TWO-DIMENSIONAL SOURCE POTENTIALS IN A TWO-FLUID MEDIUM FOR THE MODIFIED HELMHOLTZ'S EQUATION

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ABSTRACT.   Velocity potentials describing the irrotational infinitesimal motion of two superposed inviscid and incompressible fluids under gravity with a horizontal plane of mean surface of separation, are derived due to a vertical line source present in either of the fluids, whose strength, besides being harmonic in time, varies sinusiodally along its length.   The technique of deriving the potentials here is an extension of the technique used for the case of only time harmonic vertical line source.   The present case is concerned with the two-dimensional modified Helmholtz's equation while the previous is concerned with the two-dimensional Laplace's equation.

KEY WORDS AND PHRASES.   Modified Helmholtz's equation, two-fluid medium, source potentials, surface of separation.
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1.   INTRODUCTION.

Velocity potentials due to the presence of different types of singularities in an incompressible and inviscid one-fluid medium, assuming irrotational motion of small amplitudes, play an important role in dealing with problems involving radiation or scattering of surface waves by obstacles present in the medium. These problems can be reduced to equivalent problems of solving some singular integral equations of second kind in general, by a suitable use of Green's integral theorem in the fluid medium with the help of these singular potentials (generally called Green's function). Thorne [1] gave a survey of the fundamental singularities submerged in an one-fluid medium and Rhodes-Robinson [2] modified it to include the effect of surface tension at the free surface (FS).   Gorgui and Kassem [3] considered a two-fluid medium and obtained potentials due to oscillating line and point singularities submerged in either of the fluids.   The upper fluid of the two-fluid medium considered in [3] is extended infinitely upwards and the lower fluid is of either infinite or finite depth below the mean surface of separation (SS).   Later the model is modified to include a number of generalizations, e.g. presence of interfacial surface tension in the SS (cf. Rhodes-Robinson [4], Mandal [5]) upper fluid of finite depth with a free surface with or without surface tension (cf. Chakrabarti and Mandal [6], Chakrabarti [7], Kassem [8]).

In problems dealing with the scattering of obliquely incident surface waves in an one-fluid medium by horizontal plane barriers (cf. Heins [9], Green and Heins [10]

etc.) or vertical plane barriers (cf. Mandal and Goswami [11], [12], [13]), half-immersed or fully submerged infinitely long circular cylinder (cf. Mandal and Goswami [14], Levine [15], by exploiting the geometry of the obstacles, the velocity potential can be assumed to have a harmonic variation in the lateral (z) direction, same as the incident wave field. Thus the potential function satisfies a two-dimensional reduced Helmholtz's equation. Hence the problems are essentially boundary value problems (BVP) involving the Helmholtz's equation, and the construction of a two-dimensional source potential (the Green's function) is necessary to reduce the BVP's to equivalent integral equations. Both for infinite and finite constant depth of fluid, this source potential can be constructed by the method of Fourier transform (in x) (cf. Heins [9], Levine [15], Miles [16] etc.) or by the method of separation of variables (cf. Rhodes-Robinson [17] where the effect of surface tension of FS is included), thereby obtaining a linear combination of potentials due to the source in an unbounded fluid together with an 'image' potential in the FS boundary condition.

In the present paper we consider a two-fluid medium and derive velocity potentials due to a vertical line source present in either of the fluids whose strength varies harmonically with time and also with the co-ordinate measured along its length. This is the same as deriving the source potentials in a two-fluid medium for the reduced two dimensional Helmholtz's equation. The corresponding problem for the two-dimensional Laplace's equation was considered in [4]. When the strength of the line source is made independent of the co-ordinate along its length, known results for a two-fluid medium are recovered. When the density of the upper fluid is made zero, the results derived here reduce to corresponding known results for an on-fluid medium.

2. STATEMENT OF THE PROBLEM.

We consider a two-fluid medium, both the fluids being incompressible and inviscid. The mean SS is horizontal and taken as the xz-plane y-axis pointing vertically downwards. A line source is assumed to be present in either of the fluids and the y-axis is chosen to pass through the singular point so that the point of singularity is situated either at $(0,\eta)$ or $(0,-\eta)$ $(\eta > 0)$. The strength of the line source is assumed to vary sinusoidally with time as well as with z. Let ρ_1, ρ_2 be the densities of the lower and upper fluids respectively so that $\rho_1 > \rho_2$.
The motion is assumed to be irrotational and is of small amplitude, and can be described, by velocity potentials Re $\{\varphi_j(x,y,z) \exp(-i\omega t)\}$ (j=1,2), where ω is the circular frequency. φ's satisfy the three-dimensional Laplace's equation in respective fluid regions except at the point of singularity where it exists. The linearized SS conditions are

$$K\varphi_1 + \frac{\partial \varphi_1}{\partial y} = s\left(K\varphi_2 + \frac{\partial \varphi_2}{\partial y}\right), \quad y = 0,$$

$$\frac{\partial \varphi_1}{\partial y} = \frac{\partial \varphi_2}{\partial y} \qquad on \qquad y = 0,,$$

where $K = \omega^2/g$, g being the gravity and $s = \rho_2/\rho_1 < 1$. If the lower fluid is of depth 'h' below the mean SS, then

$$\frac{\partial \not{b}_1}{\partial y} = 0 \quad \text{on} \quad y = h,$$

otherwise,

$$\left| \text{grad } \not{b}_1 \right| \rightarrow 0 \quad \text{as} \quad y \rightarrow -\infty.$$

Also $\quad \left| \text{grad } \not{b}_2 \right| \rightarrow 0 \quad \text{as} \quad y \rightarrow -\infty.$

Further, $\not{b}_1, \not{b}_2$ satisfy the radiation condition that both represent outgoing waves in the far fiels as $\left| x \right| \rightarrow \infty.$

Assuming the z–variation of the strength of the line source as $\exp(i \nu z)$, it is possible to extract the z–variation completely from the functions $\not{b}_j (x,y,z)$. Thus we can write

$$\not{b}_j (x,y,z) = \phi_j (x,y) \exp (i \nu z) \qquad j = 1,2$$

where now ϕ_j's satisfy the two–dimensional modified Helmholtz's equation

$$(\nabla^2 - \nu^2) \, \phi_j = 0 \quad \text{in } D_j \tag{2.1}$$

except at a point of singularity, where D_1, D_2 denote respectively the regions occupied by lower and upper fluids and ∇^2 is the two–dimensional Laplacian operator. Near a point of singularity the potential behave as $K_0 (\nu R)$ which is a typical singular solution of Helmholtz's equation, $K_0 (z)$ being the modified Bessel function of second kind and R being the distance from the point. The boundary conditions are

$$K \phi_1 + \frac{\partial \phi_1}{\partial y} = s \, (K \phi_2 + \frac{\partial \phi_2}{\partial y}) \, , \, y = 0; \tag{2.2}$$

$$\frac{\partial \phi_1}{\partial y} = \frac{\partial \phi_2}{\partial y}, \qquad\qquad y = 0; \tag{2.3}$$

$$\frac{\partial \phi_1}{\partial y} = 0, \qquad\qquad y = h \tag{2.4a}$$

when the lower fluid is of finite depth, otherwise,

$$\left| \nabla \phi_1 \right| \rightarrow 0 \text{ as } y \rightarrow \infty \tag{2.4b}$$

when the lower fluid is of infinite depth; also

$$\left| \nabla \phi_2 \right| \rightarrow 0 \text{ as } y \rightarrow -\infty; \tag{2.5}$$

and finally, ϕ_1, ϕ_2 satisfy the radiation condition in the far field as

$$\left| x \right| \rightarrow \infty. \tag{2.6}$$

Thus ϕ_1, ϕ_2 satisfy a boundary value problem (BVP) described by (2.1) to (2.6). In section 3 we will decompose this BVP into two BVP's by defining two sets of component potentials where the first set accounts for the singularity in the medium but die out in the far field while the second set is non–singular but accounts for the radiation condition in the far field as $\left| x \right| \rightarrow \infty.$ In sections 4 and 5 we will obtain solutions to these BVP's assuming the lower fluid to be of infinite and finite depth respectively, thereby deriving the source potentials in the two fluids completely.

3. DECOMPOSITION INTO TWO BOUNDARY VALUE PROBLEMS.

We define potentials ψ_j (x,y), χ_j $(x,y)(j=1.2)$ such that

$$\phi_j = \psi_j + \chi_j \quad j = 1,2 \tag{3.1}$$

where ψ_j satisfy

$$(\nabla^2 - \nu^2) \psi_j = 0 \tag{3.2}$$

in D_j except at a point of singularity, and near a singularity the appropriate conditions are

$$\psi_j \to K_0(\nu R) \text{ as } R \to 0. \tag{3.3}$$

$$\psi_1 = s \psi_2, \qquad y = 0, \tag{3.4}$$

$$\frac{\partial \psi_1}{\partial y} = \frac{\partial \psi_2}{\partial y}, \qquad y = 0, \tag{3.5}$$

$$\psi_1, \psi_2 \to 0 \text{ as } (x^2 + y^2)^{1/2} \to \infty \tag{3.6}$$

in D_1, D_2 respectively. Thus ψ_1, ψ_2 satisfy the BVP described by (3.2) to (3.6) (hereinafter P1). Then χ_1, χ_2 satisfy the BVP (hereinafter P2) described by

$$(\nabla^2 - \nu^2) \chi_j = 0 \qquad \text{in} \qquad D_j \text{ } (j = 1,2), \tag{3.7}$$

$$K \chi_1 + \frac{\partial}{\partial y} (\psi_1 + \chi_1) = s \{K\chi_2 + \frac{\partial}{\partial y} (\psi_2 + \chi_2)\}, \text{ } y = 0; \tag{3.8}$$

$$\frac{\partial \chi_1}{\partial y} = \frac{\partial \chi_2}{\partial y}, \text{ } y = 0 \tag{3.9}$$

$$\frac{\partial \chi_1}{\partial y} = - \frac{\partial \psi_1}{\partial y}, \qquad y = h \tag{3.10a}$$

if there is a bottom to the lower fluid, otherwise,

$$|\nabla \chi_1| \to 0 \quad \text{as} \quad y \to \infty, \tag{3.10b}$$

$$|\nabla \chi_2| \to 0 \quad \text{as} \quad y \to -\infty, \tag{3.11}$$

and finally, χ_1, χ_2 satisfy the radiation condition in the far field as

$$|x| \to \infty. \tag{3.12}$$

In the conditions (3.8) and (3.10a), ψ_1 and ψ_2 are assumed to be known (solution of P1).

4. LOWER FLUID OF INFINITE DEPTH.

(i) <u>Wave Source in the Lower Fluid</u>. In this case we seek a solution to the BVP described by (2.1) to (2.6) where $\phi_1 \to K_0(\nu r)$ as $r = \{x^2 + (y-\eta)^2\}^{1/2} \to 0$. Thus in P1 the precise form of (3.2) and the condition (3.3) are

$$(\nabla^2 - \nu^2) \psi_1 = 0, \text{ } y > 0 \qquad \text{except at } (0,\eta), \tag{4.1}$$

$$(\nabla^2 - \nu^2) \psi_2 = 0, \text{ } y < 0, \psi_1 \to K_0(\nu r) \text{ as } r \to 0 \tag{4.2}$$

Let
$$\psi_1 = K_0(\nu r) + c_1 \, K_0(\nu r^*) \quad \psi_2 = c_2 \, K_0(\nu r)$$

where $r^* = \{x^2 + (y + \eta)^2\}^{1/2}$ is the distance from the image point and c_1, c_2 are unknown constants. Clearly ψ_1, ψ_2 as given above satisy the equations (4.1) and the conditions (4.2) and (3.6). We choose c_1 and c_2 such that the conditions (3.4) and (3.5) are satisfied.

The following integral representations will be needed in our calculation

$$\frac{\partial}{\partial y} K_0(\nu r) = \mp \int_\nu^\infty k \, \zeta^{-1} \cos \zeta x \, \exp \{\mp k(y-\eta)\} \, dk, \qquad y \gtrless \eta,$$

$$\frac{\partial}{\partial y} K_0(\nu r^*) = \mp \int_\nu^\infty k \, \zeta^{-1} \cos \zeta x \, \exp \{\mp k(y-\eta)\} \, dk, \qquad y \gtrless -\eta,$$

where $\zeta = (k^2 - \nu^2)^{1/2}$ and the upper (lower) sign is for $y >(<)\eta$.

Thus

$$\frac{\partial}{\partial y} \left. \begin{matrix} K_0(\nu r) \\ K_0(\nu r^*) \end{matrix} \right|_{y=0} = \mp \int_\nu^\infty k\zeta^{-1} \cos \zeta x \, \exp(-k\eta) \, dk,$$

$$\frac{\partial}{\partial y} \left. \begin{matrix} K_0(\nu r) \\ K_0(\nu r^*) \end{matrix} \right|_{y=h} = -\int_\nu^\infty k\zeta^{-1} \cos \zeta x \, \exp \{-k(h\mp\eta)\} \, dk.$$

Conditions (3.4) and (3.5) give after making use of appropriate integral representations given above

$$1 + c_1 = s \, c_2, \quad 1 - c_1 = c_2,$$

from which we obtain

$$c_1 = (1-s)(1+s)^{-1}, \quad c_2 = 2(1+s)^{-1}.$$

Hence

$$\psi_1 = K_0(\nu r) - \frac{1-s}{1+s} K_0(\nu r^*)$$

$$\psi_2 = \frac{2}{1+s} K_0(\nu r) \qquad\qquad\qquad (4.3)$$

Again, let

$$\chi_1 = \int_\nu^\infty A \, \zeta^{-1} \cos \zeta x \, \exp(-ky) dk, \quad y > 0, \qquad\qquad (4.4)$$

$$\chi_2 = \int_\nu^\infty B \, \zeta^{-1} \cos \zeta x \, \exp(ky) \, dk, \quad y < 0,$$

where A, B are unknown functions of k. Clearly χ_1, χ_2 given above satisfy (3.7), (3.10b) and (3.11). The contour in the integrals is to be chosen in such a way that the radiation condition is satisfied automatically. This will be shown in the sequel. The conditions (3.8) and (3.9) lead to

$$(K-k) A - s(K+k) B = 2(1-s)(1+s)^{-1} \exp(-k\eta),$$
$$A + B = 0.$$

Thus

$$A, B = \pm 2(1+s)^{-1} k(k-M)^{-1} \exp(-k\eta)$$

where

$$M = (1+s)(1-s)^{-1} K. \tag{4.5}$$

Hence

$$\phi_1 = K_0(\nu r) - \frac{1-s}{1+s} K_0(\nu r^*) + \frac{2}{1+s} \int_\nu^\infty \frac{k}{(k-M)\varsigma} \cos\varsigma x \exp\{-k(y+\eta)\}\, dk \tag{4.6}$$

$$\phi_2 = \frac{2}{1+s} K_0(\nu r) - \frac{2}{1+s} \int_\nu^\infty \frac{k}{(k-M)\varsigma} \cos\varsigma x \exp(k(y-\eta))\, dk \tag{4.7}$$

where the contour is indented below the pole at $k = M$ to ensure the radiation condition at infinity. To establish this, we replace $2\cos\varsigma x$ in the integrals by $\exp(i\varsigma|x|) + \exp(-i\varsigma|x|)$. The contour in the integral involving $\exp(i\varsigma|x|)$ is deformed into a line from ν to X (where X is a large positive number) on the real axis with an indentation below the pole at $k = M$, the quarter of a circle of radius X in the first quadrant, the imaginary axis from iX to 0 and a line from 0 to ν just above the cut from $k = -\nu$ to ν in the complex k-plane. It is being assumed that $\nu < M$. (In fact if we assume an incident wave field represented by $\phi_1^{inc} = -\exp\{-My + i(M\cos\alpha x + M\sin\alpha z)\}$, $y < 0$ then $\nu = M\sin\alpha$. However see section 6). In this case there will be a contribution from the pole at $k = M$. As $X \to \infty$, the contribution from the circular arc will be exponentially small. Similarly in the integral involving $\exp(-i\varsigma|x|)$ the contour is deformed into a line from 0 to ν below the cut, a line from ν to X on the real line with an indentation below the point $k = M$, the quarter of a circle of radius X in the fourth quadrant, and the imaginary axis from $-iX$ to 0. In this case as the point $k = M$ lies outside the closed contour, there will be no contribution to the integral from this. As $X \to \infty$, the contribution from the circular arc will be exponentially small. The contribution from the real line from 0 to ν above and below the cut from $k = -\nu$ to ν will cancel out. Comb-ining the two integrals we will finally obtain the alternative representations (which account for the radiation condition in the far field as $|x| \to \infty$ for ϕ_1, ϕ_2 as

$$\phi_1 = K_0(\nu r) - \frac{1-s}{1+s} K_0(\nu r^*)$$

$$+ \frac{2}{1+s} [\pi i\, M\, N^{-1} \exp\{-M(y+\eta) + iN|x|\}$$

$$+ \int_0^\infty \frac{k\{k\cos k(y+\eta) - M\sin k(y+\eta)\} \exp\{-(k^2+\nu^2)^{1/2}|x|\}}{(k^2+M^2)(k^2+\nu^2)^{1/2}}\, dk] \tag{4.8}$$

and

$$\phi_2 = -\frac{2}{1+s} [\pi i\, M\, N^{-1} \exp\{M(y-\eta) + iN|x|\}$$

$$+ \int_0^\infty \frac{k\{k\cos k(y-\eta) - M\sin k(y-\eta)\} \exp\{-k^2+\eta^2)^{1/2}|x|\}}{k^2+M^2)(k^2+\nu^2)^{1/2}}\, dk] \tag{4.9}$$

where $N = (M^2 - \nu^2)^{1/2}$. \hfill (4.10)

(ii) **Wave Source in the Upper Fluid.** In this case

$$\psi_2 \to K_0(\nu r^*) \quad \text{as} \quad r^* \to 0 \quad \text{so that} \quad \psi_2 \to K_0(\nu r^*) \quad \text{as} \quad r^* \to 0.$$

By writing

$$\psi_1 = c_1 K_0(\nu r^*), \qquad \psi_2 = c_2 K_0(\nu r) + K_0(\nu r^*)$$

and χ_1, χ_2 the same integrals as in (4.4) (with different A and B) we will similarly obtain

$$\psi_1 = \frac{2s}{1+s} [K_0(\nu r^*) - \oint_\nu^\infty \frac{k}{(k-M)\zeta} \cos\zeta x \exp\{-k(y+\eta)\} \, dk] \qquad (4.11)$$

$$\psi_2 = K_0(\nu r^*) + \frac{1-s}{1+s} K_0(\nu r) + \frac{2s}{1+s} \oint_\nu^\infty \frac{k}{(k-M)\zeta} \cos\zeta x \exp\{k(y-\eta)\} \, dk \qquad (4.12)$$

where the contour is indented below the pole at $k = M$ to ensure the radiation condition at infinity. Alternative representation for φ_1, ϕ_2 can be obtained following the same method mentioned above as

$$\phi_1 = \frac{2s}{1+s} [K_0(\nu r^*) - \text{the terms in the square bracket in (4.8)}],$$

$$\mu_2 = K_0(\nu r^*) + \frac{1-s}{1+s} K_0(\nu r) + \frac{2s}{1+s} [\text{the terms in the square bracket in (4.9)}].$$

5. LOWER FLUID OF FINITE DEPTH.

(i) **Wave Source in the Lower Fluid.** In this case ψ_1, ψ_2 are the same as in Section 4(i), while χ_1, χ_2 satisfy P2 with the condition (3.10a) in place of (3.10b). Let

$$\chi_1 = \int_\nu^\infty \zeta^{-1} \cos \zeta x \{A \cosh k(h-y) + B \sinh ky\} \, dk, \quad 0 < y < h,$$

$$\chi_2 = \int_\nu^\infty C \zeta^{-1} \cos \zeta x \exp (ky) \, dk, \quad y < 0.$$

$$(5.1)$$

χ_1, χ_2 given above obviously satisfy (3.7) and (3.11). The SS conditions (3.8), (3.9) and the bottom condition (3.10a) yield the following three equations for the derivation of A, B, C.

$$A(K\cosh kh - k\sinh kh) + kB - s(K+k) C = - 2(1-s)(1+s)^{-1} k \exp(-k\eta),$$

$$A \sinh kh - B + C = 0,$$

$$B \cosh kh = \exp \{-k(h-\eta)\} -(1-s)(1+s)^{-1} \exp \{k(y+\eta)\}.$$

Solving for A, B, C we obtain

$$\phi_1 = K_0(\nu r) - \frac{1-s}{1+s} K_0(\nu r^*)$$

$$+ \frac{2}{1+s} \int_\nu^\infty [\frac{1}{\Delta} \exp(-kh) \{s(K+k)-k\} (\sinh k\eta + s \cosh K\eta) \operatorname{sech} kh$$

$$- (1-s) \exp(-k\eta) \cosh k(h-y)$$

$$+ \exp(-kh)(\sinh k\eta + s \cosh k\eta) \operatorname{sech} kh \sinh ky] \frac{\cos\zeta x}{\zeta} \, dk \qquad (5.2)$$

$$\varphi_2 = \frac{2}{1+s} K_0(\nu r)$$

$$+ \frac{2}{1+s} \int_\nu^\infty [(\sinh k\eta + s \cosh k\eta) \operatorname{sech} kh \exp\{-k(h-y)\}$$

$$- \frac{1}{\Delta} \{\exp(-kh) \{s(K+k)-k\} (\sinh k\eta + s \cosh k\eta) \operatorname{sech} kh$$

$$- (1-s)k \exp(-k\eta)\} \sinh kh \exp(ky)] \frac{\cos\zeta x}{\zeta} dk, \qquad (5.3)$$

where $\Delta(k) = K \cosh kh + \{s(K+k)-k\} \sinh kh,$ \qquad (5.4)

and the contour in the integrals is indented below the pole at $k=k_0$ which is the only real positive zero of $\Delta(k)$, to ensure the satisfaction of the radiation condition at infinity.

Note that poles do not occur at $kh = \pm \frac{\pi i}{2}, \pm \frac{3}{2} \pi i, \ldots$. The only poles occur at the zeros of $\Delta(k)$. $\Delta(k)$ has two real zeros, one is positive, k_0 say, and the other is negative. When $s = 0$, magnitudes of these real zeros become the same. The remaining zeros of $\Delta(k)$ are complex in general. When $s = 0$, the complex zeros become purely imaginary (see Rhodes-Robinson [18] with surface tension put equal to zero). To ensure the radiation condition in the far field as $|x| \to \infty$, the same steps of section 4 (i) can be followed in the deformation of the contours in the first and fourth quadrants with the modification that the contours are indented below the pole at $k=k_0$, and the large radius of the circular arc in the first and fourth quadrants is chosen in such a way that no complex zero of $\Delta(k)$ is crossed. The far field behaviour will come only from the contribution to the integrals at $k=k_0$ (when the contour is deformed in the first quadrant), other contributions from the imaginary axis, from the poles at complex zeros lying in the first and fourth quadrants will die out in the far field as $|x| \to \infty$. The contribution from the real line from 0 to ν above and below the cut from $k = -\nu$ to ν will cancel out. Thus as $|x| \to \infty$

$$\psi_1 \to D_1 \frac{\cosh k_0 (h-y)}{\sinh k_0 h} N_0^{-1} \exp(i N_0 |x|),$$

$$\qquad (5.5)$$

$$\psi_2 \to - D_1 \exp(k_0 h) N_0^{-1} \exp(i N_0 |x|),$$

where $N_0 = (k_0^2 - \nu^2)^{1/2},$ \qquad (5.6)

and

$$D_1 = \frac{2}{1+s} \pi i [\exp(-k_0 h) \{s(k_0 + K) - k_0\} (\sinh k_0 \eta + s \cosh k_0 \eta) \operatorname{sech} k_0 h$$

$$- (1-s) k_0 \exp(-k_0 \eta)] \frac{\sinh k_0 h}{\Delta'(k_0)} \qquad (5.7)$$

where $\Delta'(k_0) = \frac{d}{dk} \Delta(k) \Big|_{k=k_0}$

(ii) <u>Wave Source Submerged in the Upper Fluid.</u> In this case ψ_1, ψ_2 are the same as given in Section 4 (ii) while χ_1, χ_2 may be assumed to have the same type of representation given in (5.1) (with different A, B, C). Then A, B, C satisfy

$$A \sinh kh - B + C = 0,$$

$$B \cosh kh = 2s (1+s)^{-1} \exp \{-k(h+n)\}$$

Thus we will finally obtain

$$\phi_1 = \frac{2s}{1+s} K_0(\nu r^*) + \frac{2s}{1+s} \int_\nu^\infty \{\frac{(1-s)k+ \{sK-(1+s)k\} \exp(-kh)\text{sech } kh}{\Delta}$$

$$\exp(-kn)\cosh k(h-y) + \exp \{-k(h+n)\} \text{ sech } kh \sinh ky\} \frac{\cos\zeta x}{\zeta} dk, \qquad (5.8)$$

$$\psi_2 = K_0(\nu r^*) + \frac{1-s}{1+s} K_0(\nu r) + \frac{2s}{1+s} \int_\nu^\infty [\exp (-kh) \text{ sech } kh$$

$$- \frac{1}{\Delta} \{(1-s)k+ \{sK-(1+s)k\} \exp(-kh)\text{sech } kh\} \qquad \sinh kh \exp k (y-n)] \frac{\cos\zeta x}{\zeta} dk \qquad (5.9)$$

As $|x| \to \infty$,

$$\varphi_1 \to D_2 \frac{\cosh k_0(h-y)}{\sinh k_0 h} N_0^{-1} \exp (i N_0 |x|),$$

$$(5.10)$$

$$\psi_2 \to -D_2 \exp (k_0 y) N_0^{-1} \exp (i N_0 |x|),$$

where

$$D_2 = 2s (1+s)^{-1} \pi i [(1-s) k_0 + \{sK-(1+s)k_0\} \exp(-k_0 h) \text{ sech } k_0 h]$$

$$\frac{\sinh k_0 h}{\Delta'(k_0)} \exp(-k_0 n), \qquad (5.11)$$

and N_0 is given by (5.6).

6. CONCLUSION.

   We have derived in the present paper source potentials for the two-dimensional modified Helmholtz's equation in a two-fluid medium. The parameter ν in the Helmholtz's equation has been assumed to be less than the wave parameter M (for infinite depth of the lower fluid) or k_0 (for finite depth of the lower fluid). However, if ν is greater than the wave parameter then the potentials will no longer represent outgoing waves in the far field, rather they will die out in the far field (see the corresponding one-fluid case with surface tension in the FS in [16]).

   Making $s \to 0$ in the above results, source potentials in an one-fluid medium ([16] with surface tension put equal to zero) can be recovered. Making $\nu \to 0$, potentials due to only time-harmonic line source in a two-fluid medium [3] can be recovered. One can also include the effect of surface tension of the SS in these results.

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