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THIRTY-NINE PERFECT NUMBERS AND THEIR DIVISORS

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ABSTRACT. The following results concerning even perfect numbers and their divisors are proved: (1) A positive integer n of the form $2^{p-1}(2^p-1)$, where 2^p-1 is prime, is a perfect number; (2) every even perfect number is a triangular number; (3) $\tau(n) = 2p$, where $\tau(n)$ is the number of positive divisors of n; (4) the product of the positive divisors of n is n^p; and (5) the sum of the reciprocals of the positive divisors of n is 2. Values of p for which 30 even perfect numbers have been found so far are also given.

KEY WORDS AND PHRASES. Perfect number; Marsenne prime; Triangular number.
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1. INTRODUCTION.

A positive integer n is called a perfect number if $\sigma(n) = 2n$, where $\sigma(n)$ is the sum of the positive divisors of n. The last digit of the first five perfect numbers (6, 28, 496, 8128, and 33 550 336) alternates between 6 and 8. This pattern does not continue as the next three perfect numbers are 8 589 869 056, 137 438 691 328 and 2 305 843 008 139 952 128. However, it has been proved in [1] that an even perfect number ends in 6 or 28. It is interesting to observe that these are the first two perfect numbers.

2. EVEN PERFECT NUMBERS.

It is well known that positive integers n of the form $2^{p-1}(2^p-1)$, where 2^p-1 is prime, are perfect numbers. This can be proved using a theorem from elementary number theory [2] which states that if

$n = \prod_{i=1}^{k} p_i^{\alpha_i}$, where the p_i's are distinct primes and the α_i's are positive integers, then

$$\sigma(n) = \prod_{i=1}^{k} \frac{p_i^{\alpha_i+1}-1}{p_i-1}.$$

If $n = 2^{p-1}(2^p-1) = 2^{p-1}\cdot q$, where $q = 2^p-1$ is prime, it follows from the above theorem that

$$\sigma(n) = \frac{2^p-1}{2-1} \cdot \frac{q^2-1}{q-1} = (2^p-1)(q+1) = (2^p-1) \cdot 2^p = 2n,$$

which proves that n is a perfect number.

It has been proved in [2] that an even perfect number is of the form $2^{p-1}(2^p-1)$, where 2^p-1 is prime. It can be easily shown that p is prime whenever 2^p-1 is prime, but the converse is false ($2^{11}-1 = 23 \cdot 89$ is not prime). Primes of the form 2^p-1 are called Mersenne primes.

Thirty-nine even perfect numbers have been found so far [2 - 3] corresponding to p = 2, 3, 5, 7, 13, 17, 19, 31, 61, 89, 107, 127, 521, 607, 1279, 2203, 2281, 3217, 4253, 4423, 9689, 9941, 11213, 19937, 21701, 23209, 44497, 86243, 132049, and 216091. No odd perfect number has yet been discovered.

3. EVEN PERFECT NUMBERS AND TRIANGULAR NUMBERS.

The k^{th} triangular number is defined as

$T_k = \sum_{i=1}^{k} i = \frac{1}{2} k(k+1)$. Every even perfect number is a triangular number.

This is proved by noting that

$$n = 2^{p-1}(2^p-1) = \tfrac{1}{2}(2^p-1) \cdot 2^p = \tfrac{1}{2} k(k+1) = T_k, \text{ where } k = 2^p-1.$$

4. DIVISORS OF AN EVEN PERFECT NUMBER.

If n is an even perfect number, then $\tau(n) = 2p$, where $\tau(n)$ is the number of positive divisors of n. This can be easily proved using a theorem from elementary number theory [2] which states that if

$n = \prod_{i=1}^{k} p_i^{\alpha_i}$, where the p_i's are distinct primes and the α_i's are positive integers, then $\tau(n) = \prod_{i=1}^{k} (\alpha_i + 1)$

The product of the positive divisors of an even perfect number n is n^p. This is obtained by using still another result from elementary number theory [1], namely,

$\prod_{i=1}^{\tau(n)} d_i = n^{\frac{1}{2}\tau(n)}$, and the value of $\tau(n) = 2p$,

where the d_i's are positive divisors of n.

Finally, the sum of the reciprocals of the positive divisors of an even perfect number is 2. This follows from

$\sum_{i=1}^{\tau(n)} \frac{1}{d_i} = \frac{\sigma(n)}{n}$, where the d_i's are positive divisors of n, a result from elementary number theory [1].

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