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UNIFORM DISTRIBUTION OF HASSE INVARIANTS

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ABSTRACT. I. Schur's study of simple algebras around the turn of the century, and subsequent investigations by R. Brauer, E. Witt and others, were later reformulated in terms of what is now called the Schur subgroup of the Brauer group. During the last twenty years this group has generated substantial interest and numerous palatable results have ensued. Among these is the discovery that elements of the Schur group satisfy uniform distribution of Hasse invariants. It is the purpose of this paper to continue an investigation of the latter concept and to highlight certain applications of these results, not only to the Schur group, but also to embeddings of simple algebras and extensions of automorphisms, among others.

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1. INTRODUCTION:

For a field K of characteristic zero, the Schur subgroup S(K) of the Brauer group B(K) consists of all equivalence classes [A] which contain a K-isomorphic copy of a simple summand of the group algebra KG for some finite group G. We restrict our attention to fields of characteristic zero since S(K) is trivial when K has non-zero characteristic. This follows from Weddeburn's well-known theorem which states that a finite-dimensional division algebra over a finite field is itself a field.

It was shown in Benard et al [1, Theorem 1, p.380] that when K/Q is finite abelian and $[A] \in S(K)$ with index m then: (1.1) ϵ_m, a primitive m^{th} root of unity, is in K, and (1.2) suppose $\sigma \in G(K/Q)$, the Galois group of K/Q, with $\epsilon_m^\sigma = \epsilon_m^{b_\sigma}$. If P is K-prime then the following relationship holds between the Hasse invariants of A: $\mathrm{inv}_P(A) \equiv b_\sigma \mathrm{inv}_{P^\sigma}(A) \pmod 1$.

Algebras satisfying (1.1)-(1.2) are called <u>algebras with uniformly distributed Hasse invariants</u>. Such algebras can be shown to form a subgroup U(K) of B(K), and therefore we have S(K) as a subgroup of U(K). For K/Q abelian we have studied the relationship between S(K) and U(K) in Mollin [2,9]. However the study of U(K) has proved valuable from several vantage points not directly related to the group itself. In particular certain tools have been developed in the aforementioned papers which have proved to be valuable in answering related open questions. Some examples are as follows. In

Mollin [10] we proved a conjecture which stated that if the center of a finite dimensional division algebra, D, contains no nontrivial odd order roots of unity then all finite odd order subgroups of the multiplicative group of D are cyclic. This problem is related to a conjecture of I.N. Heistein and results of S. Amitsur. Moreover in Mollin [11] we provided sufficient conditions for the existence of a splitting field L of an absolutely irreducible character χ of a finite group of exponent n, such that $L \subseteq Q(\epsilon_n)$ and $|L:Q(\chi)| = m_Q(\chi)$, the Schur index of χ over Q. This result is a natural outcropping of R. Brauer's well-known theorem stating that $Q(\epsilon_n)$ splits χ. Furthermore, in Mollin [12 - 14] we used the aforementioned techniques to answer specific questions pertaining to the structures of division algebras, the Schur index, and class field theory proper.

Given the above, it is natural to seek a more general context for uniform distribution. In Mollin [15] we observed that $U_F(K)$, for K/F a finite Galois extension of number fields, could be defined as those elements of B(K) satisfying (1.1) and (1.2), where Q is replaced by F. Moreover, as with the abelian case, we have that $U_F(K)$ is a subgroup of B(K) and S(K) is again a subgroup of $U_F(K)$. Furthermore another property which was proved in Bernard [16] to hold for S(K) carries over to $U_F(K)$. We isolate this property since it will be of independent interest later: (1.3) Let K/F be normal and suppose $[A] \in U_F(K)$. If $\hat{P}$ and $\hat{Q}$ are K-primes above an F-prime P then $A \otimes_K K_{\hat{P}}$ and $A \otimes_K K_{\hat{Q}}$ have the same index, where $K_{\hat{P}}$ (respectively $K_{\hat{Q}}$) denotes the completion of K at $\hat{P}$ (respectively $\hat{Q}$). The common value of indices $A \otimes_K K_{\hat{P}}$ for all K-primes $\hat{P}$ above P is called the <u>P-local index of A</u>, denoted $\mathrm{ind}_P(A)$.

Now we ask whether $U_F(K)$ may somehow be salvaged when K/F is any (not necessarily normal) extension of number fields. We define $U_F(K)$ to consist of those $[A] \in B(K)$ such that $[A \otimes_K L] \in U_F(L)$, where L is the normal closure of K over F. We call $U_F(K)$ <u>the group of algebras with uniformly distributed invariants over K relative to F</u>. It is straightforward to check that $U_F(K)$ is a subgroup of B(K) and that S(K) is in turn a subgroup of $U_F(K)$. The latter fact follows from the fact that $S(K) \otimes_K L$ is in $U_F(L)$. However when $K \neq L$, $U_F(K)$ differs markedly from $U_F(L)$. In particular the following example shows that (1.3) fails to hold for $U_F(K)$.

EXAMPLE 1.4. Let $K = Q(\theta,\epsilon_4)$ where θ is a real root of $f(x) = x^3 - 2$ and let $F = Q$. Then the normal closure of K over F is $L = Q(\theta,\epsilon_{12})$. It can be verified that the prime 29 splits into four unramified K-primes P_1,P_2,P_3 and P_4 with P_i for $i = 1,2$ having inertial degrees equal to one over Q, whereas P_i for $i = 3,4$ have inertial degrees equal to two over Q.

We now define a central simple K-algebra, A, as follows: let $\mathrm{inv}_{P_i}(A) = 1/2$ for $i = 3,4$; $\mathrm{inv}_{P_1}(A) = 1/4$ and $\mathrm{inv}_{P_2}(A) = 3/4$ while $\mathrm{inv}_{Q}(A) = 0$ for all K-primes $Q \neq P_i$ where $i = 1,2,3,4$. By the Hasse sum theorem (see Reiner [17]) we are guaranteed that $[A] \in B(K)$. Now let $\hat{P}_i$ be an L-prime above P_i for $i = 1,2,3,4$. It can be verified that $\hat{P}_i$ for $i = 1,2$ have inertial degrees equal to 2 in L over K and $\hat{P}_i$ for $i = 3,4$ have inertial degrees equal to 1 in L over K. Hence we obtain that $\mathrm{inv}_{\hat{P}}(A \otimes_K L) = 1/2$ for $i = 1,2,3,4$. Therefore by (1.1)-(1.2) we have that $[A \otimes_K L] \in U_Q(L)$, and so $[A] \in U_Q(K)$. However the index of $A \otimes_K K_{P_2}$ is four, whereas the index of $A \otimes_K K_{P_3}$

is two, contradicting (1.3). This completes the example.

The first main result of this paper is to provide a generalization of a theorem of E. Witt using $U_F(K)$ as a tool. From this result we will see that (1.1)-(1.2) fail to hold for $U_F(K)$ when $K \neq L$. Furthermore we use this generalized Witt theorem to develop further properties of $U_F(K)$. Moreover we obtain necessary and sufficient conditions, in terms of the arithmetic of the fields under consideration, for an element to exist in $U_F(K)$ with a given index. Several related properties are also developed.

Finally we generalize the concept of K-adequacy introduced in Fein et al [18] and link it to $U_F(K)$ via the arithmetic of the underlying fields in a sequence of results.

2. BASIC REFERENCES

For basic properties of number fields used in this paper we refer the reader to Marcus [19]. For fundamental results concerning the Schur subgroup of the Brauer group the reader should consult Yamada [20]. For information pertaining to properties of algebras used herein, and especially the classification of the Brauer group of a number field via Hasse invariants see the beautifully written Reiner [17]. Any concepts not described in greater detail in this paper may be found in earlier work Mollin [2 - 15].

3. UNIFORM DISTRIBUTION

THEOREM 3.1. Suppose $[A] \in U_F(K)$ where K/F is an extension of number fields. Let P be a K-prime with $inv_P(A) > 0$, and let $\hat{P}$ be an L-prime above P where L is the normal closure of K/F. Set $g = g.c.d.\ (m, |L_{\hat{P}}:K_P|)$, where m is the index of $A \otimes_K K_P$. Then $\epsilon_{m/g}$ is in K and $P \cap F$ is completely split in $F(\epsilon_{m/g})$.

PROOF. Let $\sigma \in G(L/K)$ and let $\hat{P}$ be an L-prime with $\hat{P} \cap K = P$. Therefore $inv_{\hat{P}}(A \otimes_K L) = b_\sigma\ inv_{\hat{P}^\sigma}(A \otimes_K L)$ where $\epsilon_m^\sigma = \epsilon_m^{b_\sigma}$. Hence:
$|L_{\hat{P}}:K_P|\ inv_P(A) \equiv b_\sigma |L_{\hat{P}^\sigma}:K_P| inv_P(A) \pmod 1$. Thus $b_\sigma \equiv 1 \pmod{m/g}$, and so σ fixes $\epsilon_{m/g}$ for all $\sigma \in G(L/K)$; i.e. $\epsilon_{m/g}$ is in K.

Since we have: $inv_{\hat{P}}(A \otimes_K L) \equiv |L_{\hat{P}}:K_P| inv_P(A) \pmod 1$ then $ind_P(A \otimes_K L) = m/g$. Hence by Mollin [15, Theorem 2.3, p.251], $P \cap F$ is completely split in $F(\epsilon_{m/g})$.

Q.E.D.

For completeness sake we state Witt's results [21, Satz 10, Satz 11, p.243] in succinct form as a corollary which is immediate from the theorem.

COROLLARY 3.2. If K/Q is finite abelian, $[A] \in S(K)$ and p is an odd prime with $ind_p(A) = m$ then $p \equiv 1 \pmod m$. If $p = 2$ then $ind_p(A) = 1$ or 2.

We maintain the notation of the Theorem 3.1 in the following results. The first result, which is immediate, generalizes Mollin [15, Corollary 2.4, p.254].

COROLLARY 3.3 Let $m/g = p^a$ where p is a prime and suppose ϵ_{p^n} is the largest p-power root of unity in K. If P is completely split in $F(\epsilon_{p^c})$ but not in $F(\epsilon_{p^{c+1}})$ then $a \leq \min\{n,c\}$.

The next result which will prove to be useful later in the paper generalizes Mollin [15, Corollary 2.5, p.254].

COROLLARY 3.4. Suppose P and Q are K-primes with $P \cap F = Q \cap F$ and $|L_{\hat{P}}:K_P| = |L_{\hat{Q}}:K_Q|$ where $\hat{P}$ (respectively $\hat{Q}$) is any L-prime above P(respectively Q). If $[A] \in U_F(K)$ then $inv_P(A) = inv_Q(A)$ if and only if $P \cap F(\epsilon_{m/g}) = Q \cap F(\epsilon_{m/g})$.

PROOF. Suppose $\sigma \in G(L/F)$ such that $\hat{P}^{\sigma} = \hat{Q}$. We have $\mathrm{inv}_{\hat{P}}(A \otimes_K L) \equiv b_\sigma$ $\mathrm{inv}_{\hat{P}^{\sigma}}(A \otimes_K L)$ (mod 1) where $\epsilon^{\sigma}_{m/g} = \epsilon^{b_\sigma}_{m/g}$. Thus we have by Mollin [4, (2.3), p.276] that: $|L_{\hat{P}}:K_P|\,\mathrm{inv}_P(A) \equiv b_\sigma |L_{\hat{Q}}:K_Q|\,\mathrm{inv}_Q(A)$ (mod 1).
Since $|L_{\hat{P}}:K_P| = |L_{\hat{Q}}:K_Q|$ then we have that $\mathrm{inv}_P(A) = \mathrm{inv}_Q(A)$ if and only if $b_\sigma \equiv 1$ (mod m/g) which in turn holds if and only if $\sigma \in G(L/F(\epsilon_{m/g}))$. However by theorem 3.1, $P \cap F$ is completely split in $F(\epsilon_{m/g})$. Hence $\sigma \in G(L/F(\epsilon_{m/g}))$ if and only if $P \cap F(\epsilon_{m/g}) = P^{\sigma} \cap F(\epsilon_{/g})$. Q.E.D.

We note that Theorem 3.1 shows that (1.1) does not hold for $U_F(K)$ when K/F is non-normal. Moreover (1.2) does not generalize to $U_F(K)$ where σ of (1.1) is interpreted as an embedding of K into the complex field C. This may be illustrated by considering example (1.4) with $\sigma \in C(L/K)$ and $\hat{P}^{\sigma}_2 = \hat{P}_3$. However $\mathrm{inv}_{P_2}(A) = 3/4 \not\equiv \mathrm{inv}_{P_3}(A)$ $\equiv 1/2$ (mod 1).

Now, Theorem 3.1 is the "best possible" generalization of Witt's results in the sense that we cannot hope for $P \cap F$ to be completely split in $F(\epsilon_n)$ for $m > n > m/g$ in general. The following example depicts this fact.

EXAMPLE 3.5. Let θ be a real root of $f(x) = x^{16} - 2$, and let $F = Q(\theta)$. If $K = Q(\theta, \epsilon_8)$ then $L = Q(\theta, \epsilon_{16})$ is the normal closure of K over F. Let P be an F-prime above 5. Then it can be verified that P splits into two unramified K-primes $\hat{P}_1$ and $\hat{P}_2$ each having inertial degree two over P. Define: $\mathrm{inv}_{\hat{P}_1}(A) = 1/8$, and $\mathrm{inv}_{\hat{P}_2}(A) = -1/8$ while $\mathrm{inv}_{\hat{Q}}(A) = 0$ for all K-primes $\hat{Q} \neq \hat{P}_1, \hat{P}_2$. Then by the Hasse sum Theorem $[A] \in B(K)$. Now since there exists exactly one L-prime $\hat{Q}_i$ above $\hat{P}_i$ for i = 1,2, each with inertial degree equal to 2 then $\mathrm{inv}_{\hat{Q}_1}(A \otimes_K L) = 1/4$ and $\mathrm{inv}_{\hat{Q}_2}(A \otimes_K L) = -1/4$. Thus by construction $[A] \in U_F(K)$. However, although the index of $A \otimes_K K_{\hat{P}}$ is 8, P is not completely split in $F(\epsilon_8) = K$. This completes the example.

It is natural to ask whether the converse of Theorem 3.1 holds since we would then have a criterion, in terms of the arithmetic of K and F, for the existence of an element in $U_F(K)$. Unfortunately the converse fails to hold as the following counter-example illustrates.

EXAMPLE 3.6. Let $K = Q(\theta_1)$, $F = Q$, and $L = Q(\theta_2, \epsilon_3)$ where θ_1 is a real root of $f(x) = x^3 - 2$ and θ_2 is a real root of $g(x) = x^3 - 11$. Then 2 splits into two K-primes P_1 and P_2 with inertial degrees one and two respectively over F. Hence P_1 splits into two L-primes $\hat{P}_1$ and $\hat{P}_2$ each with inertial degree 2 over K; and P_2 has one L-prime $\hat{P}_3$ above it with inertial degree one over K. Now, if there exists $[A] \in U_F(K)$ with $\mathrm{inv}_{P_2}(A) = 1/2$ then $[A \otimes_K L] \in U_F(L)$. Therefore: $\mathrm{inv}_{\hat{P}_3}(A \otimes_K L) \equiv |L_{\hat{P}_3}: K_{P_2}|\,\mathrm{inv}_{P_2}(A)$ $\equiv \mathrm{inv}_{P_2}(A) \equiv 1/2$ (mod 1). Therefore $\mathrm{ind}_2(A \otimes_K L) = 2$ is forced. Thus: $\mathrm{inv}_{\hat{P}_i}(A \otimes_K L) = 1/2$ for i = 1,2,3. However by Mollin [15, Lemma 2.8, p.259] we must have $\sum_{i=1}^{3} \mathrm{inv}_{\hat{P}_i}(A \otimes_K L) \equiv 0$ (mod 1), a contradiction which completes the example.

Now we demonstrate that under a suitable restriction we do get sufficient conditions in terms of the arithmetic of F,K and L to guarantee the existence of an element in $U_F(K)$.

THEOREM 3.7. Let L be the normal closure of K/F, an extension of number fields. Suppose that ϵ_n is the largest root of unity in F with $n \not\equiv 2$ (mod 4). If we have:

(1) ϵ_m is in K where n divides m and;

(2) $\hat{P}$ is a K-prime such that $\hat{P} \cap F = P$ is completely split in $F(\epsilon_m)$, and

(3) g.c.d. $(r, |L:K|) = 1$, where $r = \begin{cases} m/n & \text{if } m > n \\ m & \text{if } m = n \end{cases}$.

then there exists $[A] \in U_F(K)$ with the index of $A \otimes_K K_{\hat{P}}$ equal to r.

PROOF. By [15, Theorem 2.7, p.256] we have the existence of an element $[B] \in U_F(L)$ with $\mathrm{ind}_P(B) = r$. Now, let $I(B) = \{F\text{-primes } P\colon \mathrm{ind}_P(B) > 1\}$. Suppose that $\mathrm{inv}_{\check{Q}}(B) = a(\check{Q})/r$ where $\check{Q}$ is an L-prime above $P \in I(B)$. By Mollin [15, Corollary 2.5, p.254] we have that, for any L-prime $\check{R}$ such that $\check{Q} \cap K = \check{R} \cap K = \hat{Q}$ then, $\mathrm{inv}_{\check{R}}(B) = a(\check{Q})/r$. Therefore we set $a(\hat{Q}) = a(\check{Q})$ for all such L-primes $\check{Q}$ above a K-prime $\hat{Q}$ which lies above a given $P \in I(B)$. Now we define a K-central simple algebra A as follows. For each $P \in I(B)$, let $\mathrm{inv}_{\hat{Q}}(A) = a(\hat{Q})\ell(\hat{Q})/r$, for all K-primes $\hat{Q}$ above P, and let $\mathrm{inv}_{\hat{S}}(A) = 0$ for all K-primes $\hat{S}$ not above primes in $I(B)$, where $\ell(\hat{Q}) = |L:K|/|L_{\check{Q}}:K_{\hat{Q}}|$. By (3) of the hypothesis we have that the index of $A \otimes_K K_{\hat{Q}}$ is equal to r. Moreover:

$$0 \equiv \sum_{P \in I(B)} \sum_{\check{Q}|P} \mathrm{inv}_{\check{Q}}(B) \equiv \sum_{P \in I(B)} \sum_{\check{Q}|P} a(\check{Q})/r$$

$$\equiv \sum_{P \in I(B)} \sum_{\hat{Q}|P} a(\hat{Q})\ell(\hat{Q})/r \equiv \sum_{P \in I(B)} \sum_{\hat{Q}|P} \mathrm{inv}_{\hat{Q}}(A) \pmod 1.$$

Therefore $[A] \in B(K)$. Now, if $\sigma \in G(L/F)$ then:

$$\mathrm{inv}_{\check{Q}}(A \otimes_K L) \equiv |L:K|a(\check{Q})/r \equiv |L:K|\mathrm{inv}_{\check{Q}}(B)$$

$$\equiv |L:K|b_\sigma \mathrm{inv}_{\check{Q}^\sigma}(B) \equiv |L:K|b_\sigma a(\hat{Q}^\sigma)/r \equiv b_\sigma \mathrm{inv}_{\check{Q}^\sigma}(A \otimes_K L) \pmod 1$$

where $\hat{Q}^\sigma$ is a K-prime below $\check{Q}^\sigma$. Hence $[A] \in U_F(K)$. Q.E.D.

We note that Theorem 3.7 generalizes Mollin [15, Theorem 2.7, p.256].

The final result of this section is an interesting result which generalizes Mollin [15, Lemma 3.1, p.262] and simplifies the proof thereof. Moreover we note that it is possible to use the following result to generalize Mollin [15, Theorem 2.10, p.260].

THEOREM 3.8. Let K_i/F for $i = 1,2$ be extensions of number fields with N_1 being the normal closure of K_1/F and N_2 being the normal closure of K_1K_2/F. If $[A] \in U_F(K_1)$ with index m and g.c.d. $(m, |N_1N_2:N_2|) = 1$ then $[A \otimes_{K_1} K_1K_2] \in U_F(K_1K_2)$.

PROOF. Let $g = \text{g.c.d.}\,(m, |N_1N_2:K_1|)$, and let $\sigma \in G(N_2/F)$ with $\epsilon_{m/g}^{\sigma} = \epsilon_{m/g}^{b}$. Denote an extension of σ to $G(N_1N_2/F)$ by $\hat{\sigma}$. Since $[A \otimes_{K_1} N_1] \in U_F(N_1)$ implies that $[A \otimes_{K_1} N_1N_2] \in U_F(N_1N_2)$ then we have $\mathrm{inv}_{\hat{P}}(A \otimes_{K_1} N_1N_2) \equiv b\ \mathrm{inv}_{\hat{P}^{\hat{\sigma}}}(A \otimes_{K_1} N_1N_2) \pmod 1$ where $\hat{P}$ is an N_1N_2-prime. Therefore $|(N_1N_2)_{\hat{P}}:(N_2)_P|\mathrm{inv}_P(A \otimes_{K_1} N_2) \equiv b|(N_1N_2)_{\hat{P}^{\hat{\sigma}}}:(N_2)_{P^\sigma}|\mathrm{inv}_{P^\sigma}(A \otimes_{K_1} N_2) \pmod 1$. But since N_1N_2/N_2 and N_2/F are normal and g.c.d. $(m, |N_1N_2:N_2|) = 1$ then $\mathrm{inv}_P(A \otimes_{K_1} N_2) \equiv b\ \mathrm{inv}_{P^\sigma}(A \otimes_{K_1} N_2) \pmod 1$. This means $[A \otimes_{K_1} N_2] \in U_F(N_2)$; i.e. $[A \otimes_{K_1} K_1K_2] \in U_F(K_1K_2)$. Q.E.D.

4. EMBEDDINGS IN SIMPLE ALGEBRAS AND EXTENSIONS OF AUTOMORPHISMS.

Let K/F be an extension of number fields, and let n be a fixed positive integer. If D is a division algebra with $[D] \in B(K)$ then we say that D is (n,F)-<u>adequate</u> if there exists an F-division algebra B with $F\cdot I \subseteq D \subseteq M_n(B)$ where I is the identity matrix of $M_n(B)$, the full ring of $n \times n$ matrices with entries from B. This concept generalizes that of Fein et al [18] which is the $n = 1$ case. We now proceed to obtain results linking $U_F(K)$ and (n,F)-adequacy via the arithmetic of F and K. The first result is a generalization of Fein [22, Proposition 3, p.438].

LEMMA 4.1. Let K/F be an extension of number fields with L being the normal closure of K/F. Suppose $[D] \in B(K)$ where D is a division ring of index m such that g.c.d. $(m, |L:K|) = 1$. If D is (n,F)-adequate for a given positive integer n then $\mathrm{inv}_{\hat{P}}(D \otimes_K L) = \mathrm{inv}_{\hat{Q}}(D \otimes_K L)$ for all L-primes $\hat{P}$ and $\hat{Q}$ such that $\hat{P} \cap F = \hat{Q} \cap F$.

PROOF. Let $A = M_n(B)$ where D is embedded in A. If $C = C_A(K)$ denotes the centralizer of K in A then by Reiner [17, Corollary 7.14, p.96] we have $[C] = [B \otimes_F K]$. Moreover by Albert [23, Theorem 13, p.53] we have $[C] = [D \otimes_K D_1]$ where D_1 is a division ring with $[D_1] \in B(K)$. Thus $[B \otimes_F K] = [D \otimes_K D_1]$ which implies $[B \otimes_F L] = [D \otimes_K L][D_1 \otimes_K L]$. Since L/F is normal then it follows from Mollin [4, (2.2)-(2.3), p.276] that: $\mathrm{inv}_{\hat{P}}(D \otimes_K L) - \mathrm{inv}_{\hat{Q}}(D \otimes_K L) = \mathrm{inv}_{\hat{Q}}(D_1 \otimes_K L) - \mathrm{inv}_{\hat{P}}(D_1 \otimes_K L)$. Since g.c.d. $(|L:K|, m) = 1$ then $\mathrm{inv}_{\hat{P}}(D \otimes_K L) \neq \mathrm{inv}_{Q}(D \otimes_K L)$ implies g.c.d. $(|D:K|, |D_1:K|) \neq 1$ which contradicts that $D \otimes_K D_1$ is a division algebra, (see (Reiner [17].) We note that the above proof essentially uses the idea of Fein [22].

Q.E.D.

Now we use Lemma 4.1 to obtain a generalization of Mollin [15, Theorem 3.2, p.263].

THEOREM 4.2. Suppose K_i/F for $i = 1,2$ are extensions of number fields with N_1 being the normal closure of K_1/F and assume that K_1K_2/F is normal. Suppose D is a division algebra with $[D] \in U_F(K_1)$, and which has index m. Suppose D is (n,K_2)-adequate for a given positive integer n such that g.c.d. $(m, |N_1K_2:K_1K_2|) = 1$. If $g =$ g.c.d. $(|N_1:K_1|, m)$ then $\epsilon_{m/g}$ is in K_2.

PROOF. By Theorem 3.8 we have that $[D \otimes_{K_1} K_1K_2] \in U_F(K_1K_2)$, and so we have $U_F(K_1K_2) \subseteq U_{K_2}(K_1K_2)$. Now let P and Q be any two K_1K_2-primes with $P \cap K_2 = Q \cap K_2$. Now we invoke Lemma 4.1 to get that $\mathrm{inv}_P(D \otimes_{K_1} K_1K_2) = \mathrm{inv}_Q(D \otimes_{K_1} K_1K_2)$. Therefore since K_1K_2/F is normal then we may invoke Corollary 3.4 to get $P \cap K_2(\epsilon_{m/g}) = Q \cap K_2(\epsilon_{m/g})$. However P and Q were arbitrarily chosen subject only to $P \cap K_2 = Q \cap K_2$. By Theorem 3.1, $P \cap K_2$ is completely split in $K_2(\epsilon_{m/g})$. Hence $\epsilon_{m/g}$ is in K_2. Q.E.D.

We note that immediate consequences of Theorem 4.2 are Mollin [8, Theorem 3.1, p.175], and Fein et al [24, Theorem 1, p.305]. Moreover the above proof is shorter and more straightforward than the latter two cases.

Finally we present the following generalization of Mollin [10, Theorem 4.5, p.245]. The result is virtually immediate but we present it since it may be of independent interest. Aut(K) (resp. Aut(A)) refers to the automorphism group of K(resp. A).

THEOREM 4.3. Let K/F be an extension of number fields and assume that the fixed field of Aut(K) is contained in F. If $[A] \in U_F(K)$ with index n then $\sigma \in \mathrm{Aut}(K)$ extends to Aut(A) if and only if σ fixes ϵ_n.

PROOF. Since $U_{F'}(K) \subseteq U_F(K)$ where F' is the fixed field of Aut(K) then the result follows immediately from Mollin [10, Theorem 4.5, p.245]. Q.E.D.

We conclude with a note that it is an open question as to whether other results may be generalized to $U_F(K)$ when K/F is not normal. Some such examples are Mollin [6, Theorem 4.5, p.476], [7, Theorem 2.1, p.202], [2, Theorem 1.2, p.262], and [12, Theorem 1, p.1075]. It should also be noted that recently Greenfield [25] has done some work on uniform distribution from a different perspective modelled after Mollin [15].

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