Internat. J. Math. & Math. Sci.
Vol. 7 No. 4 (1984) 719-730

ON NEW CLASSES OF ANALYTIC FUNCTIONS WITH NEGATIVE COEFFICIENTS

SHIGEYOSHI OWA

Department of Mathematics
Faculty of Science and Technology
Kinki University
Higashi-Osaka, Osaka
Japan

(Received November 11, 1983)

ABSTRACT. We introduce the classes K_n^* of analytic functions with negative coefficients by using the nth order Ruscheweyh derivative. The object of the present paper is to show coefficient inequalities and some closure theorems for functions f(z) in K_n^*. Further we consider the modified Hadamard product of functions $f_i(z)$ in $K_{n_i}^*$ $(n = 1, 2, \cdots, m)$.

KEY WORDS AND PHRASES. *Ruscheweyh derivative, Analytic functions, Negative coefficients, coefficient inequalities and Hadamard product.*

AMS(MOS) SUBJECT CLASSIFICATION (1980) CODE. 30C45

I. INTRODUCTION.

Let A denote the class of functions f(z) analytic in the unit disk $U = \{z: |z| < 1\}$ and normalized by f(0) = 0 and f'(0) = 1. Ruscheweyh [10] introduced the classes K_n of functions $f(z) \in A$ satisfying

$$\mathrm{Re}\left\{\frac{(z^n f(z))^{(n+1)}}{(z^{n-1}f(z))^{(n)}}\right\} > \frac{n+1}{2} \tag{1.1}$$

for $n \in N \cup \{0\}$ and $z \in U$, where $N = \{1, 2, 3, \cdots\}$. Ruscheweyh [10] showed the basic property

$$K_{n+1} \subset K_n \tag{1.2}$$

for each $n \in N \cup \{0\}$. Note that K_0 is the class $S^*(1/2)$ of starlike functions of order 1/2.

Let

$$D^n f(z) = z(z^{n-1}f(z))^{(n)}/n! \tag{1.3}$$

for $n \in N \cup \{0\}$. This symbol $D^n f(z)$ was named the nth order Ruscheweyh derivative of f(z) by Al-Amiri [1]. We note that $D^0f(z) = f(z)$ and $D^1f(z) = zf'(z)$. The Hadamard product of two functions $f(z) \in A$ and $g(z) \in A$ will be denoted by $f*g(z)$, that is, if f(z) and g(z) are given by

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \tag{1.4}$$

and

$$g(z) = \sum_{n=0}^{\infty} b_n z^n , \tag{1.5}$$

respectively, then

$$f*g(z) = \sum_{n=0}^{\infty} a_n b_n z^n . \tag{1.6}$$

Using Hadamard product, Ruscheweyh [10] observed that if

$$D^{\alpha}f(z) = \frac{z}{(1 - z)^{\alpha+1}} *f(z), \qquad (\alpha \geqq -1) \tag{1.7}$$

then (1.3) is equivalent to (1.7) when $\alpha = n \in N \cup \{0\}$.
Thus it follows from (1.1) that the necessary and sufficient condition for $f(z) \in A$ to belong to K_n is

$$\mathrm{Re}\left\{ \frac{D^{n+1}f(z)}{D^n f(z)} \right\} > \frac{1}{2}, \qquad (z \in U). \tag{1.8}$$

Note that K_{-1} is the class of functions $f(z) \in A$ satisfying

$$\mathrm{Re}\left\{ \frac{f(z)}{z} \right\} > \frac{1}{2}, \qquad (z \in U). \tag{1.9}$$

For further information about the Hadamard products, the reader is advised to consult Ruscheweyh [11].

Recently many classes defined by using the nth order Ruscheweyh derivative of f(z) were studied by Al-Amiri [2], [3], Bulboaca [4], Goel and Sohi [5], [6], Owa [8], [9], and Singh and Singh [13].

In this paper we introduce the following classes by using the nth order Ruscheweyh derivative of f(z). The method of proofs in section 2 follow closely the one used by Silverman [12]. Also several particular results obtained by Silverman [12] and Merkes, Robertson and Scott [7] can be deduced as special cases of our results in section 2.

DEFINITION. We say that f(z) is in the class K_n^* $(n \in \mathbb{N} \cup \{0\})$, if f(z) defined by

$$f(z) = z - \sum_{k=2}^{\infty} a_k z^k \qquad (a_k \geqq 0) \tag{1.10}$$

satisfies (1.8) for $n \in \mathbb{N} \cup \{0\}$.

2. COEFFICIENT INEQUALITIES AND APPLICATIONS.

THEOREM 1. Let the function f(z) be defined by (1.10). Then f(z) is in the class K_n^* if and only if

$$\sum_{k=2}^{\infty} \frac{(k+n-1)!(2k+n-1)}{(k-1)!} a_k \leqq (n+1)! \,. \tag{2.1}$$

Equality holds for the function defined by

$$f(z) = z - \frac{(n+1)!(k-1)!}{(k+n-1)!(2k+n-1)} z^k, \qquad (k \geqq 2). \tag{2.2}$$

PROOF. We use a method of Silverman [12]. Assume that the inequality (2.1) holds. Then we have

$$\left| \frac{D^{n+1}f(z)}{D^n f(z)} - 1 \right|$$

$$= \left| \frac{\sum_{k=2}^{\infty} (k+n-1)(k+n-2)\cdots k(k-1)a_k z^{k-1}}{(n+1)! - (n+1)\sum_{k=2}^{\infty}(k+n-1)(k+n-2)\cdots k a_k z^{k-1}} \right|$$

$$\leq \frac{\sum_{k=2}^{\infty}(k+n-1)(k+n-2)\cdots k(k-1)a_k|z|^{k-1}}{(n+1)! - (n+1)\sum_{k=2}^{\infty}(k+n-1)(k+n-2)\cdots ka_k|z|^{k-1}}$$

$$\leq \frac{\sum_{k=2}^{\infty}(k+n-1)(k+n-2)\cdots k(k-1)a_k}{(n+1)! - (n+1)\sum_{k=2}^{\infty}(k+n-1)(k+n-2)\cdots ka_k}$$

$$\leq \frac{1}{2}. \tag{2.3}$$

This shows that the values of $D^{n+1}f(z)/D^nf(z)$ lie in a circle centered at $w = 1$ whose radius is $1/2$. Consequently we can see that the function $f(z)$ satisfies (1.8), hence further, $f(z) \in K_n^*$.

For the converse, assume that the function $f(z)$ is in the class K_n^*. Then we get

$$\operatorname{Re}\left\{\frac{D^{n+1}f(z)}{D^nf(z)}\right\}$$

$$= \operatorname{Re}\left\{\frac{(n+1)! - \sum_{k=2}^{\infty}(k+n)(k+n-1)\cdots ka_kz^{k-1}}{(n+1)! - (n+1)\sum_{k=2}^{\infty}(k+n-1)(k+n-2)\cdots ka_kz^{k-1}}\right\}$$

$$> \frac{1}{2} \tag{2.4}$$

for $z \in U$. Choose values of z on the real axis so that $D^{n+1}f(z)/D^nf(z)$ is real. Upon clearing the denominator in (2.4) and letting $z \to 1^-$ through real values, we obtain

$$(n+1)! - \sum_{k=2}^{\infty}(k+n)(k+n-1)\cdots ka_k$$

$$\geq \frac{1}{2}\left\{(n+1)! - (n+1)\sum_{k=2}^{\infty}(k+n-1)(k+n-2)\cdots ka_k\right\} \tag{2.5}$$

which implies (2.1).

Finally we can see that the function f(z) defined by (2.2) is an extreme one for the theorem. This completes the proof of the theorem.

COROLLARY 1. Let the function f(z) defined by (1.10) be in the class K_n^*. Then

$$a_k \leqq \frac{(n+1)!(k-1)!}{(k+n-1)!(2k+n-1)} \tag{2.6}$$

for $k \geqq 2$. The equality holds for the function f(z) of the form

$$f(z) = z - \frac{(n+1)!(k-1)!}{(k+n-1)!(2k+n-1)} z^k . \tag{2.7}$$

THEOREM 2. Let the function f(z) defined by (1.10) be in the class K_n^*. Then

$$|f(z)| \geqq |z| - \left(\frac{1}{n+3}\right)|z|^2 \tag{2.8}$$

and

$$|f(z)| \leqq |z| + \left(\frac{1}{n+3}\right)|z|^2 \tag{2.9}$$

for $z \in U$. The results are sharp.

PROOF. Since $(k+n-1)!(2k+n-1)/(k-1)!$ is increasing in k ($k \geqq 2$) and f(z) is in the class K_n^*, in view of Theorem 1, we obtain

$$(n+1)!(n+3) \sum_{k=2}^{\infty} a_k \leqq \sum_{k=2}^{\infty} \frac{(k+n-1)!(2k+n-1)}{(k-1)!} a_k$$

$$\leqq (n+1)! \tag{2.10}$$

which gives that

$$\sum_{k=2}^{\infty} a_k \leqq \frac{1}{n+3} . \tag{2.11}$$

Hence we can show that

$$|f(z)| \geqq |z| - |z|^2 \sum_{k=2}^{\infty} a_k$$

$$\geqq |z| - \left(\frac{1}{n+3}\right)|z|^2 \tag{2.12}$$

and

$$|f(z)| \leqq |z| + |z|^2 \sum_{k=2}^{\infty} a_k$$

$$\leqq |z| + \left(\frac{1}{n+3}\right)|z|^2 \tag{2.13}$$

for $z \in U$.

Further, by taking the function

$$f(z) = z - \left(\frac{1}{n+3}\right)z^2 \tag{2.14}$$

we can see that the results of the theorem are sharp.

COROLLARY 2. Let the function f(z) defined by (1.10) be in the class K_n^*. Then f(z) is included in a disk with its center at the origin and radius r given by

$$r = \frac{n+4}{n+3} \; . \tag{2.15}$$

THEOREM 3. Let the function f(z) defined by (1.10) be in the class K_n^*. Then

$$|f'(z)| \geqq 1 - \left(\frac{2}{n+3}\right)|z| \tag{2.16}$$

and

$$|f'(z)| \leqq 1 + \left(\frac{2}{n+3}\right)|z| \tag{2.17}$$

for $z \in U$. The results are sharp.

PROOF. Note that $(k + n - 1)!(2k + n - 1)/(k - 1)!$ is equal to $(k + n - 1)!k(2k + n - 1)/k!$ and $(k + n - 1)!(2k + n - 1)/k!$ is an increasing function of k $(k \geqq 2)$. Hence, by virtue of Theorem 1, we have

$$\frac{(n + 1)!(n + 3)}{2} \sum_{k=2}^{\infty} ka_k \leqq \sum_{k=2}^{\infty} \frac{(k + n - 1)!(2k + n - 1)}{(k - 1)!} a_k$$

$$\leqq (n + 1)! \tag{2.18}$$

which gives that

$$\sum_{k=2}^{\infty} ka_k \leqq \frac{2}{n + 3} . \tag{2.19}$$

Consequently, with the aid of (2.19), we can see that

$$|f'(z)| \geqq 1 - |z| \sum_{k=2}^{\infty} ka_k$$

$$\geqq 1 - \left(\frac{2}{n + 3}\right)|z| \tag{2.20}$$

and

$$|f'(z)| \leqq 1 + |z| \sum_{k=2}^{\infty} ka_k$$

$$\leqq 1 + \left(\frac{2}{n + 3}\right)|z| \tag{2.21}$$

for $z \in U$.

Further the bounds of the theorem are attained by the function $f(z)$ given by (2.14).

COROLLARY 3. Let the function $f(z)$ defined by (1.10) be in the class K_n^*. Then $f'(z)$ is included in a disk with its center at the origin and radius R given by

$$R = \frac{n + 5}{n + 3} . \tag{2.22}$$

3. CLOSURE THEOREMS.

THEOREM 4. Let the functions

$$f_i(z) = z - \sum_{k=2}^{\infty} a_{k,i} z^k \qquad (a_{k,i} \geqq 0) \tag{3.1}$$

be in the class K_n^* for every i = 1, 2, 3, ⋯, m. Then the function h(z) defined by

$$h(z) = \sum_{i=1}^{m} c_i f_i(z) \qquad (c_i \geqq 0) \tag{3.2}$$

is also in the same class K_n^*, where

$$\sum_{i=1}^{m} c_i = 1 \ . \tag{3.3}$$

PROOF. By means of the definition of h(z), we can see that

$$h(z) = z - \sum_{k=2}^{\infty} \left(\sum_{i=1}^{m} c_i a_{k,i} \right) z^k \ . \tag{3.4}$$

Further, since $f_i(z)$ are in K_n^* for every i = 1, 2, 3, ⋯, m, we obtain

$$\sum_{k=2}^{\infty} \frac{(k + n - 1)!(2k + n - 1)}{(k - 1)!} a_{k,i} \leqq (n + 1)! \tag{3.5}$$

for every i = 1, 2, 3, ⋯, m. Consequently we have

$$\sum_{k=2}^{\infty} \frac{(k + n - 1)!(2k + n - 1)}{(k - 1)!} \left(\sum_{i=1}^{m} c_i a_{k,i} \right)$$

$$= \sum_{i=1}^{m} c_i \left\{ \sum_{k=2}^{\infty} \frac{(k + n - 1)!(2k + n - 1)}{(k - 1)!} a_{k,i} \right\}$$

$$\leqq \left(\sum_{i=1}^{m} c_i \right) (n + 1)!$$

$$= (n + 1)! \tag{3.6}$$

by using (3.5). This shows that the function h(z) belongs to the class K_n^*. Thus we have the theorem.

THEOREM 5. Let

$$f_1(z) = z \tag{3.7}$$

and

$$f_k(z) = z - \frac{(k-1)!(n+1)!}{(k+n-1)!(2k+n-1)} z^k \tag{3.8}$$

for $k \in \mathbb{N} - \{1\}$. Then $f(z)$ is in the class K_n^* if and only if it can be expressed in the form

$$f(z) = \sum_{k=1}^{\infty} \lambda_k f_k(z) , \tag{3.9}$$

where $\lambda_k \geqq 0$ for $k \in \mathbb{N}$ and

$$\sum_{k=1}^{\infty} \lambda_k = 1 . \tag{3.10}$$

PROOF. Suppose that

$$f(z) = \sum_{k=1}^{\infty} \lambda_k f_k(z)$$

$$= z - \sum_{k=2}^{\infty} \frac{(k-1)!(n+1)!}{(k+n-1)!(2k+n-1)} \lambda_k z^k . \tag{3.11}$$

Then we get

$$\sum_{k=2}^{\infty} \left\{ \frac{(k+n-1)!(2k+n-1)}{(k-1)!} \frac{(k-1)!(n+1)!}{(k+n-1)!(2k+n-1)} \lambda_k \right\}$$

$$= (n+1)! \sum_{k=2}^{\infty} \lambda_k$$

$$= (n+1)!(1-\lambda_1)$$

$$\leqq (n+1)! . \tag{3.12}$$

Thus we can see that f(z) is in the class K_n^* with the aid of Theorem 1.

Conversely, suppose that f(z) is in the class K_n^*. Again, by (2.6), we obtain that

$$a_k \leqq \frac{(k-1)!(n+1)!}{(k+n-1)!(2k+n-1)} \tag{3.13}$$

for $k \in \mathbb{N} - \{1\}$. Now, setting

$$\lambda_k = \frac{(k+n-1)!(2k+n-1)}{(k-1)!(n+1)!} a_k \tag{3.14}$$

for $k \in \mathbb{N} - \{1\}$ and

$$\lambda_1 = 1 - \sum_{k=2}^{\infty} \lambda_k , \tag{3.15}$$

we have the representation (3.9). This completes the proof of the theorem.

4. MODIFIED HADAMARD PRODUCT.

Let f(z) be defined by (1.10) and g(z) be defined by

$$g(z) = z - \sum_{k=2}^{\infty} b_k z^k \qquad (b_k \geqq 0). \tag{4.1}$$

Further let f∗g(z) denote the modified Hadamard product of f(z) and g(z), that is,

$$f*g(z) = z - \sum_{k=2}^{\infty} a_k b_k z^k . \tag{4.2}$$

THEOREM 6. Let the functions $f_i(z)$ defined by (3.1) be in the classes $K_{n_i}^*$ for each i = 1, 2, 3, ⋯, m, respectively. Then the modified Hadamard product $f_1*f_2*\cdots*f_m(z)$ belongs to the class K_n^*, where $n = \underset{1\leqq i\leqq m}{\mathrm{Min}} \{n_i\}$.

PROOF. We may suppose that $n_1 = \min_{1 \leq i \leq m} \{n_i\}$. Then, by using $f_i(z) \in K^*_{n_i}$ $(i = 1, 2, 3, \cdots, m)$, we can know that (2.11) would imply

$$a_{k,i} \leq \frac{1}{n_i + 3} \qquad (i = 2, 3, 4, \cdots, m) \tag{4.3}$$

and

$$\sum_{k=2}^{\infty} \frac{(k + n_1 - 1)!(2k + n_1 - 1)}{(k - 1)!} a_{k,1} \leq (n_1 + 1)! \; . \tag{4.4}$$

Consequently, putting $n_1 = n$ we can see that

$$\sum_{k=2}^{\infty} \frac{(k + n - 1)!(2k + n - 1)}{(k - 1)!} \left(\prod_{i=1}^{m} a_{k,i} \right)$$

$$\leq \prod_{i=2}^{m} \left(\frac{1}{n_i + 3} \right) \sum_{k=2}^{\infty} \frac{(k + n - 1)!(2k + n - 1)}{(k - 1)!} a_{k,1}$$

$$\leq (n + 1)! \prod_{i=2}^{m} \left(\frac{1}{n_i + 3} \right)$$

$$\leq (n + 1)! \; . \tag{4.5}$$

Hence we have the theorem.

COROLLARY 4. Let the functions $f_i(z)$ defined by (3.1) be in the same class K^*_n for every $i = 1, 2, 3, \cdots, m$. Then the modified Hadamard product $f_1 * f_2 * \cdots * f_m(z)$ also belongs to the class K^*_n.

5. ACKNOWLEDGEMENT.

The author would like to thank the referee for his thoughtful encouragement and countless helpful advices in preparation of this paper.

REFERENCES

1. Al-Amiri, H. S. On Ruscheweyh derivatives, Ann. Polon. Math. 38(1980), 87 - 94.
2. Al-Amiri, H. S. Certain analogy of the α-convex functions, Rev. Roumaine Math. Pures Appl. 23(1978), 1449 - 1454.
3. Al-Amiri, H. S. On the Ruscheweyh-Mocanu alpha convex functions of order n, Mathematica 22(1980), 207 - 213.
4. Bulboaca, T. Asupra unor noi clase de functii analitice, Studia Univ. Babes-Bolyai Math. 26(1981), 42 - 46.
5. Goel, R. M. and Sohi, N. S. Subclasses of univalent functions, Tamkang J. Math. 11(1980), 77 - 81.
6. Goel, R. M. and Sohi, N. S. A new criterion for univalence and its applications, Glas. Mat. 16(1981), 39 - 49.
7. Merkes, E. P., Robertson, M. S. and Scott, W. I. On products of starlike functions, Proc. Amer. Math. Soc. 13(1962), 960 - 964.
8. Owa, S. On the Ruscheweyh's new criteria for univalent functions, Math. Japonica 27(1982), 77 - 96.
9. Owa, S. On new criteria for analytic functions, Tamkang J. Math. 13(1982), 201 - 213.
10. Ruscheweyh, S. New criteria for univalent functions, Proc. Amer. Math. Soc. 49(1975), 109 - 115.
11. Ruscheweyh, S. Convolutions in geometric function theory, Les Presses De l'université Montréal, Canada (1982).
12. Silverman, H. Univalent functions with negative coefficients, Proc. Amer. Math. Soc. 51(1975), 109 - 116.
13. Singh, R. and Singh, S. Integrals of certain univalent functions, Proc. Amer. Math. Soc. 77(1979), 336 - 340.