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RESEARCH NOTES

AN INTEGRAL EQUATION ASSOCIATED WITH LINEAR HOMOGENEOUS DIFFERENTIAL EQUATIONS

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ABSTRACT. Associated with each linear homogeneous differential equation $y^{(n)} = \sum_{i=0}^{n-1} a_i(x)y^{(i)}$ of order n on the real line, there is an equivalent integral equation

$$f(x) = f(x_0) + \int_{x_0}^{x} h(u)du + \int_{x_0}^{x} [\int_{x_0}^{u} G_{n-1}(u,v)a_0(v)f(v)dv]du$$

which is satisfied by each solution $f(x)$ of the differential equation.

KEY WORDS AND PHRASES. Linear homogeneous differential equations, Integral equations, Initial value problems, Variation of parameters formula, Uniform convergence.
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1. INTRODUCTION.

Let n be a positive integer, I be an interval on the real line R and $C(I)$ be the class of all functions continuous on I. Let

$$y^{(n)} = a_{n-1}(x)y^{(n-1)} + \dots + a_0(x)y , \qquad x \varepsilon I \tag{1.1}$$

be any n-th order (normalized) ordinary linear homogeneous differential equation, where $a_i(x) \varepsilon C(I)$, $i=0,1,2,\dots,(n-1)$.

The purpose of this article is to derive an equivalent integral equation satisfied by the solutions of the linear homogeneous differential equation (1.1).

2. MAIN RESULTS.

THEOREM. Let $f(x)$ be a solution of (1.1) defined on I and $x_0 \varepsilon I$. Then $f(x)$ is also a solution of the integral equation

$$f(x) = f(x_0) + \int_{x_0}^{x} h(u)du + \int_{x_0}^{x} [\int_{x_0}^{u} G_{n-1}(u,v)a_0(v)f(v)dv]du, \tag{2.1}$$

where $h(x)$ is the unique solution of the (n-1)-th order linear homogeneous differential equation

$$y^{(n-1)} = a_{n-1}(x)y^{(n-2)} + \dots + a_1(x)y , \quad x \varepsilon I \tag{2.2}$$

satisfying the initial conditions

$$y(x_0) = f'(x_0), \quad y'(x_0) = f''(x_0), \ \ldots, \ y^{(n-2)}(x_0) = f^{(n-1)}(x_0),$$

and $G_{n-1}(x,u)$ is the well-known Green's Function associated with the homogeneous equation (2.2).

PROOF OF THE THEOREM. In order to deduce the integral equation (2.1), we will use the well-known Variation of Parameters formula

$$y(x) = h(x) + \int_{x_0}^{x} G_{n-1}(x,u)\phi(u)du \tag{2.3}$$

solving uniquely the non-homogeneous initial value problem

$$y^{(n-1)} = a_{n-1}(x)y^{(n-2)} + \ldots + a_1(x)y + \phi(x)$$

$$y(x_0) = f'(x_0), \ y'(x_0) = f''(x_0), \ \ldots, \ y^{(n-2)}(x_0) = f^{(n-1)}(x_0)$$

for each $\phi(x) \in C(I)$.

Consider the sequence of functions:

$$f_1(x), \ f_2(x), \ \ldots, \ f_k(x), \ \ldots$$

defined on I, where

$$\begin{aligned} f_1(x) &= f(x_0) + \int_{x_0}^{x} h(u)du \\ f_2(x) &= f_1(x) + \int_{x_0}^{x} \left[\int_{x_0}^{u} G_{n-1}(u,v)a_0(v)f_1(v)dv\right]du \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ f_k(x) &= f_1(x) + \int_{x_0}^{x} \left[\int_{x_0}^{u} G_{n-1}(u,v)a_0(v)f_{k-1}(v)dv\right]du \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned} \tag{2.4}$$

Clearly, for each k, $f_k(x_0) = f(x_0)$, $f_k(x)$ is differentiable on I and for $k \geq 2$

$$f_k'(x) = h(x) + \int_{x_0}^{x} G_{n-1}(x,u)a_0(u)f_{k-1}(u)du. \tag{2.5}$$

Using (2.3), we conclude that, for each $k \geq 2$, $f_k'(x)$ is the unique solution of the non-homogeneous initial value problem

$$\begin{aligned} y^{(n-1)} &= a_{n-1}(x)y^{(n-2)} + \ldots + a_1(x)y + a_0(x)f_{k-1}(x) \\ y(x_0) &= f'(x_0), \ y'(x_0) = f''(x_0), \ \ldots, \ y^{(n-2)}(x_0) = f^{(n-1)}(x_0). \end{aligned} \tag{2.6}$$

Hence each $f_k(x) \in C^n(I)$. Both the sequences $\{f_k(x)\}$ and $\{f_k'(x)\}$ converge uniformly on every compact subset on the interval I. To see this, let B be a compact subset of I. Then there exists a closed and bounded interval $[a,b]$ such that $B \subset [a,b] \subset I$ and

$x_0 \in [a,b]$.

Let $M = \max |G_{n-1}(u,v)a_0(v)|$, $s = \max |f_1(v)|$ for each $u,v \in [a,b]$. One can now see very easily that for each $x \in [a,b]$,

$$|f_{k+1}(x) - f_k(x)| \le M^k s^k (b-a)^{2k}/2k!$$

$$|f'_{k+1}(x) - f'_k(x)| \le M^k s^k (b-a)^k/k!\ ,$$

using recursively the bounds for $|f_{i+1}(x) - f_i(x)|$, $i = 1,2,3,\ldots$. Since each of the series $\sum M^k s^k (b-a)^{2k}/2k!$, $\sum M^k s^k (b-a)^k/k!$ converges, we conclude by Weierstrass' M-test that each of the series of functions

$$f_1(x) + \sum_{k=1}^{\infty} (f_{k+1}(x) - f_k(x)),\ f'_1(x) + \sum_{k=1}^{\infty} (f'_{k+1}(x) - f'_k(x))$$

converges uniformly on $[a,b]$ and hence on B. Therefore there is a function $g(x) \in C^1(I)$ such that

$$f_1(x) + \sum_{k=1}^{\infty} (f_{k+1}(x) - f_k(x)) = \lim_{k\to\infty} f_k(x) = g(x)$$

$$f'_1(x) + \sum_{k=1}^{\infty} (f'_{k+1}(x) - f'_k(x)) = \lim_{k\to\infty} f'_k(x) = g'(x)$$

for all $x \in I$. In particular $g(x_0) = f(x_0)$.

Also from (2.5) we get by taking limit as $k \to \infty$

$$g'(x) = h(x) + \int_{x_0}^{x} G_{n-1}(x,u)a_0(u)g(u)du, \quad x \in I. \tag{2.7}$$

Hence

$$g(x) = f(x_0) + \int_{x_0}^{x} h(u)du + \int_{x_0}^{x} [\int_{x_0}^{u} G_{n-1}(u,v)a_0(v)g(v)dv]du \tag{2.8}$$

Again, relation (2.7) implies by (2.3) that $g'(x)$ is the unique solution of the initial value problem

$$y^{(n-1)} = a_{n-1}(x)y^{(n-2)} + \ldots + a_1(x)y + a_0(x)g(x)$$

$$y'(x_0) = f'(x_0),\ \ldots,\ y^{(n-1)}(x_0) = f^{(n-1)}(x_0).$$

Therefore $g(x) \in C^n(I)$ and

$$g^{(n)}(x) = a_{n-1}(x)g^{n-1}(x) + \ldots + a_0(x)g(x), \quad x \in I.$$

In other words, $g(x)$ is the unique solution of the homogeneous initial value problem

$$y^{(n)} = a_{n-1}(x)y^{(n-1)}(x) + \ldots + a_0(x)y, \quad x \in I.$$

$$y(x_0) = f(x_0),\ y'(x_0) = f'(x_0),\ \ldots,\ y^{(n-1)}(x_0) = f^{(n-1)}(x_0).$$

Hence $f(x) = g(x)$ for all $x \in I$. Therefore, by (2.8)

$$f(x) = f(x_0) + \int_{x_0}^{x} h(u)du + \int_{x_0}^{x} [\int_{x_0}^{u} G_{n-1}(u,v)a_0(v)f(v)dv]du.$$

This completes the proof.

REMARK. The above proof clearly shows how a solution of a linear homogeneous equation with prescribed initial values can be constructed out of a solution $h(x)$ and the Green's Function $G_{n-1}(x,u)$ of a lower order homogeneous linear equation. This is specially significant in case of second order homogeneous equations, as solutions $\{ce^{A(x)}\}$ and the Green's function $G_1(x,u) = e^{A(x)-A(u)}$, $[A(x) = \int_{x_0}^{x} a_1(u)du]$, of first order homogeneous equation $y' = a_1(x)y$ are readily available.

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