

## ON SOME NEW N-INDEPENDENT-VARIABLE DISCRETE INEQUALITIES OF THE GRONWALL TYPE

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**ABSTRACT.** In this paper we shall establish some new discrete inequalities of the Gronwall type in  $N$ -independent variables. They will have many applications for finite difference equations involving several independent variables and for numerical analysis. Their consequence for the case of  $N = 3$ , generalizes all of the known theorems obtained by Pachpatte and Singare in [1]. An example, to which those results established in [1] are inapplicable, is given here to convey the usefulness of the results obtained.

**KEY WORDS AND PHRASES.** Inequalities of the Gronwall type, difference equations.  
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**1. PRELIMINARIES.** It is well-known that the discrete inequalities of the Gronwall type played a vital role in the theory of finite difference equations and numerical analysis (see [2-8] and [9-12] and the references therein). Recently, Pachpatte and Singare [1] have established some new discrete inequalities in three independent variables. Discrete inequalities involving more than one independent variables are very useful in the study of many important problems concerning discrete versions of some partial differential and integral equations in several variables.

The aim of this paper is to obtain several  $N$ -independent-variable discrete inequalities which extend all results obtained in [1] for the case of  $N = 3$ . In what follows we shall make use of the following notations and definitions.

Let  $N_0$  be the infinite set consisting of the integers  $0, 1, 2, \dots$ , and we shall use the convention of writing

$$\sum_{j \in Z} b_j \equiv 0, \quad \prod_{j \in Z} c_j \equiv 1,$$

if  $Z$  is the empty set. For simplicity, in the sequel we will denote  $(x_1, x_2, \dots, x_n) \in N_0^n$  by  $x$ , and  $(x_1, x_2, \dots, x_j)$ ,  $(x_j, x_{j+1}, \dots, x_n)$ , and  $(x_i, x_{i+1}, \dots, x_k)$  by  $x_j$ ,  $\tilde{x}_j$ , and  $\bar{x}_{i,k}$  respectively, here  $i, k$  are integers from  $1, 2, \dots, n$  with  $i < k$ . Further, we denote the multiple-summation symbol

$$\sum_{y_j=0}^{x_j-1} \sum_{y_{j+1}=0}^{x_{j+1}-1} \cdots \sum_{y_k=0}^{x_k-1} \quad \text{by} \quad \sum_{y,j}^{x,k}$$

where  $x_i, y_i \in N_0, j \leq i \leq k$ , and  $1 \leq j \leq k \leq n$ . Moreover, we define

$$\begin{aligned} \Delta u_{x_j}(\mathbf{x}) &= u(\tilde{x}_{j-1}, x_{j+1}, \tilde{x}_{j+1}) - u(\mathbf{x}), \\ \Delta u_{x_j x_k}^2(\mathbf{x}) &= \Delta u_{x_j}(\tilde{x}_{k-1}, x_{k+1}, \tilde{x}_{k+1}) - \Delta u_{x_j}(\mathbf{x}), \\ &\dots\dots\dots \dots\dots\dots \dots\dots\dots \\ &\dots\dots\dots \dots\dots\dots \dots\dots\dots \end{aligned}$$

and so on, where  $x_j, x_k, \dots$ , are numbers from  $N_0$ , and  $j, k, \dots$ , are integers from  $1, 2, \dots, n$ . We write also here that

$$\Delta^{(r)} L(\mathbf{x}) \triangleq \Delta L_{x_1 x_2 \dots x_r}^r(\mathbf{x})$$

for any real-valued function  $L(\mathbf{x})$  on  $N_0^n$ , here  $1 \leq r \leq n, \mathbf{x} \in N_0^n$ . In addition, we shall define a class of functions on  $N_0^n$  by

$$K = \left\{ f: f(\mathbf{x}) \geq 0, \Delta^{(r)} f(\mathbf{x}) \geq 0, r=1, \dots, n-1, \text{ and } \Delta^{(n)} f(\mathbf{x}) \leq 0 \right\}.$$

It is obvious that the following properties are true:

- (1) if  $f(\mathbf{x}) \in K$  and  $c \geq 0$  is a real number, then  $cf(\mathbf{x}) \in K$ .
- (2) if  $f(\mathbf{x}), g(\mathbf{x}) \in K$  then  $f(\mathbf{x}) + g(\mathbf{x}) \in K$ .
- (3) all functions of the form

$$x_{i_1}^{r_1} x_{i_2}^{r_2} \cdots x_{i_{k-1}}^{r_{k-1}} \quad (k = 1, 2, \dots, n)$$

are in the class  $K$ , here  $r_h \geq 0$  and  $i_h (h=1, \dots, k-1)$  are integers with  $1 \leq i_1 < i_2 < \dots < i_{k-1} \leq n$ .

2. LINEAR INEQUALITIES.

THEOREM 1. Let  $u(\mathbf{x})$  and  $p(\mathbf{x})$  be real-valued non-negative functions defined for  $\mathbf{x} \in N_0^n$ , and let  $f(\mathbf{x})$  be a real-valued positive and nondecreasing function in  $K$ . Suppose further that the discrete inequality

$$u(\mathbf{x}) \leq f(\mathbf{x}) + \sum_{y,1}^{x,n} p(y)u(y) \tag{2.1}$$

is satisfied for all  $\mathbf{x} \in N_0^n$ . Then we have the inequality

$$u(\mathbf{x}) \leq f(0, \tilde{x}_2) \prod_{y_1=0}^{x_1-1} \left[ 1 + G(y_1, \tilde{x}_2) + \sum_{y,2}^{x,n} p(y) \right], \quad \mathbf{x} \in N_0^n \tag{2.2}$$

where

$$G(x) = \varepsilon_1(x_1, 0, \tilde{x}_3) + \sum_{y,2}^{x,2} \varepsilon_2(x_1, y_2, 0, \tilde{x}_4) + \dots + \sum_{y,2}^{x,n-2} \varepsilon_{n-2}(x_1, \bar{y}_{2,n-2}, 0, x_n) + \sum_{y,2}^{x,n-1} \varepsilon_{n-1}(x_1, \bar{y}_{2,n-1}, 0), \tag{2.3}$$

and herein

$$\varepsilon_k(x_k, 0, \tilde{x}_{k+2}) = \frac{\Delta^{(k)} f(x_k, 0, \tilde{x}_{k+2})}{f(x_k, 0, \tilde{x}_{k+2})} \tag{2.4}$$

for  $1 \leq k \leq n-1, x \in N_0^n$ .

PROOF. We define a function  $U(x)$  on  $N_0^n$  by the right member of (2.1), so that by definition

$$U(x_{k-1}, 0, \tilde{x}_{k+1}) = f(x_{k-1}, 0, \tilde{x}_{k+1}) > 0, \tag{2.5}$$

$$U(x_{k-1}, x_{k+1}, \tilde{x}_{k+1}) \geq U(x) > 0, x_k \in N_0, 1 \leq k \leq n.$$

since  $f(x)$  is nondecreasing.

Further, we can obtain from the definition of  $U(x)$  that,

$$\Delta^{(n)} U(x) = \Delta^{(n)} f(x) + p(x)u(x) \leq p(x)U(x_{n-1}, x_{n+1}), \tag{2.6}$$

since  $\Delta^{(n)} f(x) \leq 0, p(x) \geq 0, (2.1), \text{ and } (2.5)$ . In view of the fact that

$$\Delta^{(k)} U(x) = \Delta^{(k)} f(x) + \sum_{y,k+1}^{x,n} p(x_k, \tilde{y}_{k+1})u(x_k, \tilde{y}_{k+1}) \geq 0, \tag{2.7}$$

is valid for  $1 \leq k \leq n-1$  and  $x \in N_0^n$ . By applying (2.5) and (2.7), we derive from (2.6)

$$\frac{\Delta^{(n-1)} U(x_{n-1}, x_{n+1})}{U(x_{n-1}, x_{n+1})} - \frac{\Delta^{(n-1)} U(x)}{U(x)} \leq p(x). \tag{2.6'}$$

Keeping  $x_{n-1}$  fixed in (2.6') set  $x_n = y_n$  and sum over  $y_n = 0, 1, 2, \dots, x_{n-1}$  to obtain the inequality

$$\frac{\Delta^{(n-1)} U(x)}{U(x)} \leq \varepsilon_{n-1}(x_{n-1}, 0) + \sum_{y,n}^{x,n} p(x_{n-1}, y_n), \tag{2.8}$$

where  $g_{n-1}$  is given by (2.4). We may rewrite the last inequality as

$$\frac{\Delta^{(n-2)}U(\tilde{x}_{n-2}, \tilde{x}_{n-1}+1, x_n)}{U(\tilde{x}_{n-2}, \tilde{x}_{n-1}+1, x_n)} - \frac{\Delta^{(n-2)}U(x)}{U(x)} \leq \tag{2.9}$$

$$\leq \epsilon_{n-1}(\tilde{x}_{n-1}, 0) + \sum_{y,n}^{x,n} p(\tilde{x}_{n-1}, y_n),$$

since (2.5) and (2.7). Keeping now  $\tilde{x}_{n-2}$  and  $x_n$  fixed in (2.9), set  $x_{n-1} = y_{n-1}$  and sum over  $y_{n-1} = 0, 1, 2, \dots, x_{n-1}-1$  to get the inequality

$$\frac{\Delta^{(n-2)}U(x)}{U(x)} \leq \epsilon_{n-2}(\tilde{x}_{n-2}, 0, x_n) + \sum_{y,n-1}^{x,n-1} \epsilon_{n-1}(\tilde{x}_{n-2}, y_{n-1}, 0) \tag{2.10}$$

$$+ \sum_{y,n-1}^{x,n} p(\tilde{x}_{n-2}, \tilde{y}_{n-1}),$$

here  $g_{n-2}$  is given by (2.4). If  $n-2 > 1$ , then by using a similar argument as used above for (2.8) to (2.10), we can obtain

$$\frac{\Delta^{(n-3)}U(x)}{U(x)} \leq \epsilon_{n-3}(\tilde{x}_{n-3}, 0, \tilde{x}_{n-1}) + \sum_{y,n-2}^{x,n-2} \epsilon_{n-2}(\tilde{x}_{n-3}, y_{n-2}, 0, x_n)$$

$$+ \sum_{y,n-2}^{x,n-1} \epsilon_{n-1}(\tilde{x}_{n-3}, \tilde{y}_{n-2}, \tilde{y}_{n-1}, 0) + \sum_{y,n-2}^{x,n} p(\tilde{x}_{n-3}, \tilde{y}_{n-2}).$$

Continuing in this way then we obtain

$$\frac{\Delta^{(1)}U(x)}{U(x)} \leq G(x) + \sum_{y,2}^{x,n} p(x_1, \tilde{y}_2),$$

where  $G(x)$  is defined by (2.3). Obviously, the last inequality can be rewritten as

$$\frac{U(x_1+1, \tilde{x}_2)}{U(x)} \leq 1 + G(x) + \sum_{y,2}^{x,n} p(x_1, \tilde{y}_2). \tag{2.11}$$

Keeping  $\tilde{x}_2$  fixed in (2.11), set  $x_1 = y_1$  and then substitute  $y_1 = 0, 1, 2, \dots, x_1-1$  successively in (2.11), we then get

$$\frac{U(x)}{U(0, \tilde{x}_2)} \leq \prod_{y_1=0}^{x_1-1} \left[ 1 + G(y_1, \tilde{x}_2) + \sum_{y,2}^{x,n} p(y) \right]. \tag{2.12}$$

Thus the desired bound for  $u(x)$  in (2.2) follows from (2.1), (2.5) and (2.12) immediately.

EXAMPLE 1. Suppose that the discrete inequality

$$v(x_1, x_2, x_3) \leq a + x_2 x_3 + x_1^4 x_2 + \sum_{y,1}^{x,3} Q(y_1, y_2, y_3) v(y_1, y_2, y_3) \quad (*)$$

holds for all  $(x_1, x_2, x_3) \in N_0^3$ , where  $a > 0$  is a constant,  $v$  and  $Q$  are real-valued non-negative functions defined on  $N_0^3$ . Then, by Theorem 1 here we have the nondecreasing function

$$f(x_1, x_2, x_3) = a + x_2 x_3 + x_1^4 x_2 (> 0) \in K,$$

since the following conditions are satisfied

$$\Delta^{(1)} f(x_1, x_2, x_3) = x_2(1 + 4x_1 + 6x_1^2 + 4x_1^3) \geq 0,$$

$$\Delta^{(2)} f(x_1, x_2, x_3) = 1 + 4x_1 + 6x_1^2 + 4x_1^3 > 0, \quad \text{for } (x_1, x_2, x_3) \in N_0^3$$

$$\Delta^{(3)} f(x_1, x_2, x_3) = 0.$$

so that,

$$g_1(x_1, 0, x_3) = \frac{\Delta^{(1)} f(x_1, 0, x_3)}{f(x_1, 0, x_3)} = 0,$$

$$g_2(x_1, x_2, 0) = \frac{\Delta^{(2)} f(x_1, x_2, 0)}{f(x_1, x_2, 0)} = \frac{1 + 4x_1 + 6x_1^2 + 4x_1^3}{a + x_1^4 x_2}.$$

Hence we derive the desired bound on  $v$  from (\*) such that

$$v(x_1, x_2, x_3) \leq (a + x_2 x_3) \prod_{y_1=0}^{x_1-1} \left\{ 1 + \sum_{y,2}^{x,2} \frac{1 + 4y_1 + 6y_1^2 + 4y_1^3}{a + y_1^4 y_2} + \sum_{y,2}^{x,3} Q(y_1, y_2, y_3) \right\}, \quad \text{for } (x_1, x_2, x_3) \in N_0^3.$$

We note here that the above inequality (\*) can not be treated by means of the known results established in [1].

THEOREM 2. Let  $u(x)$ ,  $f(x)$ ,  $p(x)$  be the same as in Theorem 1, and let  $q(x)$  be a real-valued non-negative function defined for  $x \in N_0^n$ . Suppose that the inequality

$$u(x) \leq f(x) + \sum_{y,1}^{x,n} p(y) \left[ u(y) + \sum_{z,1}^{y,n} q(z) u(z) \right] \quad (2.13)$$

is satisfied for all  $x \in N_0^n$ . Then we also have

$$u(x) \leq F(x) + \sum_{y,1}^{x,n} p(y) f(0, \tilde{y}_2) \prod_{z_1=0}^{y_1-1} \left[ 1 + G(z_1, \tilde{y}_2) + \sum_{z,2}^{y,n} (r(z) + q(z)) \right], \quad (2.14)$$

for all  $x \in N_0^n$ , here  $G$  is the same as in above Theorem 1, and  $F(x)$  is given by

$$F(x) = f(0, \tilde{x}_2) + \sum_{y,1}^{x,1} \Delta^{(1)} f(y_1, 0, \tilde{x}_3) + \sum_{y,1}^{x,2} \Delta^{(2)} f(y_2, 0, \tilde{x}_4) + \dots + \sum_{y,1}^{x,n-2} \Delta^{(n-2)} f(y_{n-2}, 0, x_n) + \sum_{y,1}^{x,n-1} \Delta^{(n-1)} f(y_{n-1}, 0). \quad (2.15)$$

PROOF. We define two functions  $V(x)$  and  $W(x)$  on  $N_0^n$  by the right number of (2.13) and the following equality

$$w(x) = V(x) + \sum_{y,1}^{x,n} q(y)V(y) \quad (2.16)$$

respectively, so that by the definitions: ( $f(x)$  is nondecreasing)

$$W(x_{j-1}, 0, \tilde{x}_{j+1}) = V(x_{j-1}, 0, \tilde{x}_{j+1}) = f(x_{j-1}, 0, \tilde{x}_{j+1}) > 0, \quad (2.17)$$

$$V(x_{j-1}, x_j+1, \tilde{x}_{j+1}) \geq V(x) > 0, \quad (2.18)$$

$$W(x_{j-1}, x_j+1, \tilde{x}_{j+1}) \geq W(x) > 0, \quad (2.19)$$

where  $x_j \in N_0^n, j=1,2,\dots,n$ . In addition, we obtain here

$$\begin{aligned} \Delta^{(r)} V(x) = \Delta^{(r)} f(x) + \sum_{y,r+1}^{x,n} p(x_r, \tilde{y}_{r+1}) \left[ u(x_r, \tilde{y}_{r+1}) + \sum_{z,1}^{x,r} \sum_{z,r+1}^{y,n} q(z)u(z) \right], \end{aligned} \quad (2.20)$$

and

$$\Delta^{(r)} w(x) = \Delta^{(r)} V(x) + \sum_{y,r+1}^{x,n} q(x_r, \tilde{y}_{r+1}) V(x_r, \tilde{y}_{r+1}). \quad (2.21)$$

for  $1 \leq r \leq n, x \in N_0^n$

Letting  $r = n$  in the above (2.20) and using  $\Delta^{(n)} f(x) \leq 0$  we then derive that

$$\Delta^{(n)} V(x) \leq p(x)w(x), \quad x \in N_0^n \quad (2.22)$$

since  $p(x), q(x)$  are non-negative and  $u(x) \leq V(x) \leq W(x)$ . Now by (2.21) we obtain

$$\Delta^{(n)} w(x) \leq p(x)w(x) + q(x)V(x) \leq [p(x) + q(x)]W(x), \quad (2.23)$$

for  $x \in N_0^n$

since  $q(x) \geq 0$  and  $V(x) \leq W(x)$ . It is clear from (2.17) that,

$$\frac{\Delta^{(r)}W(\tilde{x}_r, 0, \tilde{x}_{r+2})}{W(\tilde{x}_r, 0, \tilde{x}_{r+2})} = \frac{\Delta^{(r)}V(\tilde{x}_r, 0, \tilde{x}_{r+2})}{V(\tilde{x}_r, 0, \tilde{x}_{r+2})} = g_r(\tilde{x}_r, 0, \tilde{x}_{r+2}), \quad (2.24)$$

where  $1 \leq r \leq n-1$ ,  $x \in N_o^n$ ,  $j=1,2,\dots,n$ , and  $g_r$  is given by (2.4).

Now by following the same argument as used in the proof of Theorem 1, and using (2.17), (2.24), we get from (2.23)

$$W(x) \leq f(0, \tilde{x}_2) \prod_{y_1=0}^{x_1-1} \left[ 1 + G(y_1, \tilde{x}_2) + \sum_{y_1,2}^{x_1,n} (p(y) + q(y)) \right]. \quad (2.25)$$

for  $x \in N_o^n$

Substituting this bound for  $W(x)$  in (2.22), and then rewrite it as follows

$$\Delta^{(n-1)}V(\tilde{x}_{n-1}, x_{n+1}) - \Delta^{(n-1)}V(x) \leq h(x), \quad (2.26)$$

where the function  $h(x)$  is defined by

$$h(x) = p(x)f(0, \tilde{x}_2) \prod_{y_1=0}^{x_1-1} \left[ 1 + G(y_1, \tilde{x}_2) + \sum_{y_1,2}^{x_1,n} (p(y) + q(y)) \right].$$

Keeping  $x_{n-1}$  fixed in the above (2.26), set  $x_n = y_n$  and sum over  $y_n = 0, 1, 2, \dots, x_{n-1}$  to get the estimate

$$\begin{aligned} \Delta^{(n-1)}V(x) &\leq \Delta^{(n-1)}V(\tilde{x}_{n-1}, 0) + \sum_{y,n}^{x,n} h(\tilde{x}_{n-1}, y_n) \\ &= \Delta^{(n-1)}f(\tilde{x}_{n-1}, 0) + \sum_{y,n}^{x,n} h(\tilde{x}_{n-1}, y_n). \end{aligned}$$

Keeping now  $x_{n-2}, x_n$  fixed in the last inequality, set  $x_{n-1} = y_{n-1}$  and sum over  $y_{n-1} = 0, 1, 2, \dots, x_{n-1}$  to get the inequality

$$\begin{aligned} \Delta^{(n-2)}V(x) &\leq \Delta^{(n-2)}f(\tilde{x}_{n-2}, 0, x_n) + \sum_{y,n-1}^{x,n-1} \Delta^{(n-1)}f(\tilde{x}_{n-2}, y_{n-1}, 0) \\ &\quad + \sum_{y,n-1}^{x,n} h(\tilde{x}_{n-2}, \tilde{y}_{n-1}). \end{aligned}$$

Continuing in this way then we, obtain

$$\begin{aligned} \Delta^{(1)}v(x) &\equiv v(x_1+1, \tilde{x}_2) - v(x) \\ &\leq \Delta^{(1)}f(x_1, 0, \tilde{x}_3) + \sum_{y,2}^{x,2} \Delta^{(2)}f(x_1, y_2, 0, \tilde{x}_4) + \dots + \\ &\quad \sum_{y,2}^{x,n-1} \Delta^{(n-1)}f(x_1, \bar{y}_{2,n-1}, 0) + \sum_{y,2}^{x,n} h(x_1, \tilde{y}_2). \end{aligned} \tag{2.27}$$

Keeping  $\tilde{x}_2$  fixed in (2.27), set  $x_1 = y_1$  and substitute  $y_1 = 0, 1, 2, \dots, x_1 - 1$  successively in (2.27) to derive the bound for  $V(x)$  such that

$$V(x) \leq F(x) + \sum_{y,1}^{x,n} h(y), \tag{2.28}$$

since  $V(0, \tilde{x}_2) = f(0, \tilde{x}_2)$ , where  $F(x)$  is given by (2.15). Hence the desired bound in (2.14) follows from (2.13), (2.28), and the definitions of  $V(x)$  and  $h(x)$  immediately. Q.E.D.

REMARK 1. Letting  $n = 3$  and  $f(x) = a_1(x_1) + a_2(x_2) + a_3(x_3)$  in above Theorems 1, 2, where  $a_j : N_0 \rightarrow (0, \infty)$ ,  $\Delta a_j(z) \geq 0$  for all  $z \in N_0$ ,  $j = 1, 2, 3$ , then we derive the Theorems 1, 2 of [1] respectively.

3. A NONLINEAR GENERALIZATION.

THEOREM 3. Let  $u(x)$ ,  $p(x)$ , and  $f(x)$  be the same as in above Theorem 1, and let  $u(x) \geq u_0$  be satisfied for all  $x \in N_0^n$ , where  $u_0$  is a positive number. Let  $W(z)$  be a real-valued continuous, positive and strictly increasing function defined on the interval  $(u_0, \infty)$ . Suppose further that the inequality

$$u(x) \leq f(x) + \sum_{y,1}^{x,n} p(y)W(u(y)), \tag{3.1}$$

holds for  $x \in N_0^n$ . Then for  $0 \leq x \leq X$  (this is  $0 \leq x_i \leq X_i$ ,  $i = 1, 2, \dots, n$ ) we also have the inequality

$$u(x) \leq K^{-1} \left\{ K [f(0, \tilde{x}_2)] + \sum_{y,1}^{x,1} [G^*(y_1, \tilde{x}_2) + \sum_{z,2}^{x,n} p(y_1, \tilde{z}_2)] \right\}, \tag{3.2}$$

where  $K^{-1}$  denotes the inverse of  $K$  and

$$K(r) = \int_{r_0}^r \frac{ds}{W(s)} \quad \text{for } r \geq r_0 (\geq u_0), \tag{3.3}$$

and the function  $G^*(x)$  is obtained from  $G(x)$  by replacing all of its denominators  $f(x_j, 0, \tilde{x}_{j+2})$  by  $W[f(x_j, 0, \tilde{x}_{j+2})]$  respectively, here  $j = 1, 2, \dots, n-1$ . Here  $x \in N_0^n$



is chosen so that the expression contained in the brackets {...} in (3.2) belongs to the domain of  $K^{-1}$  as long as  $0 \leq x \leq X$ .

PROOF. Define a function  $R(x)$  on  $N_0^n$  by the right member of (3.1), so that

$$R(\tilde{x}_{j-1}, 0, \tilde{x}_{j+1}) = f(\tilde{x}_{j-1}, 0, \tilde{x}_{j+1}) \geq u(\tilde{x}_{j-1}, 0, \tilde{x}_{j+1}) \geq u_0, \tag{3.4}$$

for  $j = 1, 2, \dots, n$ , and

$$\Delta^{(r)} R(x) = \Delta^{(r)} f(x) + \sum_{y, r+1}^{x, n} p(\tilde{x}_r, \tilde{y}_{r+1}) W[R(\tilde{x}_r, \tilde{y}_{r+1})] \geq 0, \tag{3.5}$$

$$1 \leq r \leq n-1, x \in N_0^n$$

$$\Delta^{(n)} R(x) = \Delta^{(n)} f(x) + p(x) W[u(x)] \leq p(x) \cdot W[R(x)], \quad x \in N_0^n, \tag{3.6}$$

since  $\Delta^{(n)} f(x) \leq 0$ ,  $p(x) \geq 0$ ,  $u(x) \leq R(x)$ , and  $W(z)$  is increasing. We can observe from (3.6) that,

$$\frac{\Delta^{(n-1)} R(\tilde{x}_{n-1}, x_{n+1})}{W[R(\tilde{x}_{n-1}, x_{n+1})]} - \frac{\Delta^{(n-1)} R(x)}{W[R(x)]} \leq p(x), \tag{3.7}$$

since (3.5),  $R(\tilde{x}_{j-1}, x_{j+1}, \tilde{x}_{j+1}) \geq u_0$ , and  $W(z)$  is increasing. Using (3.4) and keeping  $\tilde{x}_{n-1}$  fixed in (3.7), set  $x_n = y_n$  and sum over  $y_n = 0, 1, 2, \dots, x_n - 1$  to derive the inequality

$$\frac{\Delta^{(n-1)} R(x)}{W[R(x)]} \leq \frac{\Delta^{(n-1)} f(\tilde{x}_{n-1}, 0)}{W[f(\tilde{x}_{n-1}, 0)]} + \sum_{y, n}^{x, n} p(\tilde{x}_{n-1}, y_n). \tag{3.8}$$

Replacing now the left member of (3.8) by the smaller term

$$\frac{\Delta^{(n-2)} R(\tilde{x}_{n-2}, x_{n-1}+1, x_n)}{W[R(\tilde{x}_{n-2}, x_{n-1}+1, x_n)]} - \frac{\Delta^{(n-2)} R(x)}{W[R(x)]},$$

and then keeping  $\tilde{x}_{n-2}$  and  $x_n$  fixed in (3.8), set  $x_{n-1} = y_{n-1}$  and sum over  $y_{n-1} = 0, 1, 2, \dots, x_{n-1} - 1$  to get the estimate

$$\frac{\Delta^{(n-2)} R(x)}{W[R(x)]} \leq \frac{\Delta^{(n-2)} f(\tilde{x}_{n-2}, 0, x_n)}{W[f(\tilde{x}_{n-2}, 0, x_n)]} + \sum_{y, n-1}^{x, n-1} \frac{\Delta^{(n-1)} f(\tilde{x}_{n-2}, y_{n-1}, 0)}{W[f(\tilde{x}_{n-2}, y_{n-1}, 0)]}$$

$$+ \sum_{y, n-1}^{x, n} p(\tilde{x}_{n-2}, \tilde{y}_{n-1}).$$

Proceeding in this way we then obtain

$$\frac{\Delta^{(1)}_{R(x)}}{W[R(x)]} \leq G^*(x) + \sum_{y,2}^{x,n} p(x_1, \tilde{y}_2) \quad x \in N_0^n \tag{3.9}$$

where  $G^*(x)$  is obtained from  $G(x)$  by the method as described in above. Obviously, (3.9) implies the following

$$\begin{aligned} K[R(\hat{x})] - K[R(x)] &= \int_{R(x)}^{R(\hat{x})} \frac{ds}{W(s)} \leq \frac{\Delta^{(1)}_{R(x)}}{W[R(x)]} \\ &\leq G^*(x) + \sum_{z,2}^{x,n} P(x_1, \tilde{y}_2), \quad x \in N_0^n \end{aligned}$$

where  $\hat{x} = (x_1+1, \tilde{x}_2)$ . Keeping now  $\tilde{x}_2$  fixed in the last inequality, set  $x_1 = y_1$  and sum over  $y_1 = 0, 1, 2, \dots, x_1-1$  to obtain the inequality

$$K[R(x)] - K[R(0, \tilde{x}_2)] \leq \sum_{y,1}^{x,1} \left[ G^*(y_1, \tilde{x}_2) + \sum_{z,2}^{x,n} P(y_1, \tilde{z}_2) \right], \quad x \in N_0^n$$

and hence we have for  $0 \leq x \leq X$

$$R(x) \leq K^{-1} \left\{ K[f(0, \tilde{x}_2)] + \sum_{y,1}^{x,1} \left[ G^*(y_1, \tilde{x}_2) + \sum_{z,2}^{x,n} P(y_1, \tilde{z}_2) \right] \right\}. \tag{3.10}$$

Hence the desired upper bound on  $u(x)$  in (3.2) follows from (3.1), (3.10), and the definition of  $R(x)$  immediately. The choice of  $X \in N_0^n$  is obvious.

REMARK 2. By applying the same argument as used in the proofs of Theorems 2 and 3, we can easily establish an extension of the Theorem 4 of [1], which yields an upper bound for the solutions of the following inequality

$$u(x) \leq f(x) + \sum_{y,1}^{x,n} p(y) \left[ u(y) + \sum_{z,1}^{y,n} p(z) W(u(z)) \right].$$

Here  $x \in N_0^n$  and  $u(x)$ ,  $f(x)$ ,  $p(x)$ , and  $W(z)$  are the same as defined in Theorem 3. Because the proof of this result is not difficult, so we leave it here to the reader.

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