Internat. J. Math. & Math. Sci.
Vol. 7 No. 4 (1984) 663-666

THE LARGEST PROPER CONGRUENCE ON S(X)

K.D. MAGILL, JR.

Department of Mathematics
State University of New York at Buffalo
106 Diefendorf Hall
Buffalo, New York 14214-3093

(Received March 3, 1984)

ABSTRACT. S(X) denotes the semigroup of all continuous selfmaps of the topological space X. In this paper, we find, for many spaces X, necessary and sufficient conditions for a certain type of congruence to be the largest proper congruence on S(X).

KEY WORDS AND PHRASES. Semigroups of continuous selfmaps, largest proper congruence.
1980 AMS MATHEMATICS SUBJECT CLASSIFICATION CODE. 54H15, 20M20

1. INTRODUCTION.

DEFINITION 1. A space X is <u>admissible</u> if it is compact, Hausdorff, contains more than one point and every continuous function from a closed subspace of X into X can be extended to a continuous selfmap of X .

REMARK: Any nondegenerate absolute retract is admissible as is any nondegenerate compact 0-dimensional metric space [1, p. 281, Corollary 3].

DEFINITION 2. A <u>unifying family</u> $\mathcal{G}$ is any nonempty family of nonempty subsets of X such that for any $A \in \mathcal{G}$ and any $f \in S(X)$, $f[A] \in \mathcal{G}$ whenever f is injective on A .

The collections of all subarcs, compact subsets, subcontinua, etc., are all unifying families. With every unifying family $\mathcal{G}$, we associate a congruence $\sigma(\mathcal{G})$ by defining $(f,g) \in \sigma(\mathcal{G})$ iff whenever either of the functions is injective on some $A \in \mathcal{G}$, then the two functions coincide on A . The verification that $\sigma(\mathcal{G})$ is a congruence is straightforward and will be omitted.

DEFINITION 3. A congruence of the form $\sigma(\mathcal{G})$ is referred to as a <u>unifying congruence</u>.

DEFINITION 4. A subspace Y of X is a <u>quasiretract</u> of X if there exists a continuous function from X into Y which is injective on Y .

It is immediate that every retract of X is a quasiretract. However, quasiretracts which are not retracts are abundant. For example, any subspace of X which contains a copy of X is a quasiretract so that quasiretracts of connected spaces need not even be connected and, similarly, quasiretracts of compact spaces need not be compact.

DEFINITION 5. A unifying family $\mathcal{G}$ is said to be normal if each $A \in \mathcal{G}$ is a quasiretract of X and for each $A \in \mathcal{G}$ and each open subset of G of A, there exists a $B \in \mathcal{G}$ such that $B \subset G$. The corresponding congruence $\sigma(A)$ will be referred to as a normal unifying congruence.

2. MAIN RESULTS.

We are now in a position to prove the main result of this paper.

THEOREM 6. *Let X be admissible and let $\mathcal{G}$ be a normal unifying family. Then the normal unifying congruence $\sigma(\mathcal{G})$ is the largest proper congruence on $S(X)$ if and only if each nonempty open subset of each $A \in \mathcal{G}$ contains a copy of X.*

PROOF. (Sufficiency). It is immediate that all constant functions are related and that no constant function is related to the identity map so that $\sigma(\mathcal{G})$ is, indeed, a proper congruence. Let ρ be any other congruence on $S(X)$ and suppose $\rho \not\subset \sigma(\mathcal{G})$. Then there exist two functions f and g in $S(X)$ such that $(f,g) \in \rho - \sigma(\mathcal{G})$ and this implies that there exists an $A \in \mathcal{G}$ such that one of the functions (say f) is injective on A and $f(a) \neq g(a)$ for some $a \in A$. Then $G = \{x \in A: f(x) \neq g(x)\}$ is a nonempty open subset of A and therefore contains a copy Y of X. Let h be any homeomorphism from X onto Y and define a mapping k from $f[Y]$ into X by $k(x) = (f \circ h)^{-1}(x)$. Now choose any point $p \in Y$. If $g(h(p)) \in f[Y]$, extend k to a continuous selfmap t of X. This can be done since X is admissible. If $g(h(p)) \notin f[Y]$, choose any point $q \neq p$ and define a map $\hat{k}$ on $f[Y] \cup \{g(h(p))\}$ by $\hat{k}(x) = k(x)$ for $x \in f[Y]$ and $\hat{k}(g(h(p))) = q$. In this case also, $\hat{k}$ can be extended to a continuous selfmap t of X since X is admissible. It is immediate that

$$(i,\ t \circ g \circ h) = (t \circ f \circ h,\ t \circ g \circ h) \in \rho \tag{2.1}$$

where i denotes the identity map. Furthermore we assert that

$$t \circ g \circ h(p) \neq p\ . \tag{2.2}$$

This is immediate in the case where $g(h(p)) \notin f[Y]$ for then, $t(g(h(p))) = q$. As for the case where $g(h(p)) \in f[Y]$, suppose $t \circ g \circ h(p) = p$. Then $(f \circ h)^{-1} \circ g \circ h(p) = p$ which implies $g(h(p)) = f(h(p))$. But this is a contradiction since $h(p) \in Y$ and f and g differ at each point of Y. Thus, (2.2) has been verified. Now let $r = t \circ g \circ h$. Since $p \in A$ and $r(p) \neq p$, there exists an open subset H of A containing p such that $c\ell\, r[H] \cap c\ell\, H = \emptyset$. Then H contains a copy Z of X and we let α be any homeomorphism from X onto Z. Define a mapping β on $Z \cup c\ell\, r[H]$ by $\beta(x) = \alpha^{-1}(x)$ for $x \in Z$ and $\beta(x) = p$ for $x \in c\ell\, r[H]$. Since X is admissible, β has an extension to a continuous selfmap γ of X. Now $(i,r) \in \rho$ from (2.1) and this implies that

$$(i, \langle p \rangle) = (\gamma \circ \alpha,\ \gamma \circ r \circ \alpha) \in \rho \tag{2.3}$$

where $\langle p \rangle$ denotes the constant function which maps everything into the point p. Thus, for any two functions $v,w \in S(X)$ we have

$$(v,\langle p\rangle) = (i \circ v,\ \langle p\rangle \circ v) \in \rho \tag{2.4}$$

and similarly

$$(w,\langle p\rangle) = (i \circ w,\ \langle p\rangle \circ w) \in \rho\ . \tag{2.5}$$

Statements (2.4) and (2.5) together imply that $(v,w) \in \rho$. That is, ρ is the universal congruence. This completes the sufficiency portion of the proof.

(Necessity). Suppose now that $\sigma(\mathcal{G})$ is the largest proper congruence on $S(X)$ and let $\mathcal{J}$ be the family of all subspaces of X which are homeomorphic to X. Then $\mathcal{J}$ is a unifying family and the unifying congruence $\sigma(\mathcal{J})$ is a proper congruence on X. To see this, observe that for any point $p \in X$, $(i,\langle p\rangle) \notin \sigma(\mathcal{J})$ where, as before, i denotes the identity map and $\langle p\rangle$ is the function which maps everything into the point p. Thus, we have

$$\sigma(\mathcal{J}) \subset \sigma(\mathcal{G})\ . \tag{2.6}$$

Now take any $A \in \mathcal{G}$ and let G be any open subset of A. Since the unifying family $\mathcal{G}$ is normal, there exists a $B \in \mathcal{G}$ such that $B \subset G$. Furthermore, B is a quasi-retract of X so there exists a continuous function f from X into B which is injective on B. Choose any point $p \in X$ and note that $(f,\langle p\rangle) \notin \sigma(\mathcal{G})$. It then follows from this and (2.6) that $(f,\langle p\rangle) \notin \sigma(\mathcal{J})$. This means that there is some Y in $\mathcal{J}$ on which f is injective. Thus, $f[Y] \subset B \subset G$ and the proof is complete since $f[Y]$ is homeomorphic to X.

COROLLARY 6. *Let X be any N-dimensional admissible space which is a subspace of the Euclidean N-cell I^N and let $\mathcal{G}$ consist of all subspaces of X which are homeomorphic to I^N. Then $\sigma(\mathcal{G})$ is the largest proper congruence on $S(X)$.*

PROOF. It follows from Theorem IV3 [2, p. 44] that $\mathcal{G}$ is a nonempty collection and it is immediate that $\mathcal{G}$ is a normal unifying family. The conclusion now follows from Theorem 6.

EXAMPLE: Let $D = \{(x,y) \in R^2 : x^2 + y^2 \le 1\}$, let $J = \{(x,y) \in R^2 : x = 0$ and $1 \le y \le 2\}$ and let $X = D \cup J$ where the topology is that induced by the Euclidean plane. Let $\mathcal{G}_1$ consist of all subspaces of X which are homeomorphic to D, let $\mathcal{G}_2$ consist of all subspaces of X which are homeomorphic to the disjoint union of two copies of D and let $\mathcal{G}_3$ consist of all subspaces of X which are homeomorphic to either X or J. The space S is an absolute retract so it is certainly admissible. Moreover, one easily verifies that $\mathcal{G}_1$, $\mathcal{G}_2$, and $\mathcal{G}_3$ are all normal unifying families. Theorem 6 therefore applies and it follows that $\sigma(\mathcal{G}_1) = \sigma(\mathcal{G}_2)$ is the largest proper congruence on $S(X)$ while $\sigma(\mathcal{G}_3)$ is not the largest proper congruence.

In closing, we remark that the results in this paper extend and supplement some of the results in Chapter 6 of [3].

REFERENCES

1. KURATOWSKI, K., Topology, Vol. I, Academic Press, New York (1966).
2. HUREWICZ, W. and WALLMAN, J., Dimension Theory, Princeton (1941).
3. MAGILL, Jr., K. D., A survey of semigroups of continuous selfmaps, Semigroup Forum, (11) 3 (1975/76) 189-282.