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A NOTE ON THE INVERSE FUNCTION THEOREM OF NASH AND MOSER

MIKAEL LINDSTRÖM

Department of Mathematics
Åbo Akademi
SF-20500 Åbo, Finland

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ABSTRACT. The Nash-Moser inverse function theorem is proved for different kind of differentiabilities.

KEY WORDS AND PHRASES. Continuous convergence, coreflective subcategory, graded space, smoothing operators, tame map, C_α^∞*-differentiability.*
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1. INTRODUCTION

The purpose of this note is to formulate the inverse function theorem of Nash and Moser for different differentiabilities using a categorical approach. The proof is based on the inverse function theorem of Nash and Moser in the version of Hamilton [1] formulated in the category of graded Fréchet spaces which admit smoothing operators and C_c^∞-differentiable [2] tame maps. Our proof is using the same technique as Schmid [3] uses when he proves this theorem for a stronger notion of differentiability, called the Γ-differentiability, than the C_c^∞-differentiability. From our formulation it is possible to derive the inverse function theorem of Nash and Moser for natural differentiabilities stronger than the C_c^∞-differentiability.

2. THE INVERSE FUNCTION THEOREM OF NASH AND MOSER.

Let LC denote the category of locally convex limit vector spaces [2] and continuous linear mappings. Further let K_α denote a coreflective subcategory of LC which is closed under finite products and the coreflector $?^\alpha : LC \to K_\alpha$ is the identity on morphisms and such that the identity mapping $(C_c(X,F))^\alpha = C_\alpha(X,F) \to C_c(X,F)$ is continuous. Here $C_c(X,F)$ denotes the vector space of continuous mappings $X \to F$, endowed with continuous convergence [2], and X is a limit space and $F \in obj(LC)$.

For any pair $E,F \in obj(LC)$ we let $L_c^k(E,F)$ be the space of all continuous k-linear mappings from E^k into F , endowed with continuous convergence. We write $(L_c^k(E,F))^\alpha = L_\alpha^k(E,F)$.

DEFINITION. Let E and F be locally convex spaces and let U be open in E .

A mapping $f : U \to F$ is said to be *differentiable of class* C^p_α, if there exist functions

$$D^k f : U \to L^k(E,F) , \qquad k = 0,1,\dots,p ,$$

such that $D^0 f = f$ and for each $x \in U$, each $h \in E$ and each $k = 0,1,\dots,p-1$, we have

$$\lim_{t\to 0} t^{-1}(D^k f(x+th) - D^k f(x)) = D^{k+1} f(x)h ,$$

and such that for each $k \in \mathbb{N}$, $k \leq p$, the following two conditions are satisfied:

(1) $D^k f(U) \subseteq L^k(E,F)$ (2) $D^k_\alpha f : U \to L^k(E,F)$ is continuous.

f is called *differentiable of class* C^∞_α if it is differentiable of class C^p_α for every $p \in \mathbb{N}$.

By Keller [2] the chain rule is valid for C^∞_α, since α is a finer limit structure than continuous convergence. From the universal property of continuous convergence follows that for any continuous map $g : U \to L^k_\alpha(E,F)$ the associated map $\tilde{g} : U \times E^k \to F$ defined by $\tilde{g}(x,h_1,\dots,h_k) = g(x)(h_1,\dots,h_k)$, $x \in U$, $h_i \in E$, is continuous. As the limit structure α is always finer than c, we have that differentiability of class C^∞_α implies differentiability of class C^∞_c. The latter is exactly the concept of differentiability used by Hamilton [1] to prove the inverse function theorem of Nash and Moser.

We first recall some definitions that will be needed.

Let E be a Fréchet space. A grading on E is an increasing sequence of norms $(\|\cdot\|^1_r)_{r\in\mathbb{N}}$ on E which defines the topology on E. Two gradings $(\|\cdot\|^1_r)_{r\in\mathbb{N}}$ and $(\|\cdot\|^2_r)_{r\in\mathbb{N}}$ are equivalent if for some $s \in \mathbb{N}$ $\|x\|^1_r \leq c\|x\|^2_{r+s}$ and $\|x\|^2_r \leq c\|x\|^2_{r+s}$, $x \in E$, with a constant c which may depend on r. A graded space is a Fréchet space together with an equivalence class of gradings. We say that a graded space E admits smoothing operators if we can find linear maps $S_t : E \to E$, $1 \leq t < \infty$, such that for some r $\|S_t(x)\|_{i+k} \leq ct^{r+k}\|x\|_i$ and $\|S_t(x) - x\|_i \leq ct^{r-k}\|x\|_{i+k}$ for all $i,k \in \mathbb{N}$, $1 \leq t < \infty$, $x \in E$ and some constant c which may depend on i and k. Let E and F be graded spaces and U open in E. We say that a map $f : U \to F$ is tame if for every $x_0 \in U$ we can find a neighbourhood U_0 and a number $r \in \mathbb{N}$ such that for every $n \in \mathbb{N}$ we have the growth estimate $\|f(x)\|_n \leq c(\|x\|_{n+r} + 1)$ for all $x_0 \in U$, where the constant c may depend on n.

In the proof of the inverse function theorem of Nash and Moser we shall also need the following result (Lemma 2, [3]): The composition of two continuous tame maps is continuous and tame.

THEOREM. Let E and F be graded spaces which admit smoothing operators. Let U be open in E and assume that

(1) $f : U \to F$ is differentiable of class C^∞_α and tame.
(2) $\widetilde{D^k f} : U \times E^k \to F$ is tame for every $k \in \mathbb{N}$.
(3) For each $x \in U$ the derivative $Df(x) : E \to F$ is an isomorphism.
(4) The map $Vf : U \to L_\alpha(F,E)$, $Vf(x) = (Df(x))^{-1}$, is continuous.
(5) $\widetilde{Vf} : U \times F \to E$ is tame.

Then for any $x_0 \in U$ we can find open neighbourhoods of x_0 and V_0 of $f(x_0)$ such that f is a bijective map from U_0 onto V_0 and the inverse map $f^{-1} : V_0 \to U_0$ is differentiable of class C_α^∞ and the maps $\widetilde{D^k f^{-1}} : V_0 \times F^k \to E$ are tame for all $k \in \mathbb{N}$. Furthermore we have the formula $D(f^{-1})(y) = Vf(f^{-1}(y))$ for all $y \in V_0$.

PROOF. The maps $\widetilde{D^k f} : U \times E^k \to F$ are continuous and tame, since f is differentiable of class C_α^∞ and assumption (2). Further the assumptions (4) and (5) imply that also $\widetilde{Vf} : U \times F \to E$ is continuous and tame. Now we have that f is differentiable of class C_α^∞ and all $D^k f$ are tame, $Df(x) : E \to F$ is an isomorphism for every $x \in U$ and the family of inverses $\widetilde{Vf} : U \times F \to E$ are continuous and tame maps. Consequently the conditions of the inverse function theorem of Nash-Moser are fulfilled (theorem 1.1.1 p. 171 in [1]). Then for every $x_0 \in U$ there exist neighbourhoods U_0 of x_0 and V_0 of $f(x_0)$ such that $f : U_0 \to V_0$ is bijective and $f^{-1} : V_0 \to U_0$ is continuous and tame. Furthermore the formula $\lim_{t\to 0} t^{-1}(f^{-1}(y + tw) - f^{-1}(y)) = Vf(f^{-1}(y))w$ holds, for all $y \in V_0$ and $w \in F$, by the proof of theorem 1.1.1 p. 186 in [1]. By induction on k we will prove the remaining part that $f^{-1} : V_0 \to U_0$ is differentiable of class C_α^∞ and $\widetilde{D^k f^{-1}} : V_0 \times F^k \to E$ is tame for every $k \in \mathbb{N}$. From the formula $Df^{-1} = Vf \circ f^{-1}$ and assumption (4) follow that $Df^{-1} : V_0 \to L(F,E)$ is continuous. Further we have that $\widetilde{Df^{-1}} : V_0 \times F \to E$ is tame since $\widetilde{Vf}$ and f^{-1} are tame. Assume now it to be true for k. From the definition of the α-differentiability follows that the map f^{-1} is C_α^{k+1} if Df^{-1} is differentiable of class C_α^k. Since $Df^{-1} = Vf \circ f^{-1}$, $D^{k+1}f^{-1}$ is clearly tame so we only have to show that Vf is differentiable of class C^k. By induction on p. By theorem 5.3.1, p. 102 in [1] we have that Vf is weakly differentiable and that $\widetilde{D(Vf)} : U_0 \times E \times F \to E$ is continuous and the formula $[D(Vf)](x)\{u,w\} = -Vf(x)[D^2f(x)\{u,Vf(x)w\}]$ holds for all $x \in U_0$, $u \in E$ and $w \in F$. Thus the derivative $D(Vf) : U_0 \to L_\alpha(E \times F,E)$ can be factorized according to

$$U_0 \xrightarrow{(D^2f,Vf)} L_\alpha^2(E,F) \times L_\alpha(F,E) \xrightarrow{h} L_\alpha(E \times F,E) ,$$

where h is defined by $h(\phi,\psi) = -\psi \circ \phi \circ (id_E,\psi)$ for $\phi = D^2f(x)$ and $\psi = Vf(x)$. By theorem 0.3.5 in [2] h is continuous for $\alpha = c$. Since the category K_α is closed under finite products and $?^\alpha$ is a coreflector it follows that h is continuous. Thus it is true for $p = 1$. Since h is bilinear it is differentiable of class C_α^∞, and consequently the map Vf is differentiable of class C_α^∞ by induction. Thus the theorem is proved.

We shall now consider examples of coreflective subcategories of LC which are closed under finite products and the coreflectors $?^\alpha$ fulfill the assumption that the identity mapping $C_\alpha(U,F) \to C_c(U,F)$ is continuous.

EXAMPLE 1. Let K_α be the category $K_c = LC$; $?^\alpha$ is the identity functor $1_{LC} = ?^c$.

EXAMPLE 2. Let K_α be the category K_e of equable locally convex limit vector spaces [2]. The coreflector $?^e : LC \to K_e$ is the identity on morphisms and on objects E it is characterized as follows: a filter F on E converges to zero in E^e iff $VG = G \leq F$ for some filter G which converges to zero in E.

EXAMPLE 3. Let $\mathcal{K}$ be the category $\mathcal{K}_M$ of Marinescu spaces [2]. The corflector $?^M : LC \to \mathcal{K}_M$ is the identity on morphisms and on objects E it is characterized as follows: a filter $\mathcal{F}$ on E converges to zero in E^M iff $\mathcal{V}\mathcal{G} = \mathcal{G} \leq \mathcal{F}$ and $\bigcap\{\mathbb{K}G : G \in \mathcal{G}\} \in \mathcal{G}$ for some filter $\mathcal{G}$ which converges to zero in E .

EXAMPLE 4. Let $\mathcal{K}_\alpha$ be the category $\mathcal{K}_b$ of bornological locally convex limit vector spaces. The coreflector $?^b : LC \to \mathcal{K}_b$ is the identity on morphisms and on objects E it is characterized as follows: a filter $\mathcal{F}$ on E converges to zero in E^b iff $\mathcal{V}B \leq \mathcal{F}$ for some bounded subset $B \subseteq E$, i.e. some set B such that VB converges to zero in E .

Example 1 gives us the inverse function theorem of Nash and Moser by Hamilton [1]. From example 3 we derive the inverse function theorem of Nash and Moser for the differentiability of class C^∞_M (C^∞_Δ in Keller [2]). In [4] Kriegl has discussed smooth mappings between locally convex spaces, where a mapping is called smooth iff its composition with smooth curves are smooth. He has compared this concept of smoothness with different C^∞_α-differentiabilities (see [2]). From [2] and [4] follow that a mapping between Fréchet spaces is smooth iff it is C^∞_c-differentiable. Thus the inverse function theorem of Nash and Moser is valid for this concept of smoothness.

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