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RESEARCH NOTES

HOMOMORPHISMS OF COMPLETE n-PARTITE GRAPHS

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ABSTRACT. It is shown that for every homomorphism ϕ of a graph G there exists a contraction θ_ϕ on $\bar{G}$, the complement of G, such that $\overline{\phi(G)} = \theta_\phi(\bar{G})$ if and only if G is a complete n-partite graph.

KEY WORDS AND PHRASES. Homomorphisms of graphs, contractions of graphs, complete n-partite graph.
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By a graph G we mean a set V(G) of vertices together with a set E(G) of unordered pairs of distinct vertices in V(G), called edges. An elementary homomorphism of a graph G is the identification of two non-adjacent vertices of G, and a homomorphism is a sequence of elementary homomorphisms. Thus a homomorphism of a graph G onto a graph H is a function ϕ from V(G) onto V(H) such that whenever u and v are adjacent in G, $\phi(u)$ and $\phi(v)$ are adjacent in H. Likewise, an elementary contraction of a graph G is the identification of two adjacent vertices of G, and a contraction is a sequence of elementary contractions. Thus a contraction of a graph G onto a graph H is a function θ from V(G) onto V(H) such that, for every $u \in V(H)$, $\theta^{-1}(u)$ is a connected subgraph of G and for every $uv \in E(H)$, there is at least one edge in G joining a vertex of $\theta^{-1}(u)$ with one of $\theta^{-1}(v)$. Now for every homomorphism ϕ of G there is a contraction θ_ϕ of $\bar{G}$, the complement of G, that we construct as follows: ϕ is a sequence of elementary homomorphisms $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$ each of which identifies two non-adjacent vertices in G, so we let θ_ϕ be the sequence of elementary contractions $\theta_1, \theta_2, \ldots, \theta_n$ such that θ_i identifies the same vertices in $\bar{G}$ that ε_i identifies in G.

For $n \leq 1$, G is a complete n-partite graph if it is possible to partiton V(G) into n subsets $V_1, V_2, \ldots, V_n$, called partite sets, such that is $u, v \in V_i$ then $uv \notin E(G)$, but if $u_i \in V_i$ and $u_j \in V_j$ where $i \neq j$, then $u_i u_j \in E(G)$. If $n = 1$, G is the null graph $\bar{K}_p$ where p is the number of vertices in $V(G)$. If each V_i has exactly one vertex in it then G is the complete graph K_n. We also note that G is a complete n-partite graph if and only if $\bar{G}$ is a disjoint union of n complete graphs.

MAIN THEOREM. $\overline{\phi(G)} = \theta_\phi(\bar{G})$ for every homomorphism ϕ of G if and only if G is a complete n-partite graph.

Before proving this theorem we will need some additional groundwork. We note that any graph G is either connected or a union of disjoint connected graphs called components. Let $c(G)$ denote the number of components of the graph G. $\beta_o(G)$ will denote the point independence number of G, that is, the cardinality of the largest set of non-adjacent vertices in G. Clearly for any graph G, $\beta_o(G) \geq c(G)$. As usual $\chi(G)$ will denote the chromatic number of G, and Hedetniemi [2, p. 24] shows that for any graph G, $\chi(G) \geq \beta_o(\bar{G})$. Thus we have

(*) $$\chi(G) \geq c(\bar{G})$$

for any graph G.

LEMMA. $\chi(G) = c(\bar{G})$ if and only if G is a complete n-partite graph.

PROOF. If G is a complete n-partite graph, then $\chi(G) = n$ and $c(\bar{G}) = n$.

On the other hand, suppose G is not a complete n-partite graph but $\chi(G) = c(\bar{G})$. If $\bar{G}$ is connected then $c(\bar{G}) = 1$. However G must contain at least one edge (else G is a complete 1-partite graph) and so $\chi(G) \geq 2$, and we are done. So we suppose $c(\bar{G}) = n > 1$. Thus $\bar{G}$ is the union of n components at least one of which is not a complete graph. We add edges to each component of $\bar{G}$, and delete the corresponding edges of G, until each component of $\bar{G}$ becomes a complete graph, and hence G becomes a complete n-partite graph, call it G'. Now $\chi(G') = n$, and Culik [1] has shown that adding exactly one edge to a complete n-partite graph increases the chromatic number by one. However, deleting edges from any graph will either leave the chromatic number unchanged or decrease it. Therefore, if e is any edge that was deleted from G to obtain G' we have

$$n + 1 = \chi(G' + e) \leq \chi(G) = n ,$$

a contradiction, and th lemma is proved.

PROOF OF THE MAIN THEOREM. Suppose G is a complete n-partite graph with partite sets $V_1, V_2, \ldots, V_n$. Clearly the identity homomorphism and its related contraction, the null contraction, satisfy the given equation. Let ε be any elementary homomorphism of G. Then ε will identify two non-adjacent vertices and so will be restricted to some V_i. In fact $\varepsilon(G)$ will still be a complete n-partite graph with one less vertex in V_i. Likewise the related elementary contraction θ_ε will be restricted to the component of $\bar{G}$ that corresponds to V_i in G.

However every component of G is a complete graph and any elementary contraction on a complete graph yields a complete graph on one less vertex. Thus $\overline{\varepsilon(G)} = \theta_{\varepsilon}(\bar{G})$ for every elementary homomorphism of G, and so $\overline{\phi(G)} = \theta_{\phi}(\bar{G})$ for every homomorphism ϕ of G.

Now suppose G is not a complete n-partite graph but $\overline{\phi(G)} = \theta_{\phi}(\bar{G})$ for every homomorphism ϕ of G. Let $\chi(G) = m$, then we know [2, p. 10] that the homomorphic image of G with the smallest number of vertices is K_m. On the other hand $\bar{G}$ is the union of $\ell \geq 1$ components, and so the image of $\bar{G}$ with the smallest number of vertices under a contraction will be $\overline{K_\ell}$. Since G is not a complete n-partite graph, from the Lemma and (*) we have $m > \ell$. Therefore $\bar{G}$ is not contractable to $\overline{K_m}$, since any image of G under a contraction that has m vertices must have enough edges to be further contractable to $\overline{K_\ell}$. Hence for any homomorphism ϕ of G onto K_m we have $\overline{\phi(G)} \neq \theta_{\phi}(\bar{G})$ and we are done.

From the method used in the above proof it is evident that we also have the following:

THEOREM. There is a one-to-one correspondence between homomorphisms of G and the contractions of $\bar{G}$ if and only if G is a complete n-partite graph. Otherwise the number of homomorphisms of G will be less than the number of contractions of $\bar{G}$.

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