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SEPARATION AXIOMS FOR PARTIALLY ORDERED CONVERGENCE SPACES

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ABSTRACT. In partially ordered convergence spaces, separation axioms are introduced and then related to the concept of complete separatedness due to Nachbin as well as to connectedness concepts. A method to generate new separation axioms is studied.

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1. INTRODUCTION

Lately, convergence structures more general than topologies have proved to be effective tools in posets and lattices (cf. Kent [1], Erné and Weck [2], and Ball [3]). In this note we shall study some relations of interdependence between a convergence structure and a partial order, hereby concentrating upon separation axioms and related matters. This is done within the realms of partially ordered (po) convergence spaces. In the po topological case, most of the material in Section 3 is known from Nachbin [4] and Mc Cartan [5]. The interplay between separation axioms and connectivity properties, as worked out in Sections 4 and 5, has not been studied in po topological spaces. For a correspondence in topological spaces without order, reference is made to Preuss [6, Ch. 6].

Here convergence structure is used in the sense of Kent [7]; the precise definition is stated in Section 2. A partially ordered (po) convergence space is a triplet $(X,q,\leq)$, where X is a set, q a convergence structure on X and $\leq$ a partial order relation on X. Obviously, this is a generalization of the partially ordered (po) topological spaces introduced by Nachbin. We can regard every convergence space as a po convergence space, where the order in question is discrete. Every definition which we propose for po convergence spaces shall be subject to the following criteria:

(i) For po topological spaces, it reduces to the classic definition in the sense of Nachbin.

(ii) For discrete order, it coincides with a natural definition in convergence space theory.

Hence it is clear that every definition for po convergence spaces defines a natural compatibility between convergence structure and partial order.

There are natural, well-known non-topological po convergence structures, for instance order convergence on posets (cf. Kent [1], Erné and Weck [2]) and several structures defined by R.N. Ball. Before proceeding, we wish to mention the expository article Choe [8], which covers the main streams of research in po topological spaces up to recent date.

2. PRELIMINARIES

For later use, we gather a few definitions and notations concerning convergence and order. For our aim, the following basic definition is the proper one.

DEFINITION 2.1. (Kent [7]). Let X be a set. A convergence structure q on X is a map q, which assigns to every $x \in X$ a set $q(x)$ of filters on X being subject to the conditions below $(x \in X)$:

(1) $[x] \in q(x)$

(2) $\mathcal{F} \in q(x)$ and $\mathcal{G} \supseteq \mathcal{F} \Rightarrow \mathcal{G} \in q(x)$

(3) $\mathcal{F} \in q(x) \Rightarrow \mathcal{F} \cap [x] \in q(x)$.

Hereby, $[x]$ denotes the ultrafilter generated by $\{x\}$. The pair (X,q) is called a convergence space.

Obviously, this definition provides a generalization of topological structure and topological space. We do not require the filters in $q(x)$ to form full intersection ideals, since we wish to consider order convergence on posets as a special case of the convergence structures being treated here. For theory and application of convergence structures, we recommend the book Gähler [9].

If q is a convergence structure on the set X, then tq is the finest topology on X being coarser than q. The notion of open (closed) set in a space (X,q) always refers to the topological space (X,tq). Let (X,q) be a convergence space and take $A \subseteq X$. Then, $\overline{A}^q$ denotes the set of all $x \in X$ for which $q(x)$ contains some $\mathcal{F}$ with $A \in \mathcal{F}$. For (X,q) and (Y,r) given convergence spaces, a continuous map $f : (X,q) \to (Y,r)$ is a map $f : X \to Y$ for which $x \in X$, $\mathcal{F} \in q(x) \Rightarrow f(\mathcal{F}) \in r(f(x))$.

Occasionally, a po convergence space $(X,q,\leq)$ shall be denoted by X only, and then, instead of $\mathcal{F} \in q(x)$ shall be written $\mathcal{F} \to x$. The topological modification tX of a given po convergence space X is the po topological space $(X,tq,\leq)$. If X and Y are po convergence spaces, a morphism $\phi : X \to Y$ is an increasing continuous map ϕ from X to Y.

In a given poset, $x \nleq y$ denotes that $x \leq y$ is false, and $x \parallel y$ is equivalent to $x \nleq y$ and $y \nleq x$. If F is a set, then $i(F)$ $(d(F))$ denotes the smallest increasing (decreasing) set containing F, and F^* (F^+) denotes the set of all upper (lower) bounds of F. Instead of $\{a\}^*$ $(\{a\}^+)$ is written a^* (a^+). On a given poset, the interval topology is the coarsest topology for which all rays, i.e. the sets of the form a^* or a^+, are closed sets.

3. SEPARATION AXIOMS

For definitions and results in the case of po topological spaces, reference is made to Nachbin [4], Mc Cartan [5] and Ward [10]. Synonymously with T_1-ordered po topological space, however, the concepts of semi-closed partial order and semicontinuous partial order are used in Nachbin [4] and Ward [10], respectively. T_1-ordered and T_2-ordered po convergence spaces were introduced in Kent and Richardson [11].

DEFINITION 3.1. A po convergence space X is lower (upper) T_1-ordered, if for every pair $a \not\leq b$ in X and for every $\mathcal{F} \to a$ ($\mathcal{F} \to b$) there is a set $F \in \mathcal{F}$ such that $x \not\leq b$ ($a \not\leq x$) for all $x \in F$. This separation axiom is denoted by ord T_{1L} (ord T_{1U}).

THEOREM 3.2. Let X be a po convergence space. The following conditions are equivalent:

(i) X is ord T_{1L}.

(ii) For every pair $a \not\leq b$ in X, for every $\mathcal{F} \to a$ and for every $F \in \mathcal{F}$, $b \notin F^*$.

(iii) For every pair $a \not\leq b$ in X, and for every $\mathcal{F} \to a$ there is an increasing set $V \in \mathcal{F}$ with $b \notin V$.

(iv) For every $a \in X$ the ray $a\downarrow$ is a closed set.

Corresponding characterizations hold true for ord T_{1U} (with obvious changes only).

PROOF. (i) ⇒ (ii). According to (i), the filter $\mathcal{F} \to a$ in (ii) contains some F_0 with $x \not\leq b$ for all $x \in F_0$. Since every $F \in \mathcal{F}$ intersects F_0, we are through.
(ii) ⇒ (iii). Let X satisfy (ii), take $a \not\leq b$ in X and $\mathcal{F} \to a$, then write

$$\mathcal{F} = \bigcap_{k \in K} \mathcal{G}_k$$

where $\mathcal{G}_k$ $(k \in K)$ are the ultrafilters finer than $\mathcal{F}$. Now, fix $k \in K$. According to (ii), every set G_{kj} in $\mathcal{G}_k = (G_{kj})_{j \in J}$ contains an element s_j for which $s_j \not\leq b$. Denote $S_k = \{s_j | j \in J\}$. Since S_k intersects all sets in the ultrafilter $\mathcal{G}_k$, it follows $S_k \in \mathcal{G}_k$. Thus $V_k = i(S_k) \in \mathcal{G}_k$, $b \notin V_k$, and the set

$$V = \bigcup_{k \in K} V_k$$

in an increasing set in $\mathcal{F}$ with $b \notin V$.
(iii) ⇒ (iv). Assume (iii), take $a \in X$ and $x \notin a\downarrow$, i.e. $x \not\leq a$. For any $\mathcal{F} \to x$ there is an increasing set $V \in \mathcal{F}$ with $a \notin V$. Thus $V \cap a\downarrow = \emptyset$, and $X \setminus a\downarrow$ is an open set.
(iv) ⇒ (i). If X satisfies (iv), then the topological modification tX is ord T_{1L} (Mc Cartan [5]), and (i) follows.

COROLLARY 3.3. A po convergence space X is ord T_{1L} (ord T_{1U}), if and only if the topological modification tX is.

DEFINITION 3.4. A po convergence space is T_1-ordered, if it is both T_{1L}-ordered and T_{1U}-ordered. This separation axiom is denoted by ord T_1.

THEOREM 3.5. A po convergence space $(X,q,\leq)$ is ord T_1, if and only if the convergence structure q is finer than the interval topology of the po relation $\leq$. Hence, in po convergence spaces satisfying ord T_1, all maximal chains are closed sets.

REMARK 3.6. In the Introduction we stated two criteria, (i) and (ii), which new definitions in the theory of po convergence spaces should meet. The definition of ord T_{1L} (ord T_{1U}) fills (i). In case of discrete order, both ord T_{1L} and ord T_{1U}

coincide with the separation axiom T_1 for convergence spaces. Thus also (ii) is satisfied. Naturally, the definition of ord T_1 also satisfies both criteria.

Finally we note that if the po convergence space $(X,q,\leq)$ is ord T_{1L} or ord T_{1U}, then the convergence space (X,q) is T_0. Moreover, if $(X,q,\leq)$ is ord T_1, then (X,q) is T_1.

DEFINITION 3.7. A po convergence space X is T_2-ordered, if for every pair $a \nleq b$ in X and for every $\mathcal{F} \to a$ and $\mathcal{G} \to b$, there are sets $F \in \mathcal{F}$ and $G \in \mathcal{G}$ such that $x \nleq y$ for all $x \in F$, $y \in G$. This separation axiom is denoted by ord T_2.

THEOREM 3.8. (Kent and Richardson [11]). Let X be a po convergence space. The following conditions are equivalent:

(i) X is ord T_2.

(ii) For every pair $a \nleq b$ in X, for every $\mathcal{F} \to a$ and $\mathcal{G} \to b$, there is an increasing set $F \in \mathcal{F}$ and a decreasing set $G \in \mathcal{G}$ for which $F \cap G = \emptyset$.

(iii) The graph of the partial order of X is a closed set in the product convergence space $X \times X$.

REMARK 3.9. If the order relation of a po convergence space is a total order, then ord $T_1 \Leftrightarrow$ ord T_2. This follows from the corresponding statement in the po topological case (cf. Ward [10]) and from Corollary 3.3. There is a vast literature on totally ordered topological spaces satisfying ord T_1, for instance within the realms of orderability theory (cf. Eilenberg [12]). In this paper, the special case of totally ordered convergence spaces is not treated.

REMARK 3.10. For po topological spaces, the definition of ord T_2 coincides with the classic definition of closed order (Nachbin [4], Mc Cartan [5]). If the order of a po convergence space is discrete, then ord T_2 coincides with the classic separation axiom T_2 for convergence spaces. (Thus, Theorem 3.8 can be regarded as a generalization of the usual characterization "A convergence space is T_2, if and only if the diagonal is a closed set in the product space".)

Moreover, if $(X,q,\leq)$ is ord T_2, then the convergence space (X,q) is T_2. Every T_2-ordered po convergence space is also T_1-ordered. It is possible for $(X,q,\leq)$ to be ord T_2, without the topological modification $(X,tq,\leq)$ having that property.

Below, we propose two variants of regularity for po convergence spaces. For the main part, the case of lower regularity (ord T_{3L}) is treated.

DEFINITION 3.11. A po convergence space $(X,q,\leq)$ is lower T_3-ordered, if for every closed decreasing set M, for every $x \notin M$ and for every $\mathcal{G} \in q(x)$, there is a set $G \in \mathcal{G}$ for which $M \cap \overline{i(G)}^{\,q} = \emptyset$. This separation axiom is denoted by ord T_{3L}.

DEFINITION 3.12. A po convergence space $(X,q,\leq)$ is strongly lower T_3-ordered, if for every $x \in X$ and every $\mathcal{F} \in q(x)$, there is a filter $\mathcal{G} \in q(x)$ for which $\overline{i(\mathcal{F})}^{\,q} \supseteq i(\mathcal{G})$. This separation axiom is denoted by st ord T_{3L}.

By $i(\mathcal{F})$ is meant the filter on X, which is generated by the sets $i(F)$, $F \in \mathcal{F}$. It is easily verified that st ord $T_{3L} \Rightarrow$ ord T_{3L}. The reverse implication is false, even if q is a topology. Then ord T_{3L} coincides with the classic definition of lower regularity (Mc Cartan [5]), while st ord T_{3L} becomes "For every $x \in X$ and for every increasing neighborhood V of x, there is an increasing neighborhood W of x, for which $\overline{W} \subseteq V$". This deviates from the definition of Mc Cartan [5] only in the choice of the set V; in the classic definition V is taken to be an open increasing neighborhood of x.

In case of discrete order, Definition 3.11 coincides with the axiom T_{3-} for convergence spaces $(\mathcal{F} \in q(x) \Rightarrow \overline{\mathcal{F}}^q \in (tq)(x))$, while Definition 3.12 coincides with T_3 for convergence spaces $(\mathcal{F} \in q(x) \Rightarrow \overline{\mathcal{F}}^q \in q(x))$. It follows that the topological modification preserves neither ord T_{3L} nor st ord T_{3L}.

THEOREM 3.13. (cf. Mc Cartan [5, Remark 1]). In po convergence spaces, the axioms ord T_{1L} and ord T_{3L} together imply the axiom ord T_2.

A convergence space (X,q) is called strongly locally compact, if for every $x \in X$ every $\mathcal{F} \in q(x)$ contains a coarser filter $\mathcal{G} \in q(x)$ which has a filter base of compact sets. In T_2 topological spaces, it coincides with the usual definition of local compactness. We call a po convergence space ord T_3, if it satisfies both ord T_{3L} and the dual axiom ord T_{3U}. In a similar way we define st ord T_3.

THEOREM 3.14. Let $(X,q,\leq)$ be a po convergence space, whose topological modification is ord T_2. If (X,q) is strongly locally compact, then $(X,q,\leq)$ is st ord T_3.

PROOF. For $x \in X$ and $\mathcal{F} \in q(x)$ there is a coarser filter $\mathcal{G} \in q(x)$ possessing a base of compact sets. The filter $i(\mathcal{G})$ has a base of closed sets (cf. Nachbin [4, p. 44]). It follows

$$\overline{i(\mathcal{F})}^{\,q} \supseteq \overline{i(\mathcal{G})}^{\,q} = i(\mathcal{G}) ,$$

which combined with the dual reasoning gives the theorem.

COROLLARY 3.15. (Mc Cartan [5, Th. 7]). Every po topological space, which is locally compact and ord T_2, is also ord T_3.

4. SEPARATION AXIOMS AND CONNECTIVITY

In this section, the axioms of Section 3 shall be related to connectivity, the concept of complete separatedness also being involved. The results to follow are new also in the theory of po topological spaces. For a corresponding study in topological spaces without order relation, reference is made to Preuss [6, Ch. 6]. Increasing continuous maps between po convergence spaces shall be called morphisms.

Let $\mathcal{E}$ denote a family of po convergence spaces. The elements x,y of an arbitrary po convergence space X are called $(X,\mathcal{E})$-related, if $\phi(y) \leq \phi(x)$ for all $E \in \mathcal{E}$ and all morphisms $\phi : X \to E$, or if $\phi(x) \leq \phi(y)$ for all $E \in \mathcal{E}$ and all morphisms $\phi : X \to E$. Furthermore, the elements $x,y \in X$ are called $(X,\mathcal{E})$-identic, if $\phi(x) = \phi(y)$ for all $E \in \mathcal{E}$ and all morphisms $\phi : X \to E$.

DEFINITION 4.1. Let $\mathcal{E}$ be a family of po convergence spaces. A po convergence space X is called $\mathcal{E}$-orderconnected, if for every $x,y \in X$

$$x \parallel y \Rightarrow x \text{ and } y \text{ are } (X,\mathcal{E})\text{-related}$$

$$x \leq y \Rightarrow x \text{ and } y \text{ are } (X,\mathcal{E})\text{-identic.}$$

In the special case discrete order, topological space (in both the po convergence space X and the family $\mathcal{E}$), the definition above coincides with the definition of $\mathcal{E}$-connectedness for topological spaces in Preuss [6, Ch. 6].

REMARK 4.2. A po convergence space X is called strongly $\mathcal{E}$-orderconnected, if every $x,y \in X$ are $(X,\mathcal{E})$-identic. This is the natural definition of a connectedness concept in po convergence spaces. Burgess and Mc Cartan [13] used a variant of this definition in po topological spaces. Here Definition 4.1 shall be used, since from our point of view it provides the best application. However, a short comparison of the two definitions is called for. They coincide, if the po structure of X is directed (without restriction on the family $\mathcal{E}$). In general, the two definitions do not coincide. The stronger definition is applied in an example in Section 5.

DEFINITION 4.3. Let $\mathcal{E}$ be a family of po convergence spaces. A po convergence space X is called completely $\mathcal{E}$-separated, if for every pair $x \nleq y$ in X there is a space $E \in \mathcal{E}$ and a morphism $\phi : X \to E$ for which $\phi(x) \nleq \phi(y)$.

REMARK 4.4. If X is a po topological space and $\mathcal{E} = \{[0,1]\}$, then Definition 4.3 coincides with Nachbin's definition of completely separated po topological space.

REMARK 4.5. The condition of Definition 4.3 can be restated in the following way: For every pair $x \| y$ in X there are spaces $E,F \in \mathcal{E}$ and morphisms $\phi : X \to E$, $\varphi : X \to F$ for which $\phi(x) \nleq \phi(y)$ and $\varphi(y) \nleq \varphi(x)$, and furthermore, for every pair $x < y$ in X there is a space $G \in \mathcal{E}$ and a morphism $\eta : X \to G$ for which $\eta(x) < \eta(y)$. Thus, the concept of completely $\mathcal{E}$-separated is a natural disconnectedness concept related to Definition 4.1.

REMARK 4.6. In the special case discrete order and topological space, Definition 4.3 coincides with the definition of totally $\mathcal{E}$-connectedless topological space (total $\mathcal{E}$ -zusammenhangsloser topologischer Raum) in Preuss [6, Ch. 6].

For $\mathcal{E}$ a given family of po convergence spaces, the family of completely $\mathcal{E}$-separated po convergence spaces is denoted by $Q(\mathcal{E})$. It will play a crucial rôle as a key, when translating the lower separation axioms of Section 3 into connectedness concepts.

In the category of po convergence spaces and increasing continuous maps, products and subspaces are formed in the obvious way. It is easy to prove

THEOREM 4.7. For any family $\mathcal{E}$ of po convergence spaces, the related family $Q(\mathcal{E})$ is closed under formation of products and subspaces.

Denote the family of all T_i-ordered po convergence spaces by $\underline{\text{ord } T_i}$. In case of $i = 1L$, $1U$, 1 and 2, we shall determine at least one family $\mathcal{E}_i$ of po convergence spaces for which $\underline{\text{ord } T_i} = Q(\mathcal{E}_i)$. In case of $i = 3L$, $3U$ and 3, we shall later define another disconnectedness concept through which the regularity axioms shall be represented.

THEOREM 4.8. For $i = 1L$, $1U$, 1 and 2, we have $\underline{\text{ord } T_i} = Q(\underline{\text{ord } T_i})$.

PROOF. The theorem is proved for the case $i = 1L$. Start by taking a po convergence space $X \in Q(\underline{\text{ord } T_{1L}})$ and choose $a \nleq b$ in X. There is a space $E \in \underline{\text{ord } T_{1L}}$ and a morphism $\phi : x \to E$ for which $\phi(a) \nleq \phi(b)$. Hence, for any filter $\mathcal{H} \to \phi(a)$ there is a set $H \in \mathcal{H}$ such that $h \nleq \phi(b)$ for all $h \in \mathcal{H}$. Suppose there exists $\mathcal{F} \to a$ such that every $F \in \mathcal{F}$ contains some element f with $f \leq b$. Since $\phi(\mathcal{F}) \to \phi(a)$, a contradiction is obtained, and hence $X \in \underline{\text{ord } T_{1L}}$.

Then take $X \in \underline{\text{ord } T_{1L}}$. All $a \nleq b$ in X are nicely separated by the identity map $X \to X \in \underline{\text{ord } T_{1L}}$, and hence $X \in Q(\underline{\text{ord } T_{1L}})$.

COROLLARY 4.9. The separation axioms ord T_i (i = 1L, 1U, 1, 2) are closed under formation of products and subspaces in the category of po convergence spaces.

In any representation $\underline{\text{ord } T_i} = Q(\mathcal{E}_i)$ it always holds $\mathcal{E}_i \subseteq \underline{\text{ord } T_i}$, but it is not necessary for $\mathcal{E}_i$ to equal the whole family $\underline{\text{ord } T_i}$ (i = 1L, 1U, 1, 2). Endow the ordered set $\{1,2\}$ with the topology whose only non-trivial open set is $\{2\}$ ($\{1\}$), and denote the resulting po topological space by S_{1L} (S_{1U}). Furthermore, let $\mathcal{E}_1$ denote the family of all po topological spaces carrying interval topology.

THEOREM 4.10. The following representations hold: $\underline{\text{ord } T_{1L}} = Q(\{S_{1L}\})$, $\underline{\text{ord } T_{1U}} = Q(\{S_{1U}\})$ and $\underline{\text{ord } T_1} = Q(\mathcal{E}_1)$. Moreover, neither $\underline{\text{ord } T_1}$ nor $\underline{\text{ord } T_2}$ can be interpreted using one-space families $\mathcal{E}$.

REMARK 4.11. The ideas above are now applied on a new, weak separation axiom for po convergence spaces. We say a space X is T_0-ordered, if for every pair $a \nleq b$ in X at least one of the following conditions holds:

(1) For every $\mathcal{F} \to a$ there is a set $F \in \mathcal{F}$ such that $x \nleq b$ for all $x \in F$.

(2) For every $\mathcal{G} \to b$ there is a set $G \in \mathcal{G}$ such that $a \nleq y$ for all $y \in G$.

We denote this separation axiom by ord T_0. Obviously, $\underline{\text{ord } T_0} = Q(\mathcal{E}_0)$, where $\mathcal{E}_0 = \{S_{1L}, S_{1U}\}$, and hence $\underline{\text{ord } T_0}$ is closed under formation of products and subspaces in the category of po convergence spaces. A po convergence space is T_0-ordered, if and only if its topological modification is. In case of discrete order, ord T_0 coincides with the usual separation axiom T_0 for convergence spaces.

We proceed to the regularity axioms ord T_i(i = 3L, 3U, 3), starting with the definition of a new disconnectedness concept for po convergence spaces. Let M be a subset and p an element of the po convergence space X. By $M \ll p$ is indicated that there is a closed decreasing set D in X containing M but not p.

Now, for $\mathcal{E}$ a given family of po convergence spaces, let $R_L(\mathcal{E})$ be the family of po convergence spaces defined through

$X \in R_L(\mathcal{E}) \Leftrightarrow$ For every closed decreasing set $D \subseteq X$ and for every $p \notin D$ there is a space $E \in \mathcal{E}$ and a morphism $\phi : X \to E$ for which $\phi(D) \ll \phi(p)$.

THEOREM 4.12. The representation $\underline{\text{ord } T_{3L}} = R_L(\underline{\text{ord } T_{3L}})$ holds. There is no po convergence space E_{3L} for which $\underline{\text{ord } T_{3L}} = R_L(\{E_{3L}\})$.

REMARK 4.13. It is obvious how to define families $R_i(\mathcal{E})$, in order to have $\underline{\text{ord } T_i} = R_i(\underline{\text{ord } T_i})$, i = 3U,3. (These regularity axioms were defined in the remarks preceding Theorem 3.14).

REMARK 4.14. Finally, we wish to point out that the results 4.7 - 4.10 and 4.12, although stated for po convergence spaces, also hold true for po topological spaces. For topological spaces without order, these results were presented in Preuss [6, Ch. 6].

5. GENERATING NEW SEPARATION AXIOMS

In Theorem 4.8 it was shown that the lower separation axioms of Section 3 are related to the connectivity concept given in Definition 4.1. In two examples, we shall study deviating connectivity definitions, and then generate separation axioms matching these definitions.

EXAMPLE 5.1. Definition 4.1 is strong in the sense that the corresponding disconnectedness concept (Definition 4.3) allows a very weak $(X,\mathcal{E})$-separation of non-related elements (cf. Remark 4.5). Therefore, we mention the following possibility:

Definition. Let $\mathcal{E}$ be a family of po convergence spaces. A po convergence space X is called weakly $\mathcal{E}$-orderconnected, if for every $x,y \in X$, every $E \in \mathcal{E}$ and every morphism $\phi : X \to E$ holds

$$x \parallel y \Rightarrow \phi(x),\phi(y) \text{ are order related}$$
$$x \leq y \Rightarrow \phi(x) = \phi(y) .$$

We introduce the corresponding disconnectedness concept $Q'(\mathcal{E})$ by

$X \in Q'(\mathcal{E}) \Leftrightarrow$ For every $x \parallel y$ in X there is a space $E \in \mathcal{E}$ and a morphism $\phi : X \to E$ such that $\phi(x) \parallel \phi(y)$, and furthermore, for every $x < y$ in X there is a space $F \in \mathcal{E}$ and a morphism $\varphi : X \to F$ such that $\varphi(x) < \varphi(y)$.

In the special case discrete order and topological space, these definitions coincide with the definitions of $\mathcal{E}$-connected and totally $\mathcal{E}$-connectedless topological spaces, respectively (cf. Preuss [6, Ch. 6]).

It can be proved that $\underline{\text{ord } T_i} = Q'(\underline{\text{ord } T_i})$. If $\mathcal{E}_i$ is a proper subfamily of $\underline{\text{ord } T_i}$, then in general $Q'(\mathcal{E}_i)$ is a proper subfamily of $Q(\mathcal{E}_i)$. Thus, the family $Q'(\mathcal{E}_i)$ defines a stronger separation axiom than $Q(\mathcal{E}_i)$, i = 1L, IU, 1, 2. If Q is replaced by Q', Theorem 4.10 holds with the only exception that the space $S_{1L}(S_{1U})$ must be replaced by the po topological space $E_{1L}(E_{1U})$. Hereby, $E_{1L}(E_{1U})$ is defined on the set $\{a,b,c\}$, where the order is $a \parallel b$, $a \leq c$, $b \leq c$ ($a \parallel b$, $c \leq a$, $c \leq b$) and the non-trivial open sets are $\{c\}$, $\{a,c\}$, $\{b,c\}$ in both cases.

EXAMPLE 5.2. Starting with the connectivity definition of Remark 4.2 (i.e. strong $\mathcal{E}$-orderconnectedness), we write for any po convergence space X

$X \in Q''(\mathcal{E}) \Leftrightarrow$ For every $x \neq y$ in X there is a space $E \in \mathcal{E}$ and a morphism $\phi : X \to E$ for which $\phi(x) \neq \phi(y)$.

For $\mathcal{E}$ a family of po convergence spaces, in general, $Q(\mathcal{E})$ is a strict subfamily of $Q''(\mathcal{E})$, and hence $Q''(\mathcal{E})$ defines a weaker separation axiom than $Q(\mathcal{E})$. For instance, the family $Q(\{S_{1L}\})$, i.e. $\underline{\text{ord } T_{1L}}$, is a strict subfamily of $Q''(\{S_{1L}\})$. We consider $Q''(\{S_{1L}\})$ to be a separation axiom. A po convergence space is $Q''(\{S_{1L}\})$, if and only if the topological modification is. A po topological space X is $Q''(\{S_{1L}\})$, if and only if for every $x \parallel y$ in X there is an increasing open neighborhood of at least one of the two elements which does not contain the other element, and furthermore, for every $x < y$ in X there is an increasing open neighborhood of y which does not contain x .

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