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A SINGULAR BOUNDARY VALUE PROBLEM FOR NEUTRAL EQUATIONS

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ABSTRACT. Using some previous results of the author and Kirk-Schöneberg, theorems for the existence and uniqueness of an absolutely continuous solution of a singular boundary value problem for neutral equations have been proved.

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1. INTRODUCTION.

The aim of the present paper is to obtain conditions for the existence and uniqueness of an absolutely continuous solution of a singular boundary value problem for neutral equations, using the results due to Angelov [1] and Kirk-Schöneberg [2].

Let us state the singular boundary value problem: we look for an absolutely continuous solution $y(t)$ of the initial value problem

$$\begin{aligned} y'(t) &= X(\mu,t,[y(\Delta_i(t))]_{i=1}^m, [y'(\tau_l(t))]_{l=1}^n), \quad t > 0 \\ y(t) &= \psi(t,\mu), \quad y'(t) = \frac{\partial \psi(t,\mu)}{\partial t}, \quad t \leq 0 \end{aligned} \tag{1.1}$$

which satisfies the condition $\lim_{t\to\infty} y(t) = y_\infty$, where y_∞ is given. Here the unknown function $y(t)$ and the parameter μ take values in some Banach space B with a norm $\|.\|$ and

$$[y(\Delta_i(t))]_{i=1}^m = (y(\Delta_1(t)),\ldots,y(\Delta_m(t))),$$

$$[y'(\tau_l(t))]_{l=1}^n = (y'(\tau_1(t)),\ldots,y'(\tau_n(t))).$$

We shall use the standard denotations:

$R^1 = (-\infty,\infty)$; $R_+^1 = [0,\infty)$; $R_-^1 = (-\infty,0]$.

Let us set $x(t) = y'(t)$ for $t > 0$ and $\varphi(t,\mu) = \frac{\partial \Psi(t,\mu)}{\partial t}$ for $t < 0$. Then we have

$$x(t) = X(\mu,t,[\Psi(0,\mu) + \int_0^{\Delta_i(t)} x(s)ds]_{i=1}^m, [x(\tau_l(t))]_{l=1}^n), \ t > 0 \qquad (1.2)$$

$$x(t) = \varphi(t,\mu), \ t \leq 0.$$

The integrals are in a Bochner's sense. (Hille-Phillips [3]).

The method used by Angelov [1] is not immediately applicable to the singular boundary value problem for an equation(1.1). In order to illustrate this fact we shall consider the following particular case:

$$y'(t) = A(t)\mu + X(\mu,t,[y(\Delta_i(t))]_{i=1}^m, [y'(\tau_l(t))]_{l=1}^n) \qquad (1.3)$$

which occurs in the applications. Here $A(t)$ is a family of linear continuous operators. We must seek a solution of (1.3) in the space $L^1(R^1,B)$ which implies $\|A(t)\mu\| \in L^1(R^1_+)$. But then the supposition for the existence of $A^{-1}(t)$, (Seidov [4]), is not natural, because one of the conditions for the existence of an inverse operator is $\|A(t)\mu\| \geq m\|\mu\|$ (Kantorovich-Akilov [5], p. 209).

That is why we shall apply some recent results of Kirk-Schöneberg [2] for the operator equation in μ.

Let us formulate some auxiliary propositions due to Angelov [1]. If $B_i (i = 1,2)$ are Banach spaces with norms $\|.\|_i$, then:

PROPOSITON 1 [1]. Let the following conditions hold:

1. The nonlinear continuous operators $N_i : B_i \times B \to B_i$ satisfy the inequalities

$\|((1+\lambda\gamma-\lambda\alpha)x + \lambda N_i(x,\mu)) - ((1+\lambda\gamma-\lambda\alpha)y + \lambda N_i(y,\mu))\|_i \geq \|x-y\|_i$ $(i = 1,2)$ for every $x,y \in B_i$, $\lambda > 0$, $\mu \in B$, for some $\gamma > 0$, $\alpha > 0$.

2. RESULTS.

The linear map $j : B_1 \to B_2$ satisfies the condition $j(N_1(x,\mu)) = N_2(jx,\mu)$ for every $x \in B_1$ and $\mu \in B$.

Then for maps $x(\mu):B \to B_1$ and $y(\mu):B \to B_2$ connected by the condition $jx(\mu) = N_2(y(\mu),\mu) + \gamma y(\mu)$ there exists a unique map $z(\mu):B \to B_1$ for which $N_1(z(\mu),\mu) + \gamma z(\mu) = x(\mu)$ and $jz(\mu) = y(\mu)$.

PROPOSITION 2 [1]. Let the conditions of Proposition 1 hold, let the $x(\mu):B \to B_1$, $y(\mu):B \to B_2$ be continuous and let $\|N_i(x,\mu) - N_i(y,\mu)\|_i \leq M\|x-y\|_i$ $(i = 1,2)$, where $\gamma > M > 0$.

Then the map $z(\mu):B \to B_1$, satisfying the equation $N_1(z(\mu),\mu) + \gamma z(\mu) = x(\mu)$ is continuous.

PROPOSITION 3 [1]. Let the conditions of Proposition 1 hold and let $z_i(\mu)$, $(i = 1,2)$ satisfy the equalities $N_1(z_i(\mu),\mu) + \gamma z_i(\mu) = x_i(\mu)$, $jz_i(\mu) = y_i(\mu)$, $N_2(y_i(\mu),\mu) + \gamma y_i(\mu) = jx_i(\mu)$. Then $\|z_1(\mu) - z_2(\mu)\|_1 \leq \frac{1}{\alpha}\|x_1(\mu) - x_2(\mu)\|_1$.

We shall formulate the results of Kirk-Schöneberg [2] in order to apply them to our problem.

Let X and Y be complete metric spaces. Recall that: 1) Y is said to be metrically convex (Menger [6]) if for all, $u,v \in Y$ with $u \neq v$ there exists $w \in Y$, $w \neq u$, $w \neq v$, such that $\rho(u,v) = \rho(u,w) + \rho(w,v)$; 2) a mapping $T:X \to Y$ is said to be closed if $u_n \to u \in X$ and $T(u_n) \to z \in Y$ imply $T(u) = z$.

LEMMA [2]. Let $T:X \to Y$ be a closed mapping. Fix $y \in Y$. Let U be an open subset of X for which T(U) is open in Y, and suppose

$$\rho(Tu,Tv) \geq \rho(u,v)$$

for all $u,v \in \mathrm{cl}U$. Suppose also that there exists $u_0 \in U$ such that

$$\rho(Tu_0,y) < \rho(Tu,y) + \rho(Tu_0,Tu)$$

for all $u \in \partial U$.

Then there exists $u \in U$ such that $Tu = y$.

If T is a mapping of X into Y and D is an open subset of X, then T is said to be locally expansive on D if for all $u_0 \in D$ there is a neighbourhood N of u_0 contained in D such that $\rho(Tu,Tv) \geq \rho(u,v)$ for all $u,v \in N$.

PROPOSITION 4 [2]. Let $D \subset X$ be open, and suppose $T:X \to Y$ is a closed mapping which is locally expansive on D. Suppose also that T maps open subsets of D on to open subsets of Y, and there exists $x_0 \in D$ such that $\rho(Tx_0,y) \leq \rho(Tx,y)$ for all $x \in X \setminus D$. Then $y \in T(D)$.

COROLLARY [2]. Let $T:X \to Y$ be a closed mapping, which maps open subsets of X onto open subsets of Y. Suppose also that T is locally expansive on all of X. Then $T(X) = Y$.

THEOREM 1. Let the following conditions hold:

1. the functions $\Delta_i(t)$, $\tau_l(t):R^1_+ \to R^1$ are measurable, $\tau_l(t)$ have the property (S) (Angelov, [1]) and $\int_0^\infty \|f(\tau_l(t))\|dt \leq k\int_{-\infty}^\infty \|f(t)\|dt$ for some $k = \mathrm{const} > 0$ and every $f \in L^1(R^1;B)$.

2. the function $X(\mu,t,u_1,\dots,u_m,v_1,\dots,v_n):B\times R^1_+\times B^{m+n} \to B$ satisfies the Caratheodory condition and

$$\|X(\mu,t,u_1,\dots,u_m,v_1,\dots,v_n)\| \leq \frac{1}{\gamma}[\alpha(\mu,t,\|u_1\|,\dots,\|u_m\|) + \alpha_0 \sum_{l=1}^n \|v_l\|]$$

$$\|X(\mu,t,u_1,\dots,u_m,v_1,\dots,v_n) - X(\mu,t,\bar{u}_1,\dots,\bar{u}_m,\bar{v}_1,\dots,\bar{v}_n)\| \leq$$

$$\frac{m}{\gamma}[\beta(\mu,t,\|u_1-\bar{u}_1\|,\dots,\|u_m-\bar{u}_m\|) + \sum_{l=1}^n \beta_l\|v_l-\bar{v}_l\|]$$

for some constants $\gamma > m > 0$, α_o, $\beta_l > 0$; $\alpha,\beta : B\times R_+^{m+1}\to R_+^1$ are comparison functions (Angelov [1]); $\alpha(\mu,.,u_1,\dots,u_m)$, $\beta(\mu,.,u_1,\dots,u_m)\in L^1(R_+^1)$ and

$$\int_0^\infty \beta(\mu,t,v,\dots,v)dt + kv\sum_{l=1}^n \beta_l < v,\ v\in R_+^1.$$

Besides, $X(\mu,t,\dots)$ is uniformly continuous in μ with respect to the other variables (Angelov [1]).

3. the initial function $\varphi(.,\mu)\in L^1(R_-^1;B)$ for every $\mu\in B$ and

$$\lim_{\mu\to\mu_o}\int_{-\infty}^{o} \|\varphi(t,\mu) - \varphi(t,\mu_o)\|dt = 0.$$

Then there exists a unique solution $x(.,\mu)\in L^1(R^1;B)$ of the initial value problem (2), which depends continuously on μ.

Proof: Let B_1 be the Banach space $L^1(R^1;B)$ with norm $\|f\| = \int_{-\infty}^{+\infty}\|f(t)\|dt$ and B_2 be the Banach space $L^1(R_-^1;B)$ with norm $\|g\|_2 = \int_{-\infty}^{o}\|g(t)\|dt$.

Define the operators $N_i : B_i\times B\to B_i$ $(i = 1,2)$

$$N_1(f,\mu)(t) = \begin{cases} -\gamma X(\mu,t,[\Psi(0,\mu) + \int_o^{\Delta_i(t)} f(s)ds]_{i=1}^m, \\ [f(\tau_l(t))]_{l=1}^n), \quad t > 0 \\ 0 \quad\quad\quad , \ t \le 0 \end{cases}$$

$f\in B_1$, $\mu\in B$ and $N_2(g,\mu)(t) = 0$, $t \le 0$, $g\in B_2$, $\mu\in B$.

The map $j : B_1\to B_2$ is defined as in Theorem 1 (Angelov [1]).

Since $N_1(f,\mu)(t)$ is strongly measurable, then the inequalities

$$\|N_1(f,\mu)(t)\| \le \alpha(\mu,t,\|\Psi(0,\mu)\| + \|f\|_1,\dots,\ \Psi(0,\mu)\| + \|f\|_1) + \alpha_o\sum_{l=1}^n \|f(\tau_l(t))\|$$

show that $f\in B_1$ implies $N_1(f,\mu)\in B_1$ for each $\mu\in B$.

We are going to show the Lipschitz continuity of the operator N_1:

$$\int_0^\infty \|N_1(f,\mu)(t) - N_1(g,\mu)(t)\|dt \le m[\int_0^\infty \beta(\mu,t,\|f-g\|_1,\dots,\|f-g\|_1)dt + \sum_{l=1}^n \beta_l\int_0^\infty \|f(\tau_l(t)) - g(\tau_l(t))\|dt]$$

$$\le m[\int_0^\infty \beta(\mu,t,\|f-g\|_1,\dots,\|f-g\|_1)dt + k\|f-g\|_1\sum_{l=1}^n \beta_l] \le m\|f-g\|_1.$$

Further on the proof is analogous to the one of Theorem 1 from Angelov[1].

THEOREM 2. Let the conditions of Theorem 1 hold and let $x(\varphi_i,\mu)(t)$, $(i = 1,2)$ be solutions of the problem (2) with initial functions $\varphi_i(t,\mu)$. Then

$$\int_{-\infty}^{\infty}\| x(\varphi_1,\mu)(t) - x(\varphi_2,\mu)(t)\| dt < \frac{\gamma}{\gamma-m}\int_{-\infty}^{0}\|\varphi_1(t,\mu) - \varphi_2(t,\mu)\| dt.$$

THEOREM 3. Let the conditions of Theorem 1 hold true. If, in addition, we suppose:

1. $\Psi(0,.)$ is locally expansive, closed and maps open subsets of B onto open subsets of B.
2. for every $\mu\in B$ there is a neighbourhood U of μ such that

$$\| X(\mu_1,t,\dots) - X(\mu_2,t,\dots)\| \geq \xi(t)\|\mu_1-\mu_2\|$$

for $\mu_1,\mu_2\in U$, where $\xi(t)\in L^1(R^1_+)$, $\xi(t) > 0$ and the mapping $Q(\mu) = \int_0^\infty X(\mu,t,u_1,\dots,v_n)dt$ is closed and maps open subsets of B onto open subsets of B, provided the last integral exists.

Then there exists a value of the parameter μ_0 such that $x(t,\mu_0)$ is a solution of the problem (2) and the solution $y(t,\mu_0)$ of (1) satisfies the condition $\lim\limits_{t\to\infty} y(t,\mu_0) = y_\infty$.

Proof: Let B_1 be the Banach space $L^1(R^1;B)$ and B_2 be the Banach space $L^1(R^1_-;B)$.

Define the operators $N_i:B_i\times B \to B_i$ $(i = 1,2)$, $T:B\times B_1 \to B$:

$$N_1(f,\mu)(t) = \begin{cases} -\gamma X(\mu,t,[\Psi(0,\mu) + \int_0^{\Delta_i(t)} f(s)ds]_{i=1}^m, \\ [f(\tau_1(t))]_{1=1}^n), \ t > 0 \\ 0 \qquad\qquad , \ t < 0 \end{cases}$$

where $f\in B_1$, $\mu\in B$; $N_2(g,\mu)(t) = 0$, $t < 0$, $g\in B_2$, $\mu\in B$.

$$T(\mu,f) = y_\infty - \Psi(0,\mu) - \int_0^\infty X(\mu,t,[\Psi(0,\mu) + \int_0^{\Delta_i(t)} f(s)ds]_{i=1}^m,$$

$$[f(\tau_1(t))]_{1=1}^n)dt, \ f\in B_1.$$

Proposition 1 implies an existence of a unique function $x(t,\mu)$ which satisfies (2) and depends continuously on μ.

On the other hand, it is easy to verify that the operator $T(\mu,f)$ satisfies all conditions of Corollary. Indeed, $T(\mu,f)$ is closed and maps open subsets of B onto open subsets of B. Besides, for every $\mu \in B$ there is a neighbourhood U of μ such that

$$\|T(\mu_1,f) - T(\mu_2,f)\| \geq c_\Psi \|\mu_1-\mu_2\| + \int_0^\infty \xi(t)dt\|\mu_1-\mu_2\|$$

for all $\mu_1,\mu_2 \in U$. Consequently, there is $\mu_0 \in B$ such that $T(\mu_0,f) = 0$.

So we obtain

$$\lim_{t\to\infty} y(t,\mu_0) = \Psi(0,\mu_0) + \int_0^\infty x(s,\mu_0)ds =$$

$$\Psi(0,\mu_0) + \int_0^\infty X(\mu_0,s,[\Psi(0,\mu_0) + \int_0^{\Delta_i(s)} x(\theta,\mu_0)d\theta]_{i=1}^m,$$

$$[x(\tau_l(s))]_{l=1}^{l})ds = \Psi(0,\mu_0) + y_\infty - \Psi(0,\mu_0) = y_\infty.$$

Theorem 3 is thus proved.

Finally we shall note that an analogous result can be formulated for systems of the type:

$$y_i'(t) = X_i(\mu_i,t,[y_1(\Delta_{1k}(t))]_{k=1}^{m_1},\ldots,[y_s(\Delta_{sk}(t))]_{k=1}^{m_s},$$

$$[y_1'(\tau_{1k}(t))]_{k=1}^{n_1},\ldots,[y_s'(\tau_{sk}(t))]_{k=1}^{n_s}),\ t > 0$$

$$y_i(t) = \Psi_i(t,\mu_i),\ y_i'(t) = \frac{\partial \Psi_i(t,\mu_i)}{\partial t},\ t < 0$$

$(i = 1,2,\ldots,s)$.

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