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ON DUAL INTEGRAL EQUATIONS WITH HANKEL KERNEL AND AN ARBITRARY WEIGHT FUNCTION

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ABSTRACT. In this paper we deal with dual integral equations with an arbitrary weight function and Hankel kernels of distinct and general order. We propose an operational procedure, which depends on exploiting the properties of the Mellin transforms, and readily reduces the dual equations to a single equation. This then can be inverted by the Hankel inversion to give us an equation of Fredholm type, involving the unknown function. Most of the known results are then derived as special cases of our general result.

KEY WORDS AND PHRASES. Dual integral equation, Bessel functions of first kind, Mellin transforms, the Parseval theorem, Banach space $L(k-i\infty,k+i\infty)$, Hankel transforms, Hankel inversion, Fredholm equation.
AMS CLASSIFICATION CODE. 45F10, 45E10

1. INTRODUCTION.

We shall consider dual integral equations of the type

$$\int_0^\infty 2^{2\alpha}\, t^{-2\alpha}\, J_\nu(xt)[1+w(t)]\phi(t)dt = f(x),\quad 0 < x < 1 \tag{1.1}$$

$$\int_0^\infty 2^{2\beta}\, t^{-2\beta}\, J_\mu(xt)\phi(t)dt = g(x),\quad x > 1.$$

Such equations arise in the discussion of mixed boundary value problems. If $w(t) = 0$, then (1.1) become dual equation of Titchmarsh type, [1]. Different methods for solving dual integral equations have been proposed by various authors, notably, Tranter, [2], Lebedev and Uflyand [3], Noble [4] and Cook [5]. More recently, Erdelyi and Sneddon gave a solution using fractional integral operators [6]. Basically, all these methods, make use of some form of integral operator to reduce the system (1.1) to a single equation, which is then solved using standard techniques.

In this paper we use a different approach. We develop an operational procedure, which consists in exploiting the properties of the Mellin transforms. By this technique the dual equations are readily reduced to a single integral equation, which in turn can be solved using the usual Hankel inversion. A somewhat similar method has been used for solving ordinary dual integral equations by Williams [7], Tanno, [8] and Nasim and Sneddan [9].

2. KNOWN RESULTS.

First, we write down some definitions and results which are used later.

Lemma 1 [1 p.94]. Let $x^{k-1}f(x) \in L(0,\infty)$ and $F(s) \in L(k-i\infty,\ k+i\infty)$, then

$$F(s) = \int_0^\infty f(x)x^{s-1}dx,\quad s = k + i\tau,\ -\infty < \tau < \infty$$

and

$$f(x) = \frac{1}{2\pi i} \lim_{\tau\to\infty} \int_{k-i\tau}^{k+i\tau} F(s)x^{-s}ds.$$

F is then the Mellin transform of f written as

$$\mathcal{M}[f(x);s] = F(s)$$

and f is the inverse Mellin transform of F, written as

$$\mathcal{M}^{-1}[F(s);x] = f(x).$$

Lemma 2 (The Parseval theorem) [1 p.60].

If $x^{-k}g(x) \in L(0,\infty)$ and $F(s) \in L(k-i\infty,k+i\infty)$,

then

$$\int_0^\infty f(xt)g(t)dt = \frac{1}{2\pi i} \int_{k-i\infty}^{k+i\infty} F(s)G(1-s)x^{-s}ds,$$

where F and G are the Mellin transforms of f and g respectively.

3. DUAL INTEGRAL EQUATIONS.

Now, we write the system (1.1) as

$$\int_0^\infty h_1(xt)\{1+w(t)\}\phi(t)dt = x^{-2\alpha}f(x),\ 0 < x < 1 \tag{3.1a}$$

$$\int_0^\infty h_2(xt)\phi(t)dt = x^{-2\beta}g(x),\ x > 1 \tag{3.1b}$$

where $h_1(x) = 2^{2\alpha}x^{-2\alpha}J_\nu(x)$ and $h_2(x) = 2^{2\beta}x^{-2\beta}J_\mu(x)$. Then the Mellin transforms of h_1 and h_2 are respecatively,

$$H_1(s) = 2^{s-1}\frac{\Gamma(\frac{1}{2}s + \frac{1}{2}\nu - \alpha)}{\Gamma(1-\frac{1}{2}s + \frac{1}{2}\nu + \alpha)},\quad 2\alpha-\nu < \mathrm{Re}(s) < 2\alpha$$

and

$$H_2(s) = 2^{s-1}\frac{\Gamma(\frac{1}{2}s + \frac{1}{2}\mu - \beta)}{\Gamma(1-\frac{1}{2}s + \frac{1}{2}\mu + \beta)},\quad 2\beta-\mu < \mathrm{Re}(s) < 2\beta,$$

both belonging to $L(k-i\infty,k+i\infty)$, $s = k+i\tau$, [10]. The left hand sides of the equations (3.1a) and (3.1b) represent functions for all values of x and we shall denote these by $f_1(x)$ and $g_1(x)$, having Mellin transform $F_1(s)$ and $G_1(s)$, respectively.

Now we set $F_1(s)$ and $G_1(s)$ both $\in L(k-i\infty,k+i\infty)$ and put appropriate conditions on the functions $w(x)$ and $\phi(x)$. Then, due to lemma 2, the equations (3.1) give, respectively

$$\begin{aligned} H_1(s)\{\Phi(1-s) + \Psi(1-s)\} &= F_1(s) \\ H_2(s)\Phi(1-s) &= G_1(s) \end{aligned} \tag{3.2}$$

a.e. on $s = k + i\tau$, $-\infty < \tau < \infty$, where,

$$\mathcal{M}[\phi(x);s] = \Phi(s), \text{ and } \mathcal{M}[w(x)\phi(x);s) = \Psi(s).$$

Next, consider the functions

$$H_1^*(s) = \frac{\Gamma(1 - \frac{1}{2}s + \frac{1}{2}\nu + \alpha)}{\Gamma(1 - \frac{1}{2}s + \frac{1}{2}\mu + \beta)}, \quad \mathrm{Re}(s) < 2+2\alpha+\nu$$

$$H_2^*(s) = \frac{\Gamma(\frac{1}{2}s + \frac{1}{2}\nu - \alpha)}{\Gamma(\frac{1}{2}s + \frac{1}{2}\mu - \beta)}, \quad \mathrm{Re}(s) > 2\alpha\ \nu.$$

Note that

$$H_1(s)H_1^*(s) = H_2(s)H_2^*(s) = \frac{2^{s-1}\Gamma(\frac{1}{2}s + \frac{1}{2}\nu - \alpha)}{\Gamma(1 - \frac{1}{2}s + \frac{1}{2}\mu + \beta)}$$

$$\equiv K(s), \text{ say.}$$

Then [10 p.326]

$$k(x) = \mathcal{M}^{-1}[K(s);x] = (2x)^{-\Upsilon}J_\lambda(x),$$

where $\Upsilon = \frac{1}{2}\mu - \frac{1}{2}\nu + \alpha + \beta$ and $\lambda = \frac{1}{2}\nu + \frac{1}{2}\mu - \alpha + \beta$.

We now write equations (3.2) as

$$H_1(s)\Phi(1-s) = F_1(s) - H_1(s)\Psi(1-s)$$

$$H_2(s)\Phi(1-s) = G_1(s).$$

Multiply the above equations by the function $H_1^*(s)$ and $H_2^*(s)$ respectively and using the definition of $K(s)$, we obtain the following pair of functional equations,

$$K(s)\Phi(1-s) = H_1^*(s)\{F_1(s) - H_1(s)\Psi(1-s)\} \tag{3.3a}$$

$$K(s)\Phi(1-s) = H_2^*(s)G_1(s) \tag{3.3b}$$

a.e. on $s = k + i\tau$, $-\infty < \tau < \infty$.

First we consider the equation (3.3a), whence, for $x > 0$,

$$\frac{1}{2\pi i}\int_{k-i\infty}^{k+i\infty} K(s)\Phi(1-s)x^{-s}ds = \frac{1}{2\pi i}\int_{k-i\infty}^{k+i\infty} H_1^*(s)\{F_1(s) - H_1(s)\Psi(1-s)\}x^{-s}ds \tag{3.4}$$

Here $K(s) \in L(k-i\infty,k+i\infty)$ if $2\alpha-\nu < k < \frac{1}{2}\mu-\frac{1}{2}\nu+\alpha+\beta$ and if we let $x^{-k}\phi(x) \in L(0,\infty)$, then by applying Lemma 2, the left-hand side of (3.4) gives

$$\int_0^\infty k(xt)\phi(t)dt. \tag{3.5}$$

The right-hand side of (3.4) is

$$\frac{1}{2\pi i}\int_{k-i\infty}^{k+i\infty}\{F_1(s)-H_1(s)\Psi(1-s)\}\frac{\Gamma(1 - \frac{1}{2}s + \frac{1}{2}\nu + \alpha)}{\Gamma(1 - \frac{1}{2}s + \frac{1}{2}\mu + \beta)}x^{-s}ds$$

$$= \frac{1}{4\pi i}x^{1-2\alpha-\nu}\frac{d}{dx}\int_{k-i\infty}^{k+i\infty}\{F_1(s)\ H_1(s)\Psi(1-s)\}\frac{\Gamma(\alpha - \frac{1}{2}s + \frac{1}{2}\nu)}{\Gamma(1 - \frac{1}{2}s + \frac{1}{2}\mu + \beta)}x^{\nu+2\alpha\ s}ds. \tag{3.6}$$

We define,

$$\mathcal{M}^{-1}[F_1(s)-H_1(s)\Psi(1-s);x] = \mathcal{M}^{-1}[F_1(s);x] - \mathcal{M}^{-1}[H_1(s)\Psi(1-s);x]$$
$$= f_1(x) - \ell(x),$$

where
$$\ell(x) = \frac{1}{2\pi i}\int_{k-i\infty}^{k+i\infty} H_1(s)\Psi(1-s)x^{-s}ds$$
$$= \int_0^\infty h_1(xu)\omega(u)\phi(u)du \tag{3.7}$$

Here $H_1(s) \in L(k-i\infty,k+i\infty)$ if $2\alpha-\nu < k < 2\alpha$ and if we let $x^{-k}\omega(x)\ \phi(x) \in L(0,\infty)$, the result then follows from lemma 2 above. The function $\dfrac{\Gamma(\alpha - \frac{1}{2}s + \frac{1}{2}\nu)}{\Gamma(1 - \frac{1}{2}s + \frac{1}{2}\mu + \beta)} \in$ $L(k-i\infty,k+i\infty)$ if $k < 2\alpha+\nu$ and $\alpha-\beta < \frac{1}{2}\mu-\frac{1}{2}\nu$; and if we assume that $x^{-k}f_1(x)$ and $x^{-k}\ell(x)$ both both belong to $L(o,\infty)$, then by applying Lemma 2 on the s-integral, the expression (3.6) yields

$$\frac{1}{2}x^{1-2\alpha-\nu}\frac{d}{dx}\left\{x^{2\alpha+\nu}\int_0^x[f_1(t)-\ell(t)]\frac{1}{t}h_1^*(\frac{x}{t})dt\right\},\quad 0 < x < 1,$$

where

$$\mathcal{M}^{-1}\left[\frac{\Gamma(\alpha - \frac{1}{2}s + \frac{1}{2}\nu)}{\Gamma(1 - \frac{1}{2}s + \frac{1}{2}\mu + \beta)};x\right] = h_1^*(x) = \begin{cases} \dfrac{2}{\Gamma(\Upsilon-2\alpha+1)}\,x^{-\mu-2\beta}(x^2-1)^{\Upsilon-2\alpha}, & x > 1 \\ 0 & , 0 < x < 1\end{cases}$$

and $\Upsilon-2\alpha+1 > 0$, [10:p.350]. On simplifying the last expression, (3.6) then becomes

$$\frac{1}{\Gamma(\Upsilon - 2\alpha + 1)}\, x^{1-2\alpha-\nu}\frac{d}{dx}\left[x^{\nu-\mu+2\alpha-2\beta}\int_0^x[f_1(t)-\ell(t)]t^{2\alpha+\nu-1}(x^2-t^2)^{\Upsilon-2\alpha}dt\right]$$
$$\equiv m_1(x),\ 0 < x < 1. \tag{3.8}$$

Hence combining (3.5) and (3.8), the equation (3.4), finally becomes

$$\int_0^\infty k(xt)\phi(t)dt = m_1(x),\ 0 < x < 1, \tag{3.9}$$

$\alpha-\beta < \frac{1}{2}\mu-\frac{1}{2}\nu+1.$

The above analysis is justified if we consider the strip $2\alpha-\nu < k < 2\alpha$, with the condition that $\alpha-\beta < \frac{1}{2}\mu-\frac{1}{2}\nu$. The formula is actually valid for $\alpha-\beta < \frac{1}{2}\mu-\frac{1}{2}\nu+1$, and it can be extended to the full range by analytic continuation.

Now simplify the expression (3.8), by making use of the definition of the functions $f_1(t)$ and $\ell(t)$ from (3.7), we get

$$m_1(x) = \frac{1}{\Gamma(\Upsilon-2\alpha+1)}\, x^{1-2\alpha-\nu}\frac{d}{dx}\left[x^{\nu-\mu+2\alpha-2\beta}\int_0^x t^{\nu-1}(x^2-t^2)^{\Upsilon-2\alpha}f(t)dt\right]$$
$$-\frac{1}{\Gamma(\Upsilon-2\alpha+1)}x^{1-2\alpha-\nu}\frac{d}{dx}\left[x^{\nu-\mu+2\alpha-2\beta}\int_0^x t^{\nu-1}(x^2-t^2)^{\Upsilon-2\alpha}dt\int_0^\infty u^{-2\alpha}J_\nu(ut)\phi(u)\omega(u)du\right]$$
$$= I_1 - I_2,\ \text{say}.$$

On changing the order of integration in I_2, we can write the double integral as,

$$\int_0^\infty u^{2\alpha}\phi(u)\omega(u)du\int_0^x t^{\nu-1}(x^2-t^2)^{\Upsilon-2\alpha}J_\nu(ut)dt$$

$$= \frac{\beta(\gamma-2\alpha+1,\nu)}{\Gamma(\nu+1)2^{\nu+1}} x^{2(\nu+\gamma-2\alpha)}\int_0^\infty u^{\nu-2\alpha}\phi(u)\omega(u)\,{}_1F_2(\nu;\nu+1,\nu+\gamma-2\alpha+1;\ -\frac{x^2u^2}{4})du,$$

[10;p.327]. Then,

$$I_2 = \frac{\beta(\gamma-2\alpha+1,\nu)}{\Gamma(\gamma-2\alpha+1)\Gamma(\nu+1)2^{\nu+1}} x^{1-2\alpha-\nu}\frac{d}{dx}\Big[x^{2\nu}\int_0^\infty u^{\nu-2\alpha}\phi(u)\omega(u)$$
$${}_1F_2\Big[\nu;\nu+1,\nu+\gamma-2\alpha+1;\ \frac{-x^2u^2}{u}\Big]du\Big],$$

which on differenting inside the integral sign and simplifying gives,

$$I_2 = 2^{2\alpha-\nu}x^{\gamma}\int_0^\infty u^{-\gamma}\phi(u)\omega(u)J_\lambda(ux)du.$$

Hence,

$$m_1(x) = \frac{1}{\Gamma(\gamma-2\alpha+1)} x^{1-2\alpha-\nu}\frac{d}{dx}\Big[x^{\nu-\mu+2\alpha-2\beta}\int_0^x t^{\nu-1}(x^2-t^2)^{\gamma-2\alpha}f(t)dt\Big]$$
$$2^{2\alpha-\gamma}x^{-\gamma}\int_0^\infty u^{-\gamma}\phi(u)\omega(u)J_\lambda(ux)du. \tag{3.10}$$

Next we consider the equation (3.3b), whence

$$\frac{1}{2\pi i}\int_{k-i\infty}^{k+i\infty} K(s)\Phi(1-s)x^{-s}ds = \frac{1}{2\pi i}\int_{k-i\infty}^{k+i\infty} H_2^*(s)G_1(s)x^{-s}ds \tag{3.11}$$

As before

$$\frac{1}{2\pi i}\int_{k-i\infty}^{k+i\infty} K(s)\Phi(1-s)x^{-s}ds = \int_0^\infty k(xt)\phi(t)dt, \tag{3.12}$$

where $2\alpha-\nu < k < \frac{1}{2}\mu - \frac{1}{2}\nu + \alpha + \beta$.

And

$$\frac{1}{2\pi i}\int_{k-i\infty}^{k+i\infty} H_2^*(s)G_1(s)x^{-s}ds = \frac{1}{2\pi i}\int_{k-i\infty}^{k+i\infty}\frac{\Gamma(\frac{1}{2}s+\frac{1}{2}\nu-\alpha)}{\Gamma(\frac{1}{2}s+\frac{1}{2}\mu-\beta)}G_1(s)x^{-s}ds$$

$$= -\frac{1}{2}x^{\nu-2\alpha-1}\frac{d}{dx}\Big[x^{2-\nu+2\alpha}\frac{1}{2\pi i}\int_{k-i\infty}^{k+i\infty}\frac{\Gamma(\frac{1}{2}s+\frac{1}{2}\nu-\alpha-1)}{\Gamma(\frac{1}{2}s+\frac{1}{2}\mu-\beta)}G_1(s)x^{-s}ds\Big] \tag{3.13}$$

Here $\dfrac{\Gamma(\frac{1}{2}s+\frac{1}{2}\nu-\alpha-1)}{\Gamma(\frac{1}{2}s+\frac{1}{2}\mu-\beta)} \in L\,(k-i\infty,k+i\infty)$ if $k > 2+2\alpha-\nu$ and $\alpha-\beta > \frac{1}{2}\nu - \frac{1}{2}\mu$.

Further if $x^{-k}g_1(x) \in L(0,\beta)$, then by Lemma 2, the above expression gives

$$-\frac{1}{2}x^{\nu-2\alpha-1}\frac{d}{dx}\Big[x^{2-\nu+2\alpha}\int_x^\infty g_1(t)\frac{1}{t}h_2^*(\frac{x}{t})dt\Big],$$

where

$$\mathcal{M}^{-1}\Big[\frac{\Gamma(\frac{1}{2}s+\frac{1}{2}\nu-\alpha-1)}{\Gamma(\frac{1}{2}s+\frac{1}{2}\mu-\beta)};x\Big] = h_2^*(x) = \begin{cases}\frac{2}{\Gamma(\gamma-2\beta+1)}x^{\nu-2\alpha-2}(1-x^2)^{\gamma-2\beta}, & 0 < x\ 1\\ 0 & , x > 1,\end{cases}$$

$\gamma-2\beta+1 > 0$ i.e. $\frac{1}{2}\nu-\frac{1}{2}\mu-1 < \alpha-\beta$, [10, p.349].

Hence on simplifying, we have from (3.13),

$$\frac{1}{2\pi i}\int_{k-i\infty}^{k+i\infty} H_2^*(s)G_1(s)x^{-s}ds = -\frac{x^{\nu-2\alpha-1}}{\Gamma(\gamma-2\beta+1)}\frac{d}{dx}\Big[\int_x^\infty t^{-\mu}(t^2-x^2)^{\gamma-2\beta}g(t)dt\Big]$$
$$\equiv m_2(x),\ x > 1. \tag{3.14}$$

From the results (3.12) and (3.14), the equation (3.11), then gives

$$\int_0^\infty k(xt)\phi(t)dt = m_2(x), \qquad x > 1, \tag{3.15}$$

$$\tfrac{1}{2}\nu - \tfrac{1}{2}\mu - 1 < \alpha - \beta$$

The above analysis is justified if we consider the strip

$$2\alpha-\nu+2 < k < 2\beta \text{ and } \tfrac{1}{2}\nu - \tfrac{1}{2}\mu < \alpha-\beta.$$

The formula is actually valid for $\frac{1}{2}\nu - \frac{1}{2}\mu - 1 < \alpha - \beta$, and it can extended to the full range by analytic continuation.

Now combining the results (3.9) and (3.15), we have,

$$\int_0^\infty k(xt)\phi(t)dt = m(x), \qquad 0 < x < \infty,$$

where $m(x) = \begin{cases} m_1(x), & 0 < x < 1 \\ m_2(x), & x > 1 \end{cases}$, $m_1(x)$ and $m_2(x)$ defined by (3.10) and (3.14) respectively,

and
$$|\alpha - \beta| < \tfrac{1}{2}\mu - \tfrac{1}{2}\nu + 1.$$

Or,
$$\int_0^\infty (2xt)^{-\Upsilon} J_\lambda(xt)\phi(t) = m(x),$$

which by the usual Hankel inversion gives,

$$\phi(x) = 2^\Upsilon x^{\Upsilon+1} \int_0^\infty t^{\Upsilon+1} J_\lambda(xt)m(t)dt$$

$$= 2^\Upsilon x^{\Upsilon+1} \int_0^1 t^{\Upsilon+1} J_\lambda(xt)m_1(t)dt + 2^\Upsilon x^{\Upsilon+1} \int_1^\infty t^{\Upsilon+1} J_\lambda(xt)m_2(t)dt$$

Now substituting the values of $m_1(t)$ and $m_2(t)$ above, and simplifying, we obtain,

$$\phi(x) = \frac{2^\Upsilon}{\Gamma(\Upsilon-2\alpha+1)} x^{\Upsilon+1}\int_0^1 t^{2-2\nu+\lambda} J_\lambda(xt)\, d\left[t^{\nu-\mu+2\alpha-2\beta}\int_0^t u^{\nu-1}(t^2-u^2)^{\Upsilon-2\alpha} f(u)du\right]$$

$$- \frac{2^\Upsilon}{\Gamma(\Upsilon-2\beta+1)} x^{\Upsilon+1} \int_1^\infty t^\lambda J_\lambda(xt)\, d\left[\int_t^\infty u^{1-\mu}(u^2-t^2)^{\Upsilon-2\beta} g(u)du\right]$$

$$2^{2\Upsilon-2\alpha} x^{\Upsilon+1} \int_0^1 t\, J_\lambda(xt)dt \int_0^\infty u^{-\Upsilon}\phi(u)\omega(u)J_\lambda(ut)du$$

$$= I_1 - I_2 - I_3, \text{ say}$$

$$= I_1 - I_2 - 2^{2\Upsilon-2\alpha} x^{\Upsilon+1} \int_0^\infty u^{-\Upsilon}\phi(u)\omega(u)du \int_0^1 t\, J_\lambda(xt)J_\lambda(ut)dt$$

$$= I_1 - I_2 - 2^{2\Upsilon-2\alpha} x^{\Upsilon+1} \int_0^\infty u^{-\Upsilon}\phi(u)\omega(u) \frac{L(u,x)}{u^2-x^2}\, du, \tag{3.16}$$

where $L(u,x) = uJ_{\lambda+1}(u)J_\lambda(x) - x\, J_{\lambda+1}(x)J_\lambda(u)$, $\lambda = \frac{1}{2}\mu + \frac{1}{2}\nu - \alpha + \beta$, $\Upsilon = \frac{1}{2}\mu - \frac{1}{2}\nu + \alpha + \beta$, and

$$|\alpha - \beta| < \tfrac{1}{2}\mu - \tfrac{1}{2}\nu + 1.$$

This is the integral equation of the Fredholm type and can be written as

$$\phi(x) = A(x) + \int_0^\infty K(x,u)\phi(u)du.$$

Thus the solution of this single integral equation gives us the value of the unknown function $\phi(x)$, which is the solution of the system (3.1), as well.

4. SPECIAL CASES.

In particular if $\mu = \nu$, then the solution of the system

$$\int_0^\infty 2^{2\alpha}\, t^{-2\alpha}\, J_\nu(xt)[1+\omega(t)]\phi(t)dt = f(x), \quad 0 < x < 1 \tag{4.1}$$

$$\int_0^\infty 2^{2\beta}\, t^{-2\beta}\, J_\nu(xt)\phi(t)dt = g(x), \quad , \quad x > 1,$$

is the solution of the equation,

$$\phi(x) = \frac{2^{\alpha+\beta}x^{\alpha+\beta+1}}{\Gamma(\beta-\alpha+1)} \int_0^1 t^{2-\nu-\alpha+\beta} J_{\nu-\alpha+\beta}(xt)\,d\left[t^{2\alpha-2\beta}\int_0^t u^{\nu-1}(t^2-u^2)^{\beta-\alpha}f(u)du\right]$$

$$-\frac{2^{\alpha+\beta}}{\Gamma(\alpha-\beta+1)}\, x^{\alpha+\beta+1} \int_1^\infty t^{\nu-\alpha+\beta} J_{\nu-\alpha+\beta}(xt)\,d\left[\int_t^\infty u^{1-\nu}(u^2-t^2)^{\alpha-\beta}g(u)du\right]$$

$$-2^{2\beta}x^{\alpha+\beta+1}\int_0^\infty u^{-\alpha-\beta}\phi(u)\omega(u)L(u,x)\,\frac{du}{u^2-x^2}, \tag{4.2}$$

where $\qquad L(u,x) = u\, J_{\nu-\alpha+\beta+1}(u)J_{\nu-\alpha+\beta}(x) - x\, J_{\nu-\alpha+\beta+1}(x)\, J_{\nu-\alpha+\beta}(u),$

and $\qquad |\alpha-\beta| < 1$, derived as a special case from (3.16).

If we consider $0 < \alpha-\beta < 1$, then the differentiation under the integral sign in the second term of (4.2) can be carried out, and we have

$$\phi(x) = I_1 - I_2 - I_3, \tag{4.3}$$

where

$$I_2 = \frac{2^{\alpha+\beta+1}}{\Gamma(\alpha-\beta)}\, x^{\alpha+\beta+1}\int_1^\infty t^{\nu-\alpha+\beta+1}J_{\nu-\alpha+\beta}(xt)dt\int_t^\infty u^{1-\nu}(u^2-t^2)^{\alpha-\beta-1}g(u)du.$$

The special case $\mu = \nu = 0$, $\beta = 0$, $\alpha = \frac{1}{2}$ is of interest, since it arises in the discussion of certain contact problems in elasiticity. The dual equations (4.1) now become

$$\int_0^\infty t^{-1}\, J_0(xt)[1+\omega(t)]\phi(t)dt = f(x), \quad 0 < x < 1$$

$$\int_0^\infty J_0(xt)\phi(t)dt = g(x), \qquad x > 1,$$

where the unknown function ϕ satisfies the Fredholm equation, which can be derived from (4.3) to give,

$$\phi(x) = \frac{2}{\pi}\, x\int_0^1 \cos(xt)\,d\left[\int_0^t \frac{f(u)}{\sqrt{t^2-u^2}}\,du\right]$$

$$+\frac{2}{\pi}\, x\int_1^\infty \cos(xt)dt\int_t^\infty \frac{ug(u)}{\sqrt{u^2-t^2}}\,du + \int_0^\infty K(x,u)\phi(u)\omega(u)du$$

with

$$K(x,u) = \frac{x}{\pi u}\left[\frac{\sin(x+u)}{x+u} + \frac{\sin(x-u)}{x-u}\right]. \quad [6;\ 4.6.28].$$

On the other hand, if we consider $-1 < \alpha-\beta < 0$, then the differentiation under the integral sign in the first term of (4.2) can carried out, and then we have

$$\phi(x) = I_1 - I_2 - I_3, \tag{4.4}$$

where

$$I_1 = \frac{2^{\alpha+\beta+1}}{\Gamma(\beta-\alpha)} x^{\alpha+\beta+1} \int_0^1 t^{1-\nu+\alpha-\beta} J_{\nu-\alpha+\beta}(xt)dt \int_0^t u^{\nu+1}(t^2-u^2)^{\beta-\alpha-1} f(u)du.$$

One can, now deduce the special case when $\nu = 0$, $\beta = 0$ and $\alpha = -\frac{1}{2}$ easily. In this case the solution of the dual equations

$$\int_0^\infty t\, J_o(xt)[1+\omega(t)]\phi(t)dt = f(x), \quad 0 < x < 1$$

$$\int_0^\infty J_o(xt)\phi(t)dt = g(x), \quad x > 1$$

is the solution of the equation, from (4.4),

$$\phi(x) = \frac{2}{\pi}\int_0^1 \sin(xt)dt \int_0^t \frac{uf(u)}{\sqrt{t^2-u^2}} du - \frac{2}{\pi}\int_1^\infty \sin(xt)d\left[\int_t^\infty \frac{ug(u)}{\sqrt{u^2-t^2}} du\right] + \int_0^\infty K(x,u)\phi(u)du,$$

with

$$K(x,u) = \frac{1}{\pi}\left[\frac{\sin(u+x)}{u+x} - \frac{\sin(u-x)}{u-x}\right], \quad [6;\ 4,6,40].$$

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