Internat. J. Math & Math. Sci.
Vol. 9 No. 1 (1986) 29-37

AN EXTENSION OF A RESULT OF CSISZAR

P. B. CERRITO

Department of Mathematics
University of South Florida
Tampa, Florida 33620

(Received July 22, 1985)

ABSTRACT. We extend the results of Csiszar (Z. Wahr. 5(1966) 279-295) to a topological semigroup S. Let μ be a measure defined on S. We consider the value of $\alpha = \sup_{K \text{ compact}} \lim_{n\to\infty} \sup_{x\in S} \mu^n(Kx^{-1})$. First, we show that the value of α is either zero or one. If $\alpha = 1$, we show that there exists a sequence of elements $\langle a_n\rangle$ in S such that $\mu^n * \delta_{a_n}$ converges vaguely to a probability measure where δ denotes point mass. In particular, we apply the results to inverse and matrix semigroups.

KEY WORDS AND PHRASES. Topological semigroup, Infinite convolutions.
PRIMARY CLASSIFICATION. 60F17

1. INTRODUCTION.

Csiszar [1] proved the following result concerning a regular probability measure μ on a locally compact, second countable, Hausdorff group G: Either $\sup \mu^n(Kx^{-1}) \to 0$ as $n \to \infty$ for all compact sets K, where μ^n denotes the n-fold convolution of μ, or there exists a sequence of elements $\langle a_n\rangle$ such that $\mu^n * \delta_{a_n}$ converges vaguely to a probability measure where δ_{a_n} denotes point mass at a_n.

We will extend this result to probability measures defined on certain types of locally compact, second countable, Hausdorff semigroups which satisfy condition (c): If A and B are compact then so are AB^{-1} and $A^{-1}B$ where

$$AB^{-1} = \langle y: \text{there exists } z \in B \text{ such that } yz \in A\rangle.$$

We will also consider $\mu^n(Kx^{-1})$ when μ is defined on a semigroup S of $m \times m$ matrices. A matrix semigroup does not necessarily satisfy condition (c).

To each regular probability measure μ on a semigroup S we associate the value $\alpha_0 = \sup_{K \text{ compact}} \lim_{n\to\infty} \sup_{x\in S} \mu^n(Kx^{-1})$. We first show that $\alpha_0 = 0$ or $\alpha_0 = 1$. If $\alpha_0 = 0$ then $\mu^n * \delta_{a_n} \to 0$ vaguely for any sequence of elements $\langle a_n \rangle$ in S. If $\alpha_0 = 1$ we find $\langle a_n \rangle$ such that $\mu^n * \delta_{a_n}$ converges to a probability measure.

2. PRELIMINARY RESULTS.

In order to show the main results, we need the following lemma. We omit its proof since it is quite similar to an argument of Csiszar [1].

LEMMA 1. Assume S satisfies condition (c). Let μ_1 be a probability measure such that $\sup \mu_1(Kx^{-1}) \leq \alpha$ for a compact set $K \subset S$. Then there exists a compact set K_2 (depending on μ_1) such that for any other probability measure on S,

$$\mu_1 * \mu_2(Kx^{-1}) \leq \alpha - \alpha/2(1 - \mu_2(K_2x^{-1})).$$

Define $\alpha_n(K) = \sup_x \mu^n(Kx^{-1})$. Then if $k < n$,

$$\begin{aligned}\mu^n(Kx^{-1}) &= \int \mu^k(Kx^{-1}y^{-1})\mu^{n-k}(dy) \\ &\leq \alpha_k(K) \int \mu^{n-k}(dy) = \alpha_k(K).\end{aligned}$$

Therefore $\langle \alpha_n(K) \rangle$ is a nonincreasing sequence. Define $a(K) = \lim_{n\to\infty} \alpha_n(K)$ and $\alpha_0 = \sup_{K \text{ compact}} \alpha(K)$.

THEOREM 1. If S satisfies condition (c) then either $\alpha_0 = 0$ or $\alpha_0 = 1$.

PROOF. Suppose $0 < \alpha_0 < 1$. Then there exists an α such that $0 < \alpha(1+\alpha)/2 < \alpha_0 < \alpha < 1$. For any compact set K there exists a $k(K)$ such that $\sup \mu^k(Kx^{-1}) < a$. Applying Lemma 1 to $\mu_1 = \mu^k$ and $\mu_2 = \mu^{n-k}$ yields the fact that for some K_2.

$$\mu^n(Kx^{-1}) \leq \alpha - \alpha/2(1 - \mu^{n-k}(K_2x^{-1})).$$

If n is sufficiently large, $\mu^{n-k}(K_2x^{-1}) < \alpha$ for all x since

$$\sup \mu^{n-k}(K_2x^{-1}) < \alpha(K_2) \leq \alpha_0 < \alpha.$$

But then $\quad \mu^n(Kx^{-1}) \leq \alpha - \alpha/2(1-\alpha) = \alpha(1+\alpha)/2.$

Therefore $\alpha(K) \le \alpha(1+\alpha)/2$. Since K is arbitrary we have a contradiction. We conclude that $\alpha_0 = 0$ or $\alpha_0 = 1$.

QED

Before proceeding we present an example. Let $S = [0,\infty)$ with the usual topology. Define multiplication by $r \cdot s = \max(r,s)$. Let $K = [0,n]$ be a compact subset of S. Then

$$Kx^{-1} = \begin{cases} 0 & x > n \\ K & x \le n \end{cases} \quad \text{and} \quad \mu^n(Kx^{-1}) = \begin{cases} 0 & x > n \\ \mu(K)^n & x \le n. \end{cases}$$

Therefore if μ has compact support then $\alpha_0 = 1$. Otherwise, $\alpha_0 = 0$.

3. MATRIX SEMIGROUPS

Let S be the set of all m x m matrices with probability measure μ defined on S such that the support of μ generates a subsemigroup S_μ of S. We assume S has the usual topology. Define $G = \{X \in S : X \text{ is nonsingular}\}$. Then G forms a subgroup of S. We want to consider the subgroup G_μ of G generated by the set $S_\mu \cap G$. We consider the case where G_μ is locally compact. Then G_μ becomes a topological subgroup of S. If $\mu(G) = 1$ then we need only apply Csiszar [1] to show that $\alpha_0 = 0$ or $\alpha_0 = 1$. Therefore we assume $0 < \mu(G) < 1$. Define a measure μ' on G such that

$$\mu'(B) = \mu(B \cap G)/\mu(G) \text{ for } B \subset S.$$

Then $$(\mu')^2(B) = \int_S \mu'(Bx^{-1})\,\mu'(dx)$$

$$= \int_G \mu(Bx^{-1} \cap G)/\mu(G)\,\mu'(dx)$$

$$= 1/\mu(G) \int_G \mu(Bx^{-1} \cap G)/\mu(G)\,\mu(dx)$$

Now $Bx^{-1} \cap G = \{y \in G : yx \in B\} = \{y \in S : yx \in B \cap G\}$ if $x \in G$. Therefore,

$$(\mu')^2(B) = 1/\mu(G)^2 \int_G \mu((B \cap G)x^{-1})\,\mu(dx).$$

If $x \notin G$ then $(B \cap G)x^{-1} = 0$. Therefore

$$(\mu')^2(B) = 1/\mu(G)^2 \int_S \mu((B \cap G)x^{-1})\,\mu(dx)$$

$$= \mu^2(B \cap G)/\mu(G)^2.$$

By an induction argument,

$$(\mu')^n(B) = \mu^n\,(B \cap G)/\mu(G)^n.$$

Define the following notation:

$$\alpha_g = \sup_{K\subset G} \lim_n \sup_{x\in G} (\mu')^n(Kx^{-1})$$

$$= \sup_{K\subset G} \lim_n \sup_{x\in S} (\mu')^n(Kx^{-1})$$

$$= \sup_{K\subset G} \lim_n [\sup_{x\in S} \mu^n(Kx^{-1})]/\mu(G)^n.$$

Since $\mu(G) < 1$, $\mu(G)^n \to 0$ as $n \to \infty$. By Csiszar's result [1] for groups, either $\alpha_g = 0$ or $\alpha_g = 1$. However,

$$\lim[\sup \mu^n(Kx^{-1})]/\mu(G)^n < \infty.$$

This is only possible if $\lim \sup \mu^n(Kx^{-1}) = 0$ for any $K \subset G$. Henceforth, we assume that K is a compact set consisting of singular matrices. We will also exclude the zero matrix from our discussion since $0^{-1}0 = S$ reduces the problem to a triviality and it is obvious that $\alpha_0 = 1$. That is, we define

$$\alpha_0 = \sup_{K\subset S} \lim_n \sup_{\substack{x\in S\\ x\neq 0}} \mu^n(Kx^{-1}).$$

We give an example. Suppose S_μ consists of matrices with nonnegative entries such that for any $X \in S_\mu$, every entry in X is contained in the set $[\delta,\infty)$ where $\delta > 1/m$. Then

$$\mu^{n+1}(Kx^{-1}) = \int \cdots \int \mu(K(y_n\cdots y_1x)^{-1})\mu(dy_1)\cdots\mu(dy_n)$$

where $K(y_n\cdots y_1x)^{-1} = \{z\in S: zy_n\cdots y_1x \in K\}$ and

$$y_n\cdots y_1x = \begin{bmatrix} w_{11} & \cdots & w_{1m} \\ \vdots & & \\ w_{m1} & \cdots & w_{mm} \end{bmatrix}$$

where w_{ij} has minimal value $m^{n-1}\delta^n$ for all i and j. Therefore for $Z = (z_{ij}) \in K(y_n\cdots y_1x)^{-1}$, $m^{n-1}\delta^n \sum z_{ij} \in K$ so that as $n \to \infty$, $\sum z_{ij} \to 0$ for all i. Hence for any compact set K,

$$\lim_n \mu^n(Kx^{-1}) = 0$$

and $\alpha_0 = 0$. By a similar argument, if every entry of $X \in S_\mu$ is contained in $[0,1/m)$, then $\alpha_0 = 1$.

In order to state a more general result, it is necessary to define some notation. Let Δ_k be the diagonal idempotent matrix of rank K. Let $y_1, y_2, \cdots, y_n \in S_\mu$. Then

$$y_n \cdots y_1 \Delta_1 = \begin{pmatrix} \sum_j w_{1jn} & 0 & \cdots & 0 \\ \vdots & & \ddots & \\ \sum_j w_{mjn} & 0 & \cdots & 0 \end{pmatrix} \text{ where } j \in \{1, \cdots, m^{n-1}\} \text{ and}$$

w_{ijn} represents the product of n real numbers. We need to consider the distribution of $S_{ni} = \sum_j W_{ijn}$ where $j \in \{1, 2, \cdots, m^{n-1}\}$. Let F_{ijn} be the distribution function of the random variable W_{ijn}, $i = 1, 2, \cdots, m$; $j = 1, 2, \cdots, m^{n-1}$; $n = 1, 2, \cdots$. If we assume independence between the entries in the matrices then we may apply the Lindberg-Feller Theorem [2] to the double array $\{W_{ijn}\}_{jn}$ for every i.

THEOREM 2. Suppose the $\{W_{ijn}\}_{jn}$ defined above satisfy the following conditions for each i:

1. $\sum_j \operatorname{Var}(W_{ijn}) = 1$ for every n.
2. $E(W_{ijn}) = 0$ for every j,n.

If $\sum \int y^2 \, dF_{ijn}(y) \to 0$ where the integral is taken over the set $|y^2| > \delta$ for each $\delta > 0$ as $n \to \infty$ then $\alpha_0 = 1$.

PROOF: By the Lindberg-Feller Theorem, S_{ni} converges in distribution to the standard normal for every i. Therefore for n and N sufficiently large, $P(|S_{ni}| \le N) = 1 - \epsilon$ for all i where $\epsilon \to 0$ as $N \to \infty$. Therefore

$$\begin{aligned} &\mu(X = (x_{ij}): xy_1 \cdots y_n \Delta_1 \in K_k = [-k,k]) \\ &= \mu(X: |\sum_i x_{ij} S_{ni}| \le k \text{ for all } j) \\ &\ge \mu(X: |\sum_i x_{ij}| N \le k \text{ for all } j)(1-\epsilon)^m \\ &\ge (1-\epsilon)^m \mu(K_{k'}). \end{aligned}$$

Note that k' depends only on the choice of N and K and not on the choice of n. Therefore as $K_k \uparrow S$ we may also let N increase so it becomes clear that

$\alpha_0 = \sup \lim \mu^n(K\Delta_1^{-1}) = 1.$ QED

It is clear that conditions (1) and (2) may be relaxed so that $\sum_j \operatorname{Var}(W_{ijn}) < M$ for some M and $E(W_{ijk}) < \infty$ for all j,k.

We present an example. Suppose the support of the measure μ.

$$S_\mu = \left\{ \begin{bmatrix} x_1 & x_2 & x_3 & x_4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} : x_i \in R \right\}. \text{ Then}$$

for any $x = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 \\ & 0 & & \end{bmatrix} \in S\mu$, $x\Delta_1 = \begin{bmatrix} x_1 & 0 & 0 & 0 \\ 0 & & & \\ 0 & & 0 & \\ 0 & & & \end{bmatrix}$.

Therefore we need only be concerned with the probability distribution of the corner element. Suppose X_{ij} is a random variable such that

$$P(X_{ij} = 1/2) = P(X_{ij} = -1/2) = 1/2 \text{ for all } i,j.$$

Then $E(X_{1j}) = 0$ and $Var(X_{ij}) = 1/4$. Also for any n, $E(X_{1j}X_{2j}\cdots X_{nj}) = 0$ and $Var(X_{1j}x_{2j}\cdots x_{nj}) = 1/4$. Define

$$W_{1j1} = 2X_{1j},\ j = 1$$

$$W_{1j2} = 2X_{1j}X_{2j},\ j = 1,2,3,4$$

$$W_{1jn} = 2X_{1j}X_{2j}\cdots X_{nj},\ j = 1,2,\cdots,4^{n-1}. \quad \text{Then } \sum Var(W_{1jn}) = 1 \text{ and}$$

$E(W_{1jn}) = 0$. Also $\int y^2\, dF_{1jn}(y) = 0$ if n is sufficiently large. By the above theorem, $\alpha_0 = 1$.

4. The Case Where $\alpha_0 = 1$.

If $\alpha_0 = 0$ then for all compact sets K, $\lim\sup \mu^n(Kx^{-1}) = 0$ so that it is clear that for any sequence $\langle a_n \rangle$, $\mu^n * \delta_{\alpha_n}$ converges vaguely to the zero measure. Therefore we concentrate on the case where $\alpha_0 = 1$. Let S be a locally compact, second countable, Hausdorff semigroup satisfying condition (c).

LEMMA 2. If $\alpha_0 = 1$ and S is abelian then there exists a sequence $\langle x_n \rangle$ such that for any $0 \le \alpha < 1$ there exists a compact set K_α such that $\mu^n(K_\alpha x_n^{-1}) > \alpha$ for all n.

PROOF: For $\alpha = 1/2$ there exists a K_2 such that $\sup_{x \in S} \mu^n(K_2 x^{-1}) > 1/2$ for all n. Therefore there exists a sequence $\langle x_n \rangle$ such that $\mu^n(K_2 x_n^{-1}) > 1/2$ for all n. Similarly, for each $\alpha > 1/2$ there exists a K_α and a sequence $\langle x_{n\alpha} \rangle$ such that $\mu^n(K_\alpha x_{n\alpha}^{-1}) > \alpha$. Since $\alpha > 1/2$, the sets $K_2 x_n^{-1}$ and $K_\alpha x_{n\alpha}^{-1}$ cannot be disjoint so there must exist $w \in (K_2 x_n^{-1}) \cap (K_\alpha x_{n\alpha}^{-1})$. This implies that

$x_{n\alpha} \in K_\alpha w^{-1} \subset K_\alpha(K_2 x_n^{-1})^{-1}$. Therefore $K_\alpha x_{n\alpha}^{-1} \subset K_\alpha(K_\alpha(K_2 x_n^{-1}))^{-1}$. Suppose $y \in K_\alpha x_{n\alpha}^{-1}$. Then $y \in K_\alpha[K_\alpha(K_2 x_n^{-1})]^{-1}$ so there exists $z \in K_\alpha(K_2 x_n^{-1})^{-1}$ such that $yz \in K_\alpha$. Also $z \in K_\alpha(K_2 x_n^{-1})^{-1}$ implies there exists $z' \in K_2 x_n^{-1}$ such that $zz' \in K_\alpha$ and $z'x_n \in K_2$. Therefore $(yz)(zz')(z'x_n) \in K_\alpha^2 K_2$. Since S is abelian, $yx_n \in (K_\alpha^2)^{-1} K_\alpha^2 K_2$ and $y \in ((K_\alpha^2)^{-1} K_\alpha^2 K_2 x_n^{-1})$. By redefining K_α to be $(K_\alpha^2)^{-1} K_\alpha^2 K_2$, $\mu^n(K_\alpha x_n^{-1}) > \alpha$ for all n.

QED

If S is a group we can define $\nu_n = \delta_{x_{n-1}^{-1}} * \mu * \delta_{x_n}$ where the x_n's are defined in lemma 2. Then we can apply Csiszar [1] to $y_k^n = \nu_{k+1} * \nu_{k+2} * \cdots * \nu_n = \delta_{x_k^{-1}} * \mu^{n-k} * \delta_{x_n}$. Unfortunately $\delta_{x^{-1}}$ has no meaning in a semigroup and the ν_n's must be defined in some other way.

Suppose S is embeddable in an abelian group G. Then by Lemma 2 there exists a sequence $\langle x_n \rangle$ such that for any α there exists a K_α such that $\mu^n(K_\alpha x_n^{-1}) > \alpha$. We may assume that μ is a measure defined in G with support contained in S. Then $\nu_n = \delta_{x_n^{-1}} * \mu * \delta_{x_n}$ is well defined in G if we let x_0 be the identity element of G. If we write $(Kx^{-1})_S$ and $(Kx^{-1})_G$ for the respective sets defined in S and G then $(Kx^{-1})_S \subset (Kx^{-1})_G$. However since the support of μ is contained in S, $\mu^n((Kx^{-1})_S) = \mu^n((Kx^{-1})_G)$. Therefore $\alpha_0 = 1$ with respect to G. Let $y_k^n = \nu_{k+1} * \cdots * \nu_n$. Then $y_0^n(K_\alpha) = \mu^n(K_\alpha x_n^{-1}) > \alpha$ for any α. Also, by lemma 1, $y_k^n(K_\alpha^{-1} K_\alpha) \geq y_0^n(K_\alpha) + y_0^k(K_\alpha) - 1 \geq 2\alpha - 1$. Therefore it is clear that any limit point of y_k^n must be a probability measure and Csiszar [1] can be applied to this sequence. It is also clear that any limit point of y_k^n must have support contained in S and may therefore be considered a measure on S.

Next consider the case where S is an abelian inverse semigroup. S is a semigroup of this type provided for any $x \in S$ there exists a unique $x' \in S$ such that $xx'x = x$ and $x'xx' = x'$. A natural ordering can be defined on the idempotent elements of S: $e \leq f$ provided $ef = fe = e$. If S contains a minimal idempotent e then we can define $\nu_n = \delta_{x_{n-1}} * \mu * \delta_{x_n e}$ with $x_0 = e$. Then

$$\begin{aligned} y_0^n(K_\alpha \cup K_\alpha e) &= \nu_1 * \cdots \nu_n(K_\alpha \cup K_\alpha e) \\ &= \mu^n((K_\alpha \cup K_\alpha e)(x_n e)^{-1}) \\ &\geq \mu^n(K_\alpha x_n^{-1}) > \alpha \text{ for all } n. \end{aligned}$$

Therefore all limit points of y_k^n are probability measures.

If S contains a finite number of idempotents, say $e_1, e_2, \cdots, e_n$ then the product $e_1 e_2 \cdots e_n$ is minimal in S. Therefore Csiszar [1] can be applied to any abelian inverse semigroup with a finite number of idempotents.

Suppose instead that S is an inverse semigroup such that the set of idempotents can be ordered in the following manner: $f_0 > f_1 > f_2 > \cdots$. That is, suppose S is an ω-semigroup. Let $x_0 = f_0$ and consider the sequence $\langle x_n \rangle$ defined in lemma 4. Given any x_n either

a. the idempotent $x_j x_j' = e_j > x_n x_n' = e_n$ for all $j > n$ or

b. there exists some $j > n$ such that $e_j < e_n$.

If there exists some n for which (a) is true then S has a minimal idempotent. If not, there exists a subsequence $x_0, x_{i_1}, x_{i_2}, \cdots$ such that $e_{i_j} > e_{i_n}$ if $j > n$.

Define $\nu_n = \delta_{x_{i_{n-1}}} * \mu^{i_n - i_{n-1}} * \nu_{x_{i_n} e_{i_n}}$.

THEOREM 3. If $y_k^n = \nu_{k+1} * \cdots * \nu_n$ is a sequence of probability measures on S satisfying the hypotheses of Csiszar [1] then there exists a sequence $\langle w_n \rangle$ in S such that for each k, $y_k^n * \delta_{w_n}$ converges vaguely to a probability measure as $n \to \infty$.

PROOF. By Csiszar [1] there exists a sequence of integers $n_1 < n_2 < \cdots < n_j < \cdots$ such that

$$\lim_j y_k^{n_j} = \lambda_k \text{ and } \lim_j \lambda_n = \lambda_\infty$$

where the limits are defined with respect to the vague topology and λ_k is a probability measure for all $k \le \infty$. Also λ_∞ is idempotent and $\lambda_k * \lambda_\infty = \lambda_k$ for all k.

The support of any idempotent probability measure is completely simple. Let H denote the support of λ_∞. Since S is abelian, H is a group. Furthermore, λ_∞ is a Haar measure on H and H is a compact group.

The remainder of the proof, dealing with the choice of a suitable sequence $\langle w_n \rangle$, is quite similar to the argument in Csiszar [4] and will be omitted.

QED

We define $a_n = x_n w_n$ where x_n is defined in lemma 2 and w_n is defined above. If S is embeddable or an inverse semigroup with a minimal idempotent then $\lim_n y_0^n * \delta_{w_n} = \lim_n \mu^n * \delta_{a_n} = \lambda_0$ which is a probability measure. In the other two cases, the same argument can be applied to an infinite subsequence.

REFERENCES

1. CSISZAR, I. On Infinite Products of Random Elements and Infinite Convolutions of Probability Distributions on Locally Compact Groups, Z. Wahr. verw. Geb. 5, 1966, 279-295.

2. CHUNG, K. L. A Course in Probability Theory, Academic Press, New York, 1970.

3. CLIFFORD, A. H. and PRESTON, G. B. The Algebraic Theory of Semigroups, Vol. I, Mathematical Surveys 7, Amer. Math. Soc., Providience, R.I., 1961.

4. CENTER, B. and MUKHERJEA, A. More on Limit Theorems for Iterates of Probability Measures on Semigroups and Groups, Z. Wahr. verw. Geb. 46, 1979, 259-275.

5. MUKHERJEA, A. Limit Theorems for Convolution Iterates of a Probability Measure on Completely Simple or Compact Semigroups, Trans. Amer. Math. Soc., 225, 1977, 355-370.

6. YEAGER, D. Imbedding Compact Semigroups in Compact Inverse Semigroups, Semigroup Forum 10, 1975, 76-83.