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ON CERTAIN REGULAR GRAPHS OF GIRTH 5

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ABSTRACT. Let $f(v,5)$ be the number of vertices of a $(v,5)$-cage $(v \geq 3)$. We give an upper bound for $f(v,5)$ which is considerably better than the previously known upper bounds. In particular, when $v = 7$, it coincides with the well-known Hoffman-Singleton graph.

KEY WORDS AND PHRASES. Regular graph, cage, latin square.
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A graph is said to be regular of valency v if each of its vertices has valency v. A regular graph of valency v and girth g with the least possible number of vertices is called a (v,g)-cage. The number of vertices of a (v,g)-cage is denoted by $f(v,g)$. The existence of (v,g)-cages was proved by Erdös and Sachs [4]. In this paper, we consider only regular graphs of girth 5. It is easy to see that $f(v,5) \geq v^2 + 1$. Also, it is known that $f(3,5) = 10$ [1], $f(4,5) = 19$ [8], $f(5,5) = 30$ [10], $f(6,5) = 40$ [6], and $f(7,5) = 50$ [5]. For $v > 5$, Brown [2] has shown that $f(v,5) \leq 2(2v-1)(v-2)$. In [10], Wegner has shown that $f(v,5) \leq 2v(v-1)$ for primes $v \geq 3$. In Theorem 1b of [7], Parsons implicitly proved that $f(v,5) \leq 2v^3-3v+1$ when v is odd.

Notation. If two vertices x and y in a graph are adjacent, we write $x \sim y$.

We now give a better bound for $f(v,5)$.

Theorem 1. Let v (≥ 7) be an integer such that $v-2$ is a prime power. Then the following statements hold:

(a) $f(v,5) \leq 2(v-2)^2$.

(b) If n is an integer such that $3 \leq n \leq v$, then $f(n,5) \leq 2(v-2)(n-2)$.

We use the same notations as in [3, p. 169]. If $R = p^r$ is a prime power, then a set of $R-1$ mutually orthogonal latin squares of order R can be constructed. In fact, let the elements of the Galois field $GF[R]$ be denoted by $u_0 = 0$, $u_1 = 1$, $u_2 = x$, $u_3 = x^2$, ..., $u_{R-1} = x^{R-2}$, where x is a generating element of the multiplicative group of $GF[R]$ and $x^{R-1} = 1$. Then a complete set of mutually orthogonal latin squares L_1, L_2, ..., L_{R-1} can be obtained as follows. $u_i + u_j$ is the entry in the i^{th} row and j^{th} column of square L_1. As in [3], we use the symbol $u_i + u_j$ within a square to stand for the integer k, where $u_i + u_j = u_k$ $(i, j, k = 0, 1, \dots, R-1)$. The elements of the 0^{th} row of each L_i $(i = 1, 2, \dots, R-1)$ are identical and the remaining rows of L_i $(i = 2, 3, \dots, R-1)$ are obtained by permuting cyclically the remaining rows of L_1. We let

$$L_0 = \begin{bmatrix} 0 & 1 & \dots & R-1 \\ 0 & 1 & \dots & R-1 \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ 0 & 1 & \dots & R-1 \end{bmatrix}$$

to obtain R mutually orthogonal squares.

Proof of Theorem 1. (a). Let $R = v-2$ (a prime power). We give an explicit contruction of a regular graph G of girth 5 and valency v, having $2R^2$ vertices. In fact, let the vertices of G be arranged as in Figure 1.

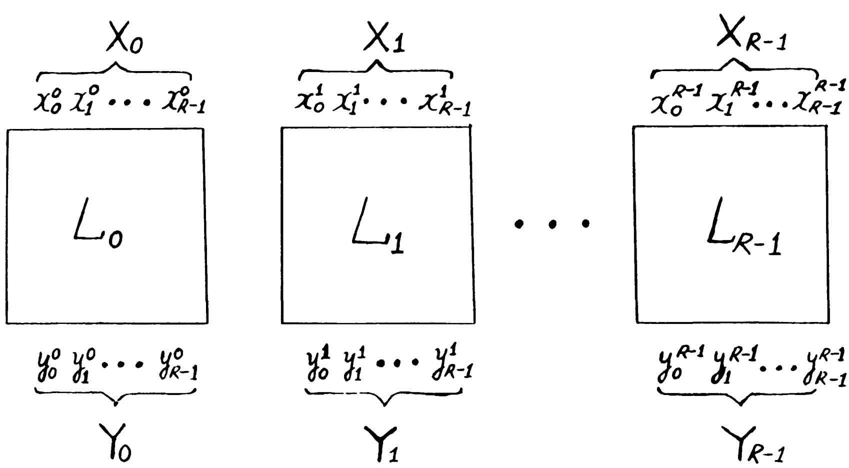

Figure 1

We say that the vertices $\{x_0^k, x_1^k, \ldots, x_{R-1}^k\}$ are in set X_k. Similarly we define set Y_i (i, k = 0, 1, ..., R-1). We join the set of vertices Y_i to the set X_k (for all i, k) according to the following rule. If n is an integer in the i^{th} row and j^{th} column of L_k, then the vertex y_n^i is adjacent to vertex x_j^k (i, j, k = 0, 1, ..., R-1). Since the squares $\{L_0, L_1, \ldots, L_{R-1}\}$ are mutually orthogonal, it is readily seen that the graph has girth 5 and valency R (= v-2). To increase this valency to v, it suffices to join each vertex x_j^k (resp. y_n^i) to two other vertices in the same set X_k (resp. Y_i) (i, k = 0, 1, ..., R-1) in such a way that the girth of the graph G remains unchanged.

For any integer n (n= 0, 1, ..., R-1), we let $\bar{A}_n$ be the set of pairs of integers $\{[u_0+u_0, u_0+u_n], [u_1+u_0, u_1+u_n], \ldots, [u_{R-1}+u_0, u_{R-1}+u_n]\}$ in L_1. The first (resp. second) terms of the pairs are integers ranging from 0 to R-1. Each row of every square contains every integer from 0 to R-1. Suppose a pair [s,t] in $\bar{A}_n$ appears in the i^{th} row and the s_1 and t_1 columns of L_k. Then we associate s and t with the two vertices $x_{s_1}^k$ and $x_{t_1}^k$ in set X_k and also with the two vertices y_s^i and y_t^i in set Y_i (i, k = 0, 1, ..., R-1). We define A_n to be the set of all pairs of vertices so associated with the set $\bar{A}_n$. The joining of a pair of vertices belong to A_n, we call an A_n-join.

We know from the construction of squares L_k (k = 2, 3, ..., R-1) from L_1, that a pair of integers [s,t] appears in some row and in columns s_1 and t_1 of L_k if and only if the same pair appears in some row and in columns s_1 and t_1 of L_1. Thus in what follows, we need only look at squares L_1.

Lemma 2. Let $[s,t] \in \bar{A}_n$ and $[v,w] \in \bar{A}_m$. Then $[v,w] \in \bar{A}_n$ if and only if v = s and w = t or v = t and w = s. Equivalently, $\bar{A}_n \cap \bar{A}_m = \{\emptyset\}$ if and only if $n \neq m$ (m, n = 1, 2, ..., R-1).

Proof. Suppose a pair of integers [s,t] appears in row i and columns s_1 and t_1 and also in row I and column S and T of square L_1. Then another pair, say [v,w], of integers appears in some row i' and columns s_1 and t_1 if and only if [v,w] also appears in some row, say I' and columns S and T. In fact, let z be the integer in column T and in the same row I' as the v which appears in column S (see Figure 2).

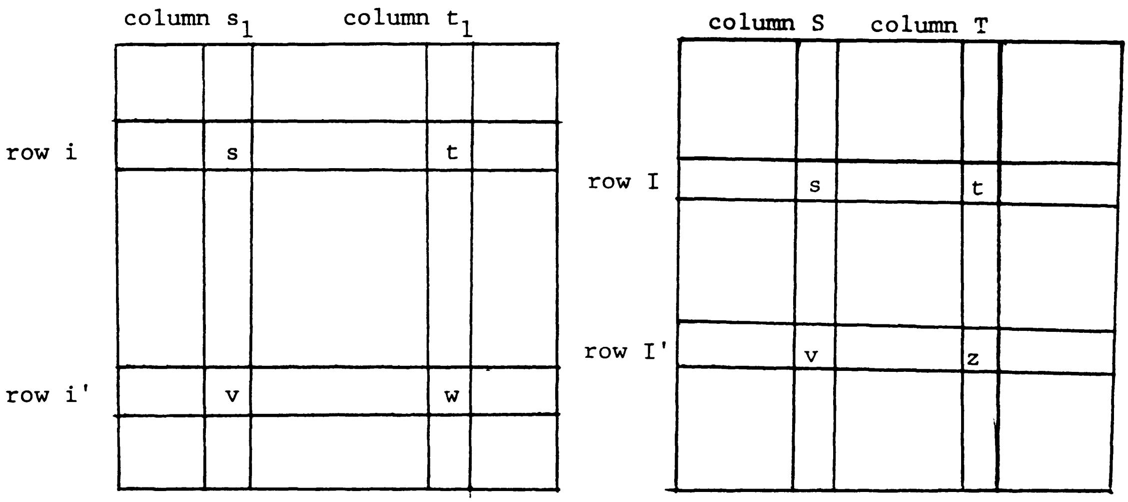

Figure 2

Then

$$
\begin{aligned}
u_s &= u_i + u_{s_1} = u_I + u_S \\
u_t &= u_i + u_{t_1} = u_I + u_T \\
u_w &= u_v + u_{t_1} - u_{s_1} \\
u_z &= u_v + u_T - u_S \\
&= u_v + (u_i + u_{t_1} - u_I) - (u_i + u_{s_1} - u_I) \\
&= u_v + u_{t_1} - u_{s_1} = u_w .
\end{aligned}
$$

It follows that if a pair of integers $[s,t]$ is in set $\bar{A}_n$, then it does not belong to set $\bar{A}_m$ for $m \neq n$. This completes the proof of the lemma.

Remark 1. If we use a collection of sets $\{\bar{A}_n : n \in N\}$ to join together the vertices of each set X_k $(k = 0, 1, \ldots, R-1)$ and a different collection of sets $\{\bar{A}_m : m \in M\}$, where $M \cap N = \{\emptyset\}$ to join together the vertices of each set Y_i $(i = 0, 1, \ldots, R-1)$, then the girth of the graph G remains five. In fact, suppose $x^k_{s_1} \sim x^k_{t_1}$, where $[x^k_{s_1}, x^k_{t_1}] \in A_n$, then by construction of G, $y^i_s \sim x^k_{s_1}$ and $y^i_t \sim x^k_{t_1}$. But by the above lemma, $[y^i_s, y^i_t] \in A_n$ and $y^i_s \not\sim y^i_t$ under an A_m-join $m \neq n$. That is $[y^i_s, y^i_t] \notin A_m$, $m \neq n$.

We divide the remaining proof of Theorem 1(a) into three saces.

Case 1. Assume $R = 2^r$ $(r \geq 3)$. The vertices of x_k $(k = 0, 1, \ldots, R-1)$ are joined together to form 2^{r-3} mutually disjoint 8-gons by using the pattern $A_1A_2A_1A_3A_1A_2A_1A_3$. Explicitly,

$$x^k_{u_0} \sim x^k_{u_0+u_1} \sim x^k_{u_0+u_1+u_2} \sim x^k_{u_0+u_2} \sim$$

$$x^k_{u_0+u_2+u_3} \sim x^k_{u_0+u_1+u_2+u_3} \sim x^k_{u_0+u_1+u_3} \sim x^k_{u_0+u_3} \quad (\sim x^k_{u_0}).$$

It is easy to see that these eight vertices are distinct and they form an 8-gon and, use the same pattern, $A_1A_2A_1A_3A_1A_2A_1A_3$, starting with $x^k_{u_s}$ to get a second 8-gon, the vertices of which are clearly distinct from those of the first. If $r > 4$, we repeat this last step until we have 2^{r-3} 8-gons. similarly, the vertices of Y_i ($i = 0, 1, \ldots, R-1$) are joined together to form 2^{r-3} mutually disjoint 8-gons by using the pattern $A_4A_5A_4A_6A_4A_5A_4A_6$.

$$y^i_{u_0} \sim y^i_{u_0+u_4} \sim y^i_{u_0+u_4+u_5} \sim y^i_{u_0+u_5} \sim$$

$$y^i_{u_0+u_5+u_6} \sim y^i_{u_0+u_4+u_5+u_6} \sim y^i_{u_0+u_4+u_6} \sim$$

$$y^i_{u_0+u_6} \sim (y^i_{u_0}) .$$

Thus G has valency v. It remains to show that G has girth 5. Since we use $A_1A_2A_1A_3A_1A_2A_1A_3$ -joins in X_k and $A_4A_5A_4A_6A_4A_5A_4A_6$ - joins in Y_i, it follows from Lemma 2 and Remark 1 that any pair of vertices in X_k which we join are not adjacent to any pair of vertices joined in Y_i ($i, k = 0, 1, \ldots, R-1$). This statement is true for X_0 and Y_0, because of the construction of L_0. Hence the graph G does not contain any 4-gons. Therefore G has girth 5. This completes the proof of Case 1.

Example. For $P = 2^3 = 8$, we have

$$\begin{array}{c} \overbrace{x_0 \; x_1 \; x_2 \; x_3 \; x_4 \; x_5 \; x_6 \; x_7}^{X_1} \\ \bullet \; \bullet \; \bullet \; \bullet \; \bullet \; \bullet \; \bullet \; \bullet \end{array}$$

$$L_1 = \begin{bmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 0 & 4 & 7 & 2 & 6 & 5 & 3 \\ 2 & 4 & 0 & 5 & 1 & 3 & 7 & 6 \\ 3 & 7 & 5 & 0 & 6 & 2 & 4 & 1 \\ 4 & 2 & 1 & 6 & 0 & 7 & 3 & 5 \\ 5 & 6 & 3 & 2 & 7 & 0 & 1 & 4 \\ 6 & 5 & 7 & 4 & 3 & 1 & 0 & 2 \\ 7 & 3 & 6 & 1 & 5 & 4 & 2 & 0 \end{bmatrix}$$

$$x_0^k \sim x_1^k \sim x_4^k \sim x_2^k \sim x_5^k \sim x_6^k \sim x_7^k \sim x_3^k \sim (x_0^k)$$

and

$$y_0^i \sim y_4^i \sim y_7^i \sim y_5^i \sim y_1^i \sim y_2^i \sim y_3^i \sim y_6^i \sim (y_0^i)$$

where $i, j = 0, 1, \ldots, 7$.

Case 2. Assume $P = 3^r$ $(r \geq 2)$. The vertices of X_k are joined together to form 3^{r-2} mutually disjoint 9-gons by using the pattern $A_1A_1A_2A_1A_1A_2A_1A_1A_2$. Explicitly,

$$x^k_{u_0} \sim x^k_{u_0+u_1} \sim x^k_{u_0+u_1+u_1} \sim x^k_{u_0+u_1+u_1+u_2} \sim$$

$$x^k_{u_0+u_2} \sim x^k_{u_0+u_1+u_2} \sim x^k_{u_0+u_1+u_2+u_2} \sim$$

$$x^k_{u_0+u_1+u_1+u_2+u_2} \sim x^k_{u_0+u_2+u_2} \quad (\sim x^k_{u_0}).$$

Repeating this pattern, we form 3^{r-2} mutually disjoint 9-gons from X_k $(k = 0, 1, \ldots, R-1)$. Similarly, the vertices of Y_i are joined together to form 3^{r-2} mutually disjoint 9-gons by using the pattern $A_3A_3A_4A_3A_3A_4A_3A_3A_4$. Thus G has valency v. It follows from Lemma 2 and Remark 1 that G has girth 5.

Case 3. Assume $R = p^r$ where $p(\geq 5)$ is a prime and $r \geq 1$. Join the vertices of X_k using the pattern $\underbrace{A_1A_1\ldots A_1}_{p}$ to produce r mutually disjoint p-gons. Explicitly, the first p-gon is

$$x^k_{u_0} \sim x^k_{u_0+u_1} \sim x^k_{u_0+u_1+u_1} \sim \cdots \sim x^k_{\underbrace{u_0+u_1+u_1+\cdots+u_1}_{p-1}} \sim (x^k_{u_0}).$$

Similarly, we join the vertices of Y_i using the pattern $\underbrace{A_2A_2\ldots A_2}_{p}$ to get r mutually disjoint p-gons $(i, j = 0, 1, \ldots, R-1)$. Therefore G has valency v. It follows from Lemma 2 and Remark 1 that G has girth 5. This completes the proof of (a).

(b). Let G be the graph constructed as in (a). The subgraph of G induced by $X_0, X_1, \ldots, X_{n-3}, Y_0, Y_1, \ldots,$ and Y_{n-3} clearly has girth 5 and valency n with order $2(v-2)(n-2)$. This completes the proof of Theorem 1.

Remark 2. Let v be an integer ≥ 3. Since there always exists a prime power R such that $R \geq v$, it follows that Theorem 1 gives an upper bound for $f(v,5)$ for any $v(\geq 3)$.

Remark 3. For $v = 7$, Theorem 1 is identical with the construction of the Hoffman-Singleton graph given in [1] and [9], and $f(7,5) = 50$. The upper bound for $f(v,5)$ given in Theorem 1 is better than the other bound mentioned previously. For example, we have $f(9,5) \leq 98$ and $f(8,5) \leq 84$.

Remark 4. For $p = v - 1$ a prime number, a set of mutually orthogonal latin squares $L_1, L_2, \ldots, L_{p-1}$ is more easily obtained by a simple rotation of the elements in the rows of L_0 which is the same as before. Explicitly

$$L_1 = \begin{bmatrix} 0 & 1 & 2 & 3 & . & . & . & p-1 \\ p-1 & 0 & 1 & 2 & . & . & . & p-2 \\ p-2 & p-1 & 0 & 1 & . & . & . & p-3 \\ . & . & . & . & & & & . \\ . & . & . & . & & & & . \\ . & . & . & . & & & & . \\ 1 & 2 & 3 & 4 & . & . & . & 0 \end{bmatrix}$$

In general, in L_k $(k = 2, 3, \ldots, p-1)$, if k appears in row i and column j, then 0 is in row $i + 1$ and column j. This simplifies the construction of the graph.

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