with respect to the variable η. It is easily shown that the function θ given by (4.135) is independent of any particular choice of primative. Without loss of generality we make the choice

$$M(\eta,\tau) = \int_{-\infty}^{\eta} m(v,\tau)dv \quad . \tag{4.37}$$

If we require the initial condition

$$H(\eta,\tau=0) = 0 \tag{4.38}$$

equation (4.36) has the unique solution

$$H(\eta,\tau) = \int_0^{\tau} du \int_{-\infty}^{\infty} dv \quad g(\eta,\tau|v,u)M(v,u)S_o(v,u) \tag{4.39}$$

where $g(\eta,\ \tau\ |\ v,\ u)$ is the usual Greens function for the heat equation. This function provides a unique solution θ for a given function m. We denote that

solution by $\theta(\eta, \tau|m)$. The mapping which associates this function with m is denoted by T_p. That is we write

$$T_p[m](\eta,\tau) = \theta(\eta,\tau|m) \quad . \tag{4.40}$$

Both T_P and T_Q are linear functionals and we can express the equation (4.23) and (4.26) in the integral equation forms

$$\theta = -T_p[h_p[\Theta_o,\Psi_o,0]] - \varepsilon T_p[G[\theta,\Psi,\varepsilon] + a\theta\theta_\eta] \tag{4.41}$$

$$\Psi = T_Q[K_o^{-1}\Psi_{o\eta}] - \varepsilon T_Q[h_Q[\Theta_o+\varepsilon\theta,\Psi,\varepsilon] + b(\theta\Theta_o)_\eta + b\varepsilon\theta\theta_\eta] \quad . \tag{4.42}$$

These equations provide the starting point for an iterative scheme for solving the equations (4.17-18).

Consider the iterative scheme defined by the equations

$$\theta^{n+1} = -T_p[h_p[\Theta_o,\Psi_o,0]] - \varepsilon T_p[G[\theta^n,\Psi^n,\varepsilon] + a\theta^n\theta^n_\eta] \tag{4.43}$$

$$\Psi^{n+1} = T_Q[K_o^{-1}\Psi_{o\eta}] - \varepsilon T_Q[h_Q[\Theta_o+\varepsilon\theta^n,\Psi^n,\varepsilon] + b(\theta^n\Theta_o)_\eta + \varepsilon\theta^n\theta^n_\eta] \tag{4.44}$$

and the initial values $\theta^o = 0$ and $\Psi^o = \Psi_o$. If we can establish the convergence of such an iterative scheme we will have a means of constructing a sequence of approximants that are known to converge and also some measure of the accuracy for a given value of n. Our principal weapon is the contraction mapping theorem. In order to use that theorem we must define an appropriate Banach space to which our solutions belong and which is mapped into itself by the mapping T defined by

$$T(\theta^n,\Psi^n) = (\theta^{n+1},\Psi^{n+1}) \tag{4.45}$$

where this is an abstract representation of the explicit relationship expressed by equations (4.43-44).

To help us decide on an appropriate function space let us examine some simple solutions of the equations (4.19-20). Consider the single soliton solution

$$\Theta_o = (k\alpha/a)(1 + \tanh(\tfrac{1}{2}(\alpha\eta - k\alpha^2\tau))) \quad . \tag{4.46}$$

This corresponds to the heat equation solution

$$S_o = 1 + \exp(\alpha\eta - k\alpha^2\tau) \quad . \tag{4.47}$$

The function Ψ_o is then found to be given by a vector combination of the two functions θ_o^2 and $\theta_{o\eta}$ given by

$$\Theta_{o\eta} = \tfrac{1}{2}(\alpha^2 k/a)\operatorname{sech}^2(\tfrac{1}{2}(\alpha\eta - k\alpha^2\tau)) \quad . \tag{4.48}$$

The most obvious source of possible trouble is the fact that θ_o does not go to zero for extreme values of its arguments. It is a shock-wave solution. However, it is bounded and its derivative (4.48) does go to zero at infinity. These special properties of our initial function suggest a possible choice of function space. Consider the space of functions K_α^n defined by

$$K_\alpha^n = \{\phi\varepsilon C^\infty(R,R^n) : ||\phi||, \exp(\alpha|\eta|)||\frac{\partial^m\phi}{\partial\eta^m}||, \text{ are bounded and continuous for all } m>1\} \tag{4.49}$$

where $||\ \ ||$ is any norm in R^n. This function space can be equipped with any of

the norms

$$||\phi||_p = \sup_{R^n} \{ ||\phi||, \exp(\alpha|\eta|)||\partial_\eta \phi||, \ldots\ldots, \exp(\alpha|\eta|)||\partial_\eta^p \phi|| \} \qquad (4.50)$$

As we will need to bound derivatives of all orders in our iterative scheme we adopt the norm,

$$||\phi||_{K_\alpha^n} = \sup \{||\phi||, ||\phi||_1, ||\phi||_2, \ldots\ldots, ||\phi||_m, \ldots\} . \qquad (4.51)$$

The functions that we are dealing with contain two independent variables and so we must consider the variable τ also. If we suppose that $k < 0$ the heat equation (4.33) defines a semi-flow for $\tau \geq 0$. We define $\hat{K}_\alpha^n$ by

$$\hat{K}_\alpha^n = \{\Psi \in C^\infty(R^2, R^n) : \Psi(r,\tau) \in K_\alpha^n \text{ for all } \tau \geq 0\} \qquad (4.52)$$

and equip it with the norm

$$||\Psi||_{\hat{K}_\alpha^n} = \sup_{\tau \geq 0} ||\Psi(\tau, *)||_{K_\alpha^n} . \qquad (4.53)$$

We can now define our space of initial function pairs for (4.42-43). Define the Banach space B_α to be the completion of the space $\hat{K}_\alpha^1 \times \hat{K}_\alpha^{n-1}$ with respect to the norm,

$$||(\theta,\Psi)||_{B_\alpha} = \max \{ ||\theta||_{\hat{K}_\alpha^1}, ||\Psi||_{\hat{K}_\alpha^{n-1}} \} . \qquad (4.54)$$

If we make this choice of initial function space it can be shown that T is a contraction mapping of B_α provided K and A are C^∞ functions. [8]. Once we have established that there is a solution of the type sought we can adopt an alternative route. We can fix $\theta \in \hat{K}_\alpha$ in (4.18) and define the sequence of functions $\Psi \in \hat{K}^{(n-1)}$ by

$$\Psi^{n+1} = T_Q [-6\theta\theta_\eta - c\theta_{\eta\eta} - \epsilon\, hq\, [\theta, \Psi^n, \epsilon]] . \qquad (4.55)$$

Again we can show that this is a contraction for sufficiently small ϵ and so (4.55) defines a functional $\Psi[\theta]$ by the limit of the above sequence,

$$\Psi[\theta] = \lim_{n\to\infty} \Psi^n . \qquad (4.56)$$

If we substitute this into (4.17) we obtain the NONLOCAL CANONICAL FORM OF (4.1),

$$\theta_t + a\theta\theta_\eta + k\theta_{\eta\eta} + \epsilon h_p [\theta, \Psi(\theta), \epsilon] = 0 . \qquad (4.57)$$

Let us now summarise the results of this final section. What we have shown is that for the general equation (4.1) there exists a 1-parameter solution determined by each sufficiently well behaved solution of the Burgers equation (4.19). The general system (4.1) contains an invariant manifold upon which it is reduced to the single equation (4.57). We regard the normal reductive perturbation procedure developed in sections two and three as a way of expressing (4.57) as a local partial

differential equation of infinite order. This is usually done in the special case relating to the single soliton solution of the Burgers equation. A similar analysis is possible for the general dispersive system (3.2) [8] .

Acknowledgements: Derek Sheil wishes to acknowledge the support of an Irish Dept. of Education post graduate award which enabled this paper to be written.

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