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GROWTH RESULTS FOR A SUBCLASS OF BAZILEVIČ FUNCTIONS

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ABSTRACT. For $\alpha > 0$, let $B(\alpha)$ be the class of regular normalized Bazilevič functions defined in the unit disc. Choosing the associated starlike function $g(z) \equiv z$ gives a proper subclass $B_1(\alpha)$ of $B(\alpha)$. For $B(\alpha)$, correct growth estimates in terms of the area function are unknown. Several results in this direction are given for $B_1(\frac{1}{2})$.

KEY WORDS AND PHRASES. Bazilevic functions, subclasses of S, functions whose derivative has positive real part, close-to-convex functions, coefficient and length-area estimates.
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1. INTRODUCTION.

Let S be the class of regular, normalized, univalent functions with power series expansion

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \qquad (1.1)$$

for $z \in D$, where $D = \{z : |z| < 1\}$.

Denote R, S^*, K and $B(\alpha)$ the subclasses of S which are functions whose derivative has positive real part [8], starlike with respect to the orgin [9 p.221], close-to-convex [6] and Bazilevic of type α [13] respectively. Following [13] we define $f \in B(\alpha)$, $\alpha > 0$ to be the class of functions f, regular and normalized in D, such that, there exist $g \in S^*$ such that for $z \in D$,

$$\mathrm{Re}\frac{zf'(z)}{f(z)^{1-\alpha}g(z)^{\alpha}} > 0. \qquad (1.2)$$

Then if $g(z) \equiv f(z)$, $B(\alpha) = S^*$ and $B(1) = K$. Let $C(r)$ denote the closed curve which is the image of D under the mapping $w = f(z)$, $L(r)$ be the length of $C(r)$ and $A(r)$ the area enclosed by the curve $C(r)$. For $f \in S^*$, it was shown [7] that, with $z = re^{i\theta}$, $0 < r < 1$,

$$L(r) = O(1)\ (M(r) \log \frac{1}{1-r}) \quad \text{as} \quad r \to 1, \qquad (1.3)$$

where $M(r) = \max_{|z|=r} |f(z)|$, and Hayman [4] gave an example to show that this estimate is best possible when f is bounded. In [14] this result was extended to starlike functions with $A(r) < A$ constant. A modification of this method also shows that for $f \in S^*$,

$$L(r) = O(1)\sqrt{A(r)}(\log \frac{1}{1-r}) \quad \text{as} \quad r \to 1. \tag{1.4}$$

Thomas [14] also showed that (1.3) holds for the class K and for the class $B(\alpha)$, $0 < \alpha \le 1$ [13]. It is apparently an open question that (1.4) is valid for $f \in K$ or $B(\alpha)$.

Pommerenke [11] showed that if $f \in S^*$, then for $n \ge 2$

$$n|a_n| \le C\sqrt{A}(1 - \frac{1}{n}), \tag{1.5}$$

where C is constant, and Noor [10] extended this to $B(\alpha)$ by showing that

$$n|a_n| \le C\, M(1 - \frac{1}{n}). \tag{1.6}$$

The question as to whether (1.5) is valid for $f \in K$ or $B(\alpha)$ is also apparently open.

In [12] the subclass $B_1(\alpha)$ of $B(\alpha)$ consisting of those functions in $B(\alpha)$ for which $g(z) \equiv z$ was considered and sharp estimates for the modules of the coefficients a_2, a_3, and a_4 were given. In [15] Thomas gave sharp estimates for the coefficients a_n in (1.1) when $\alpha = 1/N$, N a positive integer.

In this paper we shall be concerned with the class $B_1(\frac{1}{2})$ and will use the method of Clunie and Keogh [1] to establish (1.5) and hence (1.6) and the method of Thomas [14] to prove (1.4) and hence (1.3). The methods will in fact give results which are stronger for this subclass.

2. STATEMENTS OF MAIN RESULTS.

THEOREM 1. Let $f \in B_1(\frac{1}{2})$ and be given by (1.1). Then

(i) $n|a_n| \le o(1) + O(1)\sqrt{A}(1 - \frac{1}{n})$, as $n \to \infty$,

(ii) $L(r) = O(1)\{\sqrt{A(r)} \log \frac{1}{1-r}\}$ as $r \to 1$.

We shall need the following:

LEMMA 1. Let $f \in B_1(\frac{1}{2})$ and be given by (1.1). Define the function F in D by $F(z)^2 = f(z^2)$. Then

$$A(r,F) \le \frac{1}{2(\pi-2)^2} A(r^2,f).$$

PROOF. For $z = \rho e^{i\theta}$, $0 \le r < 1$,

$$A(r,F) = \int_0^{2\pi}\int_0^r |F'(z)|^2 \rho\, d\rho\, d\theta$$

$$= \int_0^{2\pi}\int_0^r \left|\frac{zf'(z^2)}{f(z^2)^{\frac{1}{2}}}\right|^2 \rho\, d\rho\, d\theta.$$

Now $\left|\frac{1}{f(z^2)}\right| \le \frac{4}{(\pi-2)^2}$ [15] and so using (1.1) we have

$$A(r,F) \le \frac{4}{(\pi-2)^2}\int_0^{2\pi}\int_0^r |zf'(z^2)|^2 \rho\, d\rho\, d\theta$$

$$= \frac{\pi}{2(\pi-2)^2}\sum_{n=1}^{\infty} n|a_n|^2 r^{4n}, \quad (\text{where } |a_1| = 1)$$

$$= \frac{1}{2(\pi-2)^2} A(r^2,f).$$

LEMMA 2. For $f \in S$

(i) the map $r \to (1-r)^6 A(r)/(1+r)^2 r^2$ is decreasing on the interval $(0,1)$,

(ii) $A(\sqrt{r}) < \frac{512}{r} A(r)$ for $0 < r < 1$.

PROOF. Since

$$A(r) = \int_0^{2\pi} \int_0^r |f'(z)|^2 \rho d\rho d\theta,$$

we have

$$rA'(r) = \int_0^{2\pi} |zf'(z)|^2 d\theta$$

$$\leq 2 \int_0^{2\pi} \int_0^r |f'(z)|^2 \left|\frac{zf''(z)}{f'(z)}\right| \rho d\rho d\theta + 2 \int_0^{2\pi} \int_0^r |f'(z)|^2 \rho d\rho d\theta.$$

The classical distortion theorem for $f \in S$ [3,p.5] gives

$$rA'(r) \leq 4r\frac{(r+2)}{1-r^2} \int_0^{2\pi} \int_0^r |f'(z)|^2 \rho d\rho d\theta + 2 \int_0^{2\pi} \int_0^r |f'(z)|^2 \rho d\rho d\theta.$$

$$= 2A(r) \left\{ \frac{r^2+4r+1}{1-r^2} \right\}.$$

Thus

$$\frac{d}{dr} (\log A(r)) \leq \frac{d}{dr}\left\{ \left(\log \frac{r^2(1+r)^2}{(1-r)^6} \right)\right\}$$

and part (i) of Lemma 2 is now obvious. Part (ii) follows immediately.

PROOF OF THEOREM 1.

(i) Since $f \in B_1(\frac{1}{2})$, we can write from (1.2)

$$zf'(z^2) = f(z^2)^{\frac{1}{2}} h(z^2) \tag{2.1}$$

where $\operatorname{Re} h(z) > 0$, for $z \in D$.

Set
$$h(z^2) = \frac{1+w(z^2)}{1-w(z^2)},$$

where $w(z^2)$ is regular, $|w(z^2)| < 1$ in D, $w(0) = 0$ and $w(z) = \sum_{n=1}^{\infty} w_n z^n$.

Then with $F(z) = f(z^2)^{\frac{1}{2}}$ and

$$F(z) = z + \sum_{n=2}^{\infty} b_{2n-1} z^{2n-1}, \text{ (2.1) gives}$$

$$(zf'(z^2) + F(z))\, w(z^2) = zf'(z^2) - F(z).$$

Thus

$$\left\{2z + \sum_{k=2}^{\infty} (ka_k + b_{2k-1}) z^{2k-1}\right\} w(z^2) = \sum_{k=2}^{\infty} (ka_k - b_{2k-1}) z^{2k-1}. \tag{2.2}$$

Equating coefficients of z^{2n-1} in (2.2), we find that for $n \geq 2$

$$na_n - b_{2n-1} = 2 w_{n-1} + (2a_2 + b_3) w_{n-2} + \dots + [(n-1)a_{n-1} + b_{2n-3}]w_1.$$

This means that the coefficient combination $na_n - b_{2n-1}$ on the left hand side of (2.2) depends only on the coefficient combinations $[2a_2 + b_3], \dots, [(n-1)a_{n-1} + b_{2n-3}]$ on the right-hand side. Hence, for $n \geq 2$ we can write

$$\left\{2z + \sum_{k=2}^{n-1} (ka_k+b_{2k-1}) z^{2k-1}\right\} w(z^2) = \sum_{k=2}^{n} (ka_k-b_{2k-1}) z^{2k-1} + \sum_{k=n+1}^{\infty} c_k z^{2k-1} \tag{2.3}$$

say. Squaring the moduli of both sides of (2.3) and integrating round $|z| = r$, we obtain, using the fact that $|w(z^2)| < 1$ for $z \in D$,

$$\sum_{k=2}^{n} |ka_k - b_{2k-1}|^2 r^{4k-2} + \sum_{k=n+1}^{\infty} |c_k|^2 r^{4k-2}$$

$$< 4 + \sum_{k=2}^{n-1} |ka_k + b_{2k-1}|^2$$

letting $r \to 1$, we have

$$\sum_{k=2}^{n} |ka_k - b_{2k-1}|^2 \leq 4 + \sum_{k=2}^{n-1} |ka_k + b_{2k-1}|^2.$$

Thus

$$|na_n - b_{2n-1}|^2 \leq 4 + \sum_{k=2}^{n-1} k|a_k|\ |b_{2k-1}|, \tag{2.4}$$

(where $|a_1| = |b_1| = 1$).

Hence

$$|na_n - b_{2n-1}|^2 \leq 4 \left(\sum_{k=1}^{n-1} k|a_k|^2\right)^{\frac{1}{2}}\left(\sum_{k=1}^{n-1} k|b_{2k-1}|^2\right)^{\frac{1}{2}}$$

$$\leq 4r^{-4n+1} \left(\sum_{k=1}^{\infty} k|a_k|^2 r^{4k}\right)^{\frac{1}{2}}\left(\sum_{k=1}^{\infty} (2k-1)|b_{2k-1}|^2 r^{4k-2}\right)^{\frac{1}{2}}$$

for $0 < r < 1$, and so

$$|na_n - b_{2n-1}|^2 \leq 4 r^{-4n+1} \sqrt{\left(\frac{A(r^2,f)}{\pi}\right)} \sqrt{\left(\frac{A(r,F)}{\pi}\right)}.$$

Lemma 1 now gives

$$|na_n - b_{2n-1}|^2 \leq \frac{4}{\sqrt{2\pi(\pi-2)}} r^{-4n+1} A(r^2,f).$$

and choosing $r^2 = 1 - \frac{1}{n}$ we obtain

$$|na_n - b_{2n-1}|^2 \leq C\, A(1 - \tfrac{1}{n}), \text{ where } C \text{ is constant.} \tag{2.5}$$

Finally, it is easy to see that from the definition of $B_1(\frac{1}{2})$, $F \in R$ and so [8] for $n \leq 2$, $|b_{2n-1}| \leq \frac{2}{2n-1}$. Thus (2.5) gives

$$n|a_n| = o(1) + O(1) \sqrt{(A(1 - \tfrac{1}{n}))} \quad \text{as} \quad n \to \infty.$$

This proves part (i) of Theorem 1.

(ii)

Since $L(r) = \int_0^{2\pi} |zf'(z)|d\theta$, and $F(z)^2 = f(z^2)$, (2.1) gives

$$L(r^2) = \int_0^{2\pi} |z^2f'(z^2)|d\theta \leq r \int_0^{2\pi} |F(z)\ h(z^2)|d\theta$$
$$\leq \int_0^{2\pi}\int_0^{r} |F'(z)\ h(z^2)|d\rho d\theta + 2r \int_0^{2\pi}\int_0^{r} |F(z)\ h'(z^2)|\rho d\rho d\theta \quad (z=\rho e^{i\theta})$$
$$= I_1(r) + I_2(r) \text{ say.}$$

Again using (2.1) we have

$$I_1(r) = r \int_0^{r}\int_0^{2\pi} |h(z^2)|^2 d\theta d\rho = 2\pi r \int_0^{r} \left(1 + \sum_{n=1}^{\infty} |h_n|^2 \rho^{4n}\right)d\rho$$

where $h(z) = 1 + \sum_{n=1}^{\infty} h_n z^n$ for $z \in D$, and since $|h_n| \leq 2$ for $n \geq 1$ [2 p. 10],

$$I_1(r) \leq 2\pi r \int_0^r (1 + 4 \sum_{n=1}^{\infty} \rho^{4n}) d\rho$$

$$= 0(1) \log(\frac{1}{1-r}) \quad \text{as} \quad r \to 1.$$

Also

$$I_2(r) \leq 2r \left(\int_0^r \int_0^{2\pi} |F(z)^2 h'(z^2)| \rho d\theta d\rho\right)^{\frac{1}{2}} \left(\int_0^r \int_0^{2\pi} |h'(z^2)| \rho d\theta d\rho\right)^{\frac{1}{2}}$$

$$= 2r \, (J_1(r))^{\frac{1}{2}} \, (J_2(r))^{\frac{1}{2}} \quad \text{say.}$$

Since $\operatorname{Re} h(z^2) > 0$, for $z \in D$, we may write

$$h(z^2) = \frac{1}{2\pi} \int_0^{2\pi} \frac{1 + z^2 e^{-it}}{1 - z^2 e^{-it}} d\mu(t), \tag{2.6}$$

where $\mu(t)$ increases and $\frac{1}{2\pi} \int_0^{2\pi} d\mu(t) = 1$. [5 p. 68].

Therefore

$$h'(z^2) = \frac{1}{\pi} \int_0^{2\pi} \frac{e^{-it}}{(1 - z^2 e^{-it})^2} d\mu(t),$$

and so

$$|h'(z^2)| \leq \frac{1}{\pi} \int_0^{2\pi} \frac{d\mu(t)}{|1 - z^2 e^{-it}|^2} .$$

Thus

$$J_1(r) \leq \frac{1}{\pi} \int_0^r \int_0^{2\pi} \int_0^{2\pi} \left|\frac{F(z)}{1 - z^2 e^{-it}}\right|^2 \rho d\theta d\mu(t) d\rho .$$

Since F is an odd function we may write

$$\frac{F(z)}{1 - z^2 e^{-it}} = \sum_{n=1}^{\infty} S_{2n-1}(t) z^{2n-1} \tag{2.7}$$

and so

$$\int_0^{2\pi} \int_0^{2\pi} \left|\frac{F(z)}{1 - z^2 e^{-it}}\right|^2 d\theta d\mu(t) = 2\pi \sum_{n=1}^{\infty} \rho^{4n-2} \int_0^{2\pi} |S_{2n-1}(t)|^2 \, d\mu(t).$$

We now show that for $n \geq 1$,

$$\int_0^{2\pi} |S_{2n-1}(t)|^2 \, d\mu(t) \leq 2\pi \sum_{j=1}^{n} j|a_j| \, |b_{2j-1}|,$$

where $F(z) = \sum_{n=1}^{\infty} b_{2n-1} z^{2n-1}$ for $z \in D$.

From (2.7) we have

$$\sum_{n=1}^{\infty} S_{2n-1}(t) \, z^{2n-1} = \left(\sum_{n=1}^{\infty} b_{2n-1} z^{2n-1}\right)\left(\sum_{n=0}^{\infty} e^{-int} z^{2n}\right)$$

and so for $n \geq 1$,

$$S_{2n-1}(t) = \sum_{k=1}^{\infty} b_{2k-1} \, e^{-i(n-k)t} . \tag{2.8}$$

Now (2.1) gives

$$\left(\sum_{n=1}^{\infty} n a_n z^{2n-1}\right) = \left(\sum_{n=1}^{\infty} b_{2n-1} z^{2n-1}\right)\left(\sum_{n=0}^{\infty} h_n z^{2n}\right)$$

(where $h_0 = 1$) and so for $n \geq 1$

$$n a_n = \sum_{\nu=1}^{n} b_{2\nu-1} h_{n-\nu}.$$

It is easy to see from (2.6) that, for $k \geq 1$,

$$h_k = \frac{1}{\pi}\int_0^{2\pi} e^{-kit}\, d\mu(t).$$

and so

$$na_n = \frac{1}{\pi}\sum_{\nu=1}^{n} b_{2\nu-1}\int_0^{2\pi} e^{-i(n-\nu)t}. \tag{2.9}$$

Now

$$|S_{2n-1}(t)|^2 = S_{2n-1}(t)\,\overline{S_{2n-1}(t)}$$

$$= 2\,\mathrm{Re}\sum_{j=1}^{n}\sum_{k=1}^{j} b_{2j-1}\overline{b}_{2k-1}\, e^{-i(k-j)t} - \sum_{\nu=1}^{n}|b_{2\nu-1}|^2$$

where we have used (2.8). Therefore, using (2.9)

$$\int_0^{2\pi}|S_{2n-1}(t)|^2\, d\mu(t) = 2\,\mathrm{Re}\sum_{j=1}^{n}\sum_{k=1}^{j} b_{2j-1}\overline{b}_{2k-1}\int_0^{2\pi} e^{-i(k-j)t}d\mu(t) - 2\pi\sum_{\nu=1}^{n}|b_{2\nu-1}|^2$$

$$= 2\pi\,\mathrm{Re}\sum_{j=1}^{n} b_{2j-1}\, j\overline{a}_j - 2\pi\sum_{\nu=1}^{n}|b_{2\nu-1}|^2$$

$$\le 2\,\mathrm{Re}\sum_{j=1}^{n} j\overline{a}_j\, b_{2j-1}$$

$$\le 2\pi\sum_{j=1}^{n} j\,|a_j|\,|b_{2j-1}|.$$

Thus

$$J_1(r) \le 4\pi\int_0^r \sum_{n=1}^{\infty}\rho^{4n-1}\Big(\sum_{j=1}^{n} j|a_j|\,|b_{2j-1}|\Big)d\rho$$

$$\le 4\pi\int_0^r \sum_{n=1}^{\infty}\rho^{4n-1}\Big(\sum_{j=1}^{n} j|a_j|^2\Big)^{\frac{1}{2}}\Big(\sum_{j=1}^{n} j|b_{2j-1}|^2\Big)^{\frac{1}{2}}d\rho$$

$$\le 4\pi\int_0^r \sum_{n=1}^{\infty}\rho^{n}\quad \sum_{j=1}^{\infty} j|a_j|^2\rho^{2j}\quad ^{\frac{1}{2}}\quad \sum_{j=1}^{\infty}(2j-1)\,|b_{2j-1}|^2\rho^{4j-2}\quad ^{\frac{1}{2}}\, d\rho$$

$$\le 4\int_0^r \frac{A(\rho,f)^{\frac{1}{2}}A(\rho,F)^{\frac{1}{2}}}{1-\rho}\,d\rho$$

$$\le \frac{4}{(\pi-2)\sqrt{2}}\int_0^r \frac{A(\rho,f)}{1-\rho}\,d\rho, \text{ by lemma 1.}$$

Since $A(\rho)$ is increasing on $(0,1)$

$$J_1(r) \le \frac{4}{(\pi-2)\sqrt{2}}A(r,f)\int_0^r\frac{d\rho}{1-\rho}$$

$$= \frac{4}{(\pi-2)\sqrt{2}}A(r,f)\log\Big(\frac{1}{1-r}\Big).$$

Now

$$J_2(r) = \int_0^r\int_0^{2\pi}|h'(z^2)|\rho\, d\rho\, d\theta,$$

and since $\quad |h'(z^2)| \le \dfrac{\mathrm{Re}\ h(z^2)}{1-r^4}$ for $0 \le r \le 1$,

$$J_2(r) \le 2\int_0^r\int_0^{2\pi}\frac{\mathrm{Re}\ h(z^2)}{1-\rho^4}\,\rho\, d\theta\, d\rho$$

$$\le 4\pi\int_0^r\frac{d\rho}{1-\rho^4}$$

since h is harmonic in D.

Thus

$$J_2(r) \le 4\pi\log\frac{1}{1-r}.$$

Combining the estimates for $J_1(r)$ and $J_2(r)$ shows that

$$I_2(r) = O(1)\ \sqrt{(A(r))}\ \log\left(\frac{1}{1-r}\right) \quad \text{as} \quad r \to 1$$

and the result is proved.

COROLLARY 1. Let $f \in B_1(\tfrac{1}{2})$, then as $n \to \infty$,

(i) $n|a_n| \le o(1) + O(1)\ P(1 - \frac{1}{n})^{\frac{1}{2}}$,

where for $0 \le r < 1$, $P(r) = \sum_{n=1}^{\infty} |a_n| r^n$

(ii) $n|a_n| \le o(1) + O(1)\ A(1 - \frac{1}{n})^{\frac{1}{4}}(\log n)^{\frac{1}{4}}$,

(iii) $L(r) = O(1)\ A(r)^{\frac{1}{4}}(\log(\frac{1}{1-r}))^{\frac{5}{4}}$ as $r \to 1$.

PROOF. From (2.4) and the fact that $|b_{2k-1}| \le \frac{2}{2k-1}$ for $k \ge 1$, it follows that, for $0 < r < 1$,

$$|na_n - b_{2n-1}|^2 \le 8 \sum_{k=1}^{n-1} \frac{k|a_k|}{2k-1}$$

$$\le 4r^{-n}P(r).$$

Choosing $r = 1 - \frac{1}{n}$ (i) follows.

(ii) follows since

$$P(r) \le \left(\sum_{n=1}^{\infty} n|a_n|^2 r^n\right)^{\frac{1}{2}}\left(\sum_{n=1}^{\infty} \frac{r^n}{n}\right)^{\frac{1}{2}}.$$

It follows trivially from(2.1) that

$$L(r) = O(1)\ M(r)^{\frac{1}{2}} \log \frac{1}{1-r} \quad \text{as} \quad r \to 1$$

and so (iii) follows at once on noting that

$$M(r)^2 \le \frac{A(\sqrt{r})}{\pi} \log\frac{1}{1-r}$$

and on using lemma 2.

REMARK. In view of Theorem 1 and Corollary 1, it is possible that for $f \in B_1(\tfrac{1}{2})$ the following conjectures are valid.

(i) $n^2 a_n^2 = O(1)\ M(1 - \frac{1}{n})$ as $n \to \infty$,

(ii) $n^4 a_n^4 = O(1)\ A(1 - \frac{1}{n})$ as $n \to \infty$,

(iii) $L(r) = O(1)\ (A(r))^{\frac{1}{4}} \log(\frac{1}{1-r})$ as $r \to 1$.

We note that (ii) is stronger that (i) and that we have proved (ii) and (iii) in the case when $A(r)$ is finite.

The following extensions to Theorem 1 support the above conjectures.

3. INTEGRAL MEANS.

For f regular in D, define for λ real,

$$I_\lambda(r,f) = \int_0^{2\pi} |f(re^{i\theta})|^\lambda d\theta.$$

THEOREM 2. For $f \in B_1(\frac{1}{2})$ and $\lambda > 1$,

$$I_\lambda(r^2, zf') \leq C(\lambda) \int_0^r \frac{M(\rho)^{\lambda/2}}{(1-\rho)^\lambda}\, d\rho,$$

where $C(\lambda)$ is a constant depending only on λ.

PROOF. (2.1) gives

$$I_\lambda(r^2, zf') = \int_0^{2\pi} |z^2 f'(z^2)|^\lambda d\theta$$

$$\leq \lambda r \int_0^r \int_0^{2\pi} |zf'(z^2)|^{\lambda-1} |F'(z)\ h\ (z^2)| d\theta d\rho$$

$$+ 2\lambda r \int_0^r \int_0^{2\pi} |zf'(z^2)|^{\lambda-1} |F(z)\ h'(z^2)| \rho d\theta d\rho$$

$$= J_1'(r) + J_2'(r) \quad \text{say.}$$

Now for $\lambda \geq 1$,

$$J_1'(r) \leq \lambda r \int_0^r (M(\rho^2, zf'))^{\lambda-1} \int_0^{2\pi} |F'(z)\ h(z^2)| d\theta \rho^{1-\lambda}\, d\rho.$$

From the proof of Theorem 1 (ii), we have with $z = \rho e^{i\theta}$,

$$\int_0^{2\pi} |F'(z)\ h(z^2)| d\theta \leq O(1) \frac{1}{1-\rho} \quad \text{as } \rho \to 1.$$

Also (2.1) and the distortion theorem for functions of positive real part [4] gives

$$M(\rho^2, zf') \leq \frac{2\rho\ M(\rho, F)}{1-\rho}.$$

Thus

$$J_1'(r) \leq C(\lambda) \int_0^r \frac{M(\rho, F)^{\lambda-1}}{(1-\rho)^\lambda}\, d\rho$$

and since $F(z)^2 = f(z^2)$,

$$J_1'(r) \leq C(\lambda) \int_0^r \frac{M(\rho, f)^{\frac{\lambda-1}{2}}}{(1-\rho)^\lambda}$$

Similarly, using the fact that h is harmonic, we have

$$J_2'(r) \leq C(\lambda) \int_0^r \frac{M(\rho, F)^\lambda}{(1-\rho)^\lambda} \rho^\lambda d\rho$$

$$\leq C(\lambda) \int_0^r \frac{M(\rho, f)^{\lambda/2}}{(1-\rho)\lambda}\, d\rho$$

Combining the results for $J_1'(r)$ and $J_2'(r)$ we obtain the result.

THEOREM 3. Let $f \in B_1(\frac{1}{2})$, then for $0 \leq r \leq 1$,

$$I_1(r^2, f) \leq I_1(r^2, f_0)$$

where

$$f_0(z^2) = \left(\int_0^z \frac{1+t^2}{1-t^2}\, dt\right)^2$$

PROOF. Since $f \in B_1(\frac{1}{2})$, then $F \in R$.
Thus

$$\frac{1}{2\pi} \int_0^{2\pi} |f(z^2)| d\theta = \frac{1}{2\pi} \int_0^{2\pi} |F(z)|^2\, d\theta$$

$$= r^2 + \sum_{n=2}^{\infty} |b_{2n-1}|^2\ r^{4n-2}$$

and since for $n \geq 1$, $|b_{2n-1}| \leq \frac{2}{2n-1}$

$$I_1(r^2,f) \leq r^2 + 4 \sum_{n=2}^{\infty} \frac{r^{4n-2}}{(2n-1)^2} = \frac{1}{2\pi} \int_0^{2\pi} |f_0(z^2)| \, d\theta$$

and the theorem is proved.

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