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ON SOME EXTENSIONS OF HARDY'S INEQUALITY

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ABSTRACT. We present in this paper some new integral inequalities which are related to Hardy's inequality, thus bringing into sharp focus some of the earlier results of the author.

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1. INTRODUCTION.

The present note is a sequel to the author's paper [1] in which the following theorem that generalizes Shum's result [2] was proved.

THEOREM 1.1. Let g be continuous and non-decreasing on $[o,\infty]$ with $g(0) = 0$, $g(x) > 0$ for $x > 0$ and $g(\infty) = \infty$. Let $p \geq 1$, $r \neq 1$ and $f(x)$ be non-negative and Lebesgue-Stieltjes integrable with respect to $g(x)$ on $[0,b]$ or $[a,\infty]$ according as $r > 1$ or $r < 1$, where $a > 0$ and $b > 0$. Suppose

$$F(x) = \begin{cases} \int_0^x f(t)dg(t) & (r > 1) \\ \int_x^\infty f(t)dg(t) & (r < 1). \end{cases} \tag{1.1}$$

Then

$$\int_0^b g(x)^{-r} F(x)^p dg(x) + [p/(r-1)]^p g(b)^{1-r} F(b)^p \leq [p/(r-1)]^p \int_0^b g(x)^{-r} [g(x) f(x)]^p dg(x) \quad (r > 1) \tag{1.2}$$

and

$$\int_a^\infty g(x)^{-r} F(x)^p dg(x) + [p/(1-r)]^p g(a)^{1-r} F(a)^p \leq [p/("1-r")]^p \int_a^\infty g(x)^{-r} [g(x) f(x)]^p dg(x) \quad (r < 1), \tag{1.3}$$

with both inequalities reversed in $0 < p \leq 1$.

Equality holds in either inequality, when either $p = 1$ or $f = 0$. The constant $[p/(r-1)]^p$ or $[p/(i-r)]^p$ is the best possible when the left side of (1.2) or (1.3) is unchanged, respectively.

The objective of this paper is to obtain a new integral inequality which is an extension of Theorem 1.1. Indeed, we shall show that Theorem 1.1 in its modified form leads us to some extensions, variants and a new generalization of a class of inequalities which are related to Hardy's integral inequality. Moreover, we shall examine the case when $r = 1$ in Theorem 1.1, a case which was not discussed in [1]. In fact, certain extensions of Hardy's inequality due to Ling-Yau Chan [3], which seemed new, are shown to be immediate consequences of the modified form of Theorem 1.1.

In Section 2 we state our main result. An immediate corollary of this result shows that the inequalities (1.2) and (1.3) continue to persist except for an added constant when $\int_0^b$ in (1.2) and $\int_a^\infty$ are replaced by $\int_a^b$, where $0 \le a < b \le \infty$.

In Section 3, we prove an elementary lemma which is then used to give the proof of our main result. In Section 4, our result is applied to give some useful variants and extensions of Hardy's inequality.

Throughout what follows, unless otherwise stated, we shall assume g is continuous, non-negative and non-decreasing on $[c,d]$ with $g(c) = 0$ and $g(d) = \infty$. Furthermore, f is assumed to be a non-negative Lebesgue-Stieltjes integrable function with respect to g on $[c,d]$. Our notations and terminologies are similar to the ones in [2]. Thus, for real numbers $p \neq 0$ and $\delta \neq 0$, the functions θ, F and $Q(.,\delta,F)$ on $[c,d]$ are defined as follows:

$$\theta(x) = \begin{cases} \int_c^x g(t)^{(p-1)(\delta+1)} f(t)^p dg(t) & (\delta < 0) \\ \int_x^d g(t)^{(p-1)(\delta+1)} f(t)^p dg(t) & (\delta > 0), \end{cases} \tag{1.4}$$

$$F(x) = \begin{cases} \int_c^x f(t)\, dg(t) & (\delta < 0) \\ \int_x^d f(t)\, dg(t) & (\delta > 0) \end{cases} \tag{1.5}$$

and

$$Q(x,\delta,F) = g(x)^{\delta}\, \theta(x) - |\delta|^{p-1}\, g(x)^{\delta p}\, F(x)^p. \tag{1.6}$$

2. STATEMENT OF MAIN RESULT.

Our main result is the following.

THEOREM 2.1. Let $f(t) \in L((c,x),dg)$ or $f(t) \in L((x,d),dg)$ for every $x \in (a,b)$ according as $\delta < 0$ or $\delta > 0$, where $c \le a < b \le d$. Suppose for $p \ge 1$ or $p < 0$, $\int_c^d g(x)^{\delta p-1}[g(x)f(x)]^p\, dg(x) < \infty$ and $\int_c^d g(x)^{\delta p-1}F(x)^p\, dg(x) < \infty$ if $0 < p \le 1$. then we have

$$\int_a^b g(x)^{\delta p-1}F(x)^p dg(x) \le \delta^{-1}[g(b)^{\delta p}F(b)^p - g(a)^{\delta p}F(a)^p] + |\delta|^{-p} \int_a^b g(x)^{\delta p-1}[g(x)f(x)]^p dg(x) \tag{2.1}$$

whenever $p \ge 1$ or $p < 0$. The inequality is reversed if $0 < p \le 1$. The inequality is strict unless either $p = 1$ or $f = 0$. The constant factor $|\delta|^{-p}$ is the best possible when the term $\delta^{-1}[g(b)^{\delta p}F(b)^p - g(a)^{\delta p}\, F(a)^p]$ remains unchanged.

REMARK 2.2. For $\delta = (1-r)/p$, $r \neq 1$ and $p \ge 1$ or $p < 0$, we obtain the following results.

$$\int_a^b g(x)^{-r}F(x)^p dg(x) \le [p/(1-r)][g(b)^{1-r}F(b)^p - g(a)^{1-r}F(a)^p] + [p/(r-1)]^p \int_a^b g(x)^{-r}[g(x)f(x)]^p dg(x) \qquad (r > 1) \tag{2.2}$$

and

$$\int_a^b g(x)^{-r}F(x)^p dg(x) \le [p/(1-r)][g(b)^{1-r}F(b)^p - g(a)^{1-r}F(a)^p] + [p/(1-r)]^p \int_a^b g(x)^{-r}[g(x)f(x)]^p dg(x) \qquad (r < 1); \tag{2.3}$$

when $0 < p \le 1$ the inequalities are reversed.

These inequalities generalize our earlier results in [1], namely Theorem 1.1.

REMARK 2.3. Suppose the hypotheses $g(c) = 0$ and $g(d) = \infty$ are relaxed. On replacing g by $\sigma o g$ and f by $[\sigma'(g)]^{-1}f$, where σ is differentiable, non-negative and non-decreasing on $[g(c) = g(d)]$ with $(\sigma o g)(c) = 0$ and $(\sigma o g)(d) = \infty$, we obtain from inequality (2.1) the following form

$$\begin{aligned}&\int_a^b \sigma'(g(x))\sigma(g(x))^{\delta p-1} G(x)^p\, dg(x)\\ &\le \delta^{-1}[\sigma,(g(b))^{\delta p} F(b)^p - \sigma(g(a))^{\delta p} F(a)^p\\ &+ |\delta|^{-p} \int_a^b \sigma'(g(x))^{1-p}\sigma(g(x))^{\delta p-1}[\sigma(g(x))f(x)]^p dg(x)\end{aligned} \tag{2.4}$$

where F is, however, still defined by (1.5) and $p \ge 1$ or $p < 0$. Inequality (2.4) is reversed if $0 < p \le 1$.

3. PROOF OF THEOREM: The proof of our main result will depend essentially on the following lemma.

LEMMA: For $p \ge 1$ or $p < 0$, the function $Q(.,\delta,F)$, $\delta \neq 0$, is positive and non-decreasing or non-increasing on $[c,d]$ according as $\delta < 0$ or $\delta > 0$. If however $0 < p \le 1$, the function $Q(.,\delta,F)$, $\delta \neq 0$, is negative and non-increasing or non-decreasing on $[c,d]$ according as $\delta < 0$ or $\delta > 0$.

PROOF: We shall show that the integrals defining $\theta(x)$ exist uncer the hypotheses of the theorem. For $p \ge 1$ or $p < 0$ and $\delta < 0$,

$$\begin{aligned}0 \le \theta(x) &= \int_c^x g(t)^{(p-1)(\delta+1)} f(t)^p dg(t)\\ &\le g(x)^{-\delta}\int_c^x g(t)^{\delta p-1}[g(t)f(t)]^p\, dg(t)\\ &\le g(x)^{-\delta}\int_c^d g(t)^{\delta p-1}[g(t)f(t)]^p\, dg(t) < \infty\end{aligned}$$

which is obtained from the non-decreasing property of g and the hypothesis of the theorem.

For $p \ge 1$ or $p < 0$ and $\delta > 0$, we have by the same token

$$\begin{aligned}0 \le \theta(x) &= \int_x^d g(t)^{(p-1)(\delta+1)} f(t)^p\, dg(t)\\ &\le g(x)^{-\delta}\int_x^d g(t)^{\delta p-1}[g(t)f(t)]^p\, dg(t)\\ &\le g(x)^{-\delta}\int_c^d g(t)^{\delta p-1}[g(t)f(t)]^p\, dg(t) < \infty.\end{aligned}$$

For $p \geq 1$ or $p < 0$, $\underline{t} = (t_1,t_2)$, $t_1,t_2 \in [c,d]$, let $h(t) = g(t_2)^{p(\delta+1)}f(t_2)^p$ and let $d\lambda(\underline{t}) = g(t_1)^{-1-p}g(t_2)^{-(\delta+1)}dg(t_1)\, dg(t_2)$. Suppose $x^* = (x,x)$, $c^* = (c,c)$ and $d^* = (d,d)$ where $x \in [c,d]$. Then, $(|\delta|p)^{-1}\, g(x)^{-\delta(p+1)}\, Q(x,\delta,F)$

$$= \begin{cases} \int_{c^*}^{x^*} h(\underline{t})d\lambda(\underline{t}) - [\int_{c^*}^{x^*} d\lambda(\underline{t})]^{1-p}[\int_{c^*}^{x^*} h(\underline{t})^{1/p}d\lambda(t)]^p & (\delta < 0) \\ \int_{x^*}^{d^*} h(\underline{t})d\lambda(\underline{t}) - [\int_{x^*}^{d^*} d\lambda(\underline{t})]^{1-p}[\int_{x^*}^{d^*} h(\underline{t})^{1/p}d\lambda(\underline{t})]^p & (\delta > 0). \end{cases}$$

Hence, the non-negative property of $Q(x,\delta,F)$, $x \in [c,d]$, $\delta \neq 0$, is a direct consequence of Jensen's inequality for convex functions.

Also from the Jensen-Steffensen inequality for sums, we have for $p \geq 1$ or $p > 0$,

$$P_n(z,q,p) = \sum_{i=1}^{n} q_i z_i - (\sum_{i=1}^{n} q_i)^{1-p}(\sum_{i=1}^{n} q_i z_i^{\;1/p})^p \geq 0,$$

where $(z_i)_{1 \leq i \leq n}$ is non-decreasing and $(q_i)_{1 \leq i \leq n}$ satisfies the conditions

$$0 \leq \sum_{i=\nu}^{n} q_i \leq \sum_{i=1}^{n} q_i \quad (1 \leq \nu \leq n) \quad \text{with} \quad \sum_{i=1}^{n} q_i > 0.$$

In particular, for $n = 2$, we have

$$0 \leq q_1z_1 + q_2z_2 - (q_1 + q_2)^{1-p}(q_1z_1^{\;1/p} + q_2z_2^{\;1/p})^p. \tag{3.1}$$

Now suppose $c \leq y \leq x \leq d$. Making use of the substitutions

$$q_1 = \int_{c^*}^{y^*} d\lambda(t), \qquad q_2 = \int_{y^*}^{x^*} d\lambda(\underline{t}),$$

$$z_1 = [\int_{c^*}^{y^*} h(\underline{t})^{1/p}d\lambda(\underline{t}) \;/ \int_{c^*}^{y^*} d\lambda(\underline{t})]^p, \quad \text{and}$$

$$z_2 = [\int_{y^*}^{x^*} h(\underline{t})^{1/p}d\lambda(\underline{t}) \;/ \int_{y^*}^{y^*} d\lambda(\underline{t})]^p$$

in inequality (3.1), we get

$$0 \leq [\int_{c^*}^{y^*} d\lambda(\underline{t})]^{1-p}[\int_{c^*}^{y^*} h(\underline{t})^{1/p}d\lambda(\underline{t})]^p$$
$$+ [\int_{y^*}^{x^*} d\lambda(\underline{t})]^{1-p}[\int_{y^*}^{x^*} h(\underline{t})^{1/p}d\lambda(\underline{t})]^p$$
$$- [\int_{c^*}^{x^*} d\lambda(\underline{t})]^{1-p}[\int_{c^*}^{x^*} h(\underline{t})^{1/p}d\lambda(\underline{t})]^p .$$

On the application of Jensen's inequality to the second summand, we obtain

$$(|\delta|p)^{-1}\, g(x)^{-\delta(p+1)}[Q(x,\delta,F) - Q(y,\delta,F)] \geq 0.$$

Hence $(|\delta|p)^{-1}g(.)^{-\delta(p+1)}\, Q(.,\delta,F)$ is positive and non-decreasing on $[c,d]$. But $(|\delta|p)^{-1}g(x)^{-\delta(p+1)}$ is zero for $x = c$ and non-decreasing for $x \geq c$ and $Q(c,\delta,F) = 0$; whence for $\delta < 0$, $Q(x,\delta,F)$ is non-negative and non-decreasing on $[c,d]$. Consequently, the assertion of the lemma is valid for $p \geq 1$ or $p < 0$. Similar argument also shows that for $p \geq 1$ or $p < 0$ and $\delta > 0 Q(\delta,F)$ is non-negative and non-decreasing on

$[c,d]$. Also, it can be proved in the same manner that $Q(.,\delta,F)$ for $0 < p \le 1$ and $\delta \neq 0$ is negative and non-increasing or non-decreasing on $[c,d]$ according as $\delta < 0$ oθ $\delta > 0$. This completes the proof of the lemma.

REMARK 3.1. We note in passing that Holder's inequality yields $Q(x,\varepsilon,F) \ge 0$ for $c \le x \le d$ when $p \ge 1$ or $p < 0$, $\delta < 0, \delta > 0$ and $0 < p \le 1$, $\delta < 0, \delta > 0$. For example, for $p \ge 1$ and $\delta < 0$ we have

$$F(x) = \int_c^x f(t)g(t)^s\, g(t)^{-s}dg(t) \le \theta(x)^{1/p}[\ {}_c^x\ g(t)^{-\delta-1}dg(t)]^{1-(1/p)}$$

where $s = (p + 1)(\delta + 1)/p$. The result follows after raising both sides to the power p and simplifying.

The method by which Theorem 1.1 is obtained in [1] will be used to prove our main result. Suppose $p \ge 1$ or $p < 0$, $\delta \neq 0$, and $c \le a < b \le d$. Using first the non-negative property of $Q(.,\delta,F)$, next an application of integration by parts, and finally the monotonicity of $Q(.,\delta,F)$ (cf. Lemma), we obtain

$$|\delta|^{p-1}\int_a^b g(x)^{\delta p-1}\, F(x)^p dg(x) \le \int_a^b g(x)^{\delta-1}\, \theta(x)\, dg(x)$$

$$= \delta^{-1}[g(b)^\delta\, \theta(b) - g(a)^\delta\, \theta(a)] + |\delta|^{-1}\int_a^b g(x)^{\delta p-1}[g(x)f(x)]^p dg(x)$$

$$\le \delta^{-1}\, |\delta|^{p-1}[g(b)^{\delta p}F(b)^p - g(a)^{\delta p}F(a)^p]$$

$$+ |\delta|^{-1}\int_a^b\, g(x)^{\delta p-1}[g(x)f(x)]^p\, dg(x).$$

Hence,

$$\int_a^b g(x)^{\delta p-1}F(x)^p dg(x) \le \delta^{-1}[g(b)^{\delta p}F(b)^p - g(a)^{\delta p}F(a)^p]$$

$$+ |\delta|^{-p}\int_a^b g(x)^{\delta p-1}[g(x)f(x)]^p\, dg(x).$$

This proves the assertion of the theorem when $p \ge 1$ or $p < 0$. The proof of the theorem when $0 < p \le 1$ is similar; hence it is omitted. We also omit the proof of the exactness of constant and the conditions for equality since the proof is similar to the one given in [2]. Thus the proof of the theorem is complete.

4. APPLICATIONS. We examine here some of the consequences of Theorem 2.1. The conditions for equality and the exactness of constant as stated in Theorem 2.1 will be tacitly assumed in all our results. The next two results yield the case $r = 1$ of Theorem 1.1.

COROLLARY 4.1. Let $p \ge 1$ or $p < 0$ and $r \neq 1$. For $c \le a < b \le d$, let $f(t) \in L((c,x)),dg)$ or $f(t) \in L((x,d),dg)$ for every $x \in (a,b)$ according as $r > 1$ or $r < 1$, where $g(c) = 1$ and $g(d) = \infty$. Suppose

$$F(x) = \begin{cases} \int_c^x f(t)\, dg(t) & (r > 1), \\ \int_x^d f(t)\, dg(t) & (r < 1). \end{cases} \tag{4.1}$$

Then we have

$$\int_a^b g(x)^{-1}[(\log g(x))^{-r/p} F(x)]^p \, dg(x)$$

$$\leq [p/(1-r)][(\log g(b))^{-1-r}F(b)^p - (\log g(a))^{1-r}F(a)^p] \tag{4.2}$$

$$+ [p/|1-r|]^p \int_a^b g(x)^{-1}[g(x)(\log g(x))^{1-(r/p)}f(x)]^p \, dg(x);$$

with the inequality reversed if $0 < p \leq 1$.

PROOF. The assertion of the Corollary follows on taking

$\sigma(u) = \log u$, $1 \leq u \leq \infty$, and $\delta = (1-r)/p$, $r \neq 1$ in inequality (2.4).

REMARK. For $g(x) = x$, $r = 0$ and $r = p$ we obtain as special cases of the Corollary, Chan's results [3, Theorem 1] and [3, Theorem 3] respectively.

COROLLARY 4.3. Let $p \geq 1$ or $p < 0$ and $r \neq 1$. For $c \leq a < b \leq d$, let $f(t) \in L((c,x),dg)$ or $f(t) \in L((x,d),dg)$ according as $r > 1$ or $r < 1$ where $x \in (a,b)$, $g(c) = 0$ and $g(d) = 1$. Suppose F is as in (4.1). Then

$$\int_a^b g(x)^{-1}[|\log g(x)|^{(r-2)/p}F(x)]^p \, dg(x)$$

$$\leq [p/(1-r)][|\log g(b)|^{r-1} F(b)^p - |\log g(a)|^{r-1} F(a)^p] \tag{4.3}$$

$$+ [p/|1-r|]^p \int_a^b g(x)^{-1}[g(x)|\log g(x)|^{1+((r-2/p)}f(x)]^p dg(x);$$

when $0 < p < 1$, the inequality is reversed.

PROOF. Take $\sigma(u) = |\log u|^{-1}$, $0 \leq u \leq 1$ and $\delta = (1-r)/p$, $r \neq 1$ in inequality (2.4) and the conclusion of the Corollary follows at once.

REMARK 4.4. Corollary 4.3 is a generalization of Chan's results, [3, Theorems 2 and 4]. Indeed, if we take $g(x) = x$, $r = 2$ and $r = 2-p$, $p > 1$ in the Corollary, we obtain Theorems 2 and 4 in [1], respectively.

Finally, we note that many interesting inequalities may be obtained by specializing any or all of the functions $f(x)$, $g(x)$ and the real number $\delta \neq 0$. For example, $f(x)$ may be replaced by $f(x)(g'(x))^{-1}$ in inequalities (2.2), (2.3), (4.2) and (4.3). Of particular interest are inequalities (2.2) and (2.3) which for $p \geq 1$ or $p < 0$ become

$$\int_a^b \sigma_o(x) \, F(x)^p dx \leq [p/(1-r)][\sigma_2(b)F(b)^p - \sigma_2(a) \, F(a)^p]$$

$$+ [p/|1-r|]^p \int_a^b \sigma_1(x)f(x)^p dx,$$

where $\sigma_o(x) \; g(x)^{-r}g'(x)$, $\sigma_1(x) = g(x)^{p-r} \, (g'(x))^{1-p}$ and

$$\sigma_2(x) = \sigma_o(x)^{1-1/p}\sigma_1(x)^{1/p} = g(x)^{1-r}.$$

Inequality (4.4) is reversed when $0 < p \leq 1$. For $r = p$, $a = c$ and $g(x) = e^{\beta(x)} - e^{\beta(c)}$, where β is non-negative and non-decreasing with $\beta(x) \to \infty$ as $x \to d$, we obtain an extension of certain inequalities considered by Kufner and Triebel [4].

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