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SPACE-TIME CAUSTICS

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ABSTRACT. The Lagrange manifold (WKB) formalism enables the determination of the asymptotic series solution of linear differential equations modelling wave propagation in spatially inhomogeneous media at caustic (turning) points. Here the formalism is adapted to determine a class of asymptotic solutions at caustic points for those equations modelling wave propagation in media with both spatial and temporal inhomogeneities. The analogous Schrodinger equation is also considered.

KEYWORDS AND PHRASES. *Wave Propagation, Lagrange manifold, Schrodinger equation, turning points.*
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1. INTRODUCTION

Scalar wave propagation in a medium with both spatial and temporal inhomogeneities is commonly represented by an equation of the form

$$\nabla^2\psi(\bar{r},t) - f(\bar{r},t)\frac{\partial^2\psi(\bar{r},t)}{\partial t^2} - g(\bar{r},t)\psi(\bar{r},t) = 0 \tag{1.1}$$

In Equation (1.1), $\psi(\bar{r},t)$ is the wave function, $\bar{r}$ refers to the spatial coordinates and t is the time. When associated with propagation in a plasma, $f(\bar{r},t)$ is related to the index of refraction and $g(\bar{r},t)$ to the plasma oscillations [1]. Physically, the spatial inhomogeneity is principally due to refractive effects. The temporal inhomogeneity occurs when the characteristic frequencies of the medium, e.g., the resonant absorption frequency of the molecules, the cyclotron frequency of the plasma, lie within the frequency range of the source. In this case, the component frequencies of the source signal are not uniformly absorbed and reradiated; the reradiated frequency components are propagated with different velocities, leading to a distortion of the waveform [2].

No general technique exists for solving equations such as Equation (1.1) exactly. Consequently, approximate solutions, each valid under specific assumptions, are often constructed. One such approach, valid for high frequency waves transmitted from a time-harmonic source, is the eikonal or geometrical optics solution [3,4], which has long been applied to problems involving propagation in media with both temporal and spatial inhomogeneities [1,2,5,6].

The approach proceeds by scaling coordinates ($\bar{r} \to \bar{r}/\lambda$, $t \to t/\lambda$; $\lambda>>0$) so that Equation (1.1) may be written as

$$\nabla^2\psi(\bar{r},t) - f(\bar{r},t)\frac{\partial^2\psi(\bar{r},t)}{\partial t^2} - \lambda^2 g(\bar{r},t)\psi(\bar{r},t) = 0. \tag{1.2}$$

Physically, this implies the regime of long distances and observation times and slowly varying $g(\bar{r},t)$. Next, a solution of the form

$$\psi(\bar{r},t) = \exp\{i\lambda S(\bar{r},t)\}A(\bar{r},t,\lambda) \tag{1.3}$$

where

$$A(\bar{r},t,\lambda) \simeq \sum_{k=0} A_k(\bar{r},t)(i\lambda)^{-k}, \; A_{-k} = 0 \tag{1.4}$$

is assumed. $S(\bar{r},t)$ may be regarded as a phase and $A(\bar{r},t,\lambda)$ as an amplitude. Substituting Equation (1.3) into Equation (1.2), followed by a re-grouping by powers of $(i\lambda)$ obtains

$$\begin{aligned}&\{(i\lambda)^2[(\nabla S)^2 - f(\bar{r},t)\left(\frac{\partial S}{\partial t}\right)^2 + g(\bar{r},t)] + \\ &(i\lambda)[\nabla^2 S - f(\bar{r},t)\frac{\partial^2 S}{\partial t^2} + 2\nabla S\cdot\nabla - 2f(\bar{r},t)\frac{\partial S}{\partial t}\frac{\partial}{\partial t}] + \\ &(i\lambda)^0[\nabla^2 - f(\bar{r},t)\frac{\partial^2}{\partial t^2}]\} \sum_{k=0} A_k(\bar{r},t)(i\lambda)^{-k} \simeq 0 \quad .\end{aligned} \tag{1.5}$$

Introducing the wavenumber and frequency,

$$\bar{p} = \nabla S \quad , \quad \omega = -\frac{\partial S}{\partial t} \tag{1.6}$$

respectively, into the coefficient of the $(i\lambda)^2$ term leads to a dispersion (the eikonal) equation

$$\bar{p}\cdot\bar{p} - f(\bar{r},t)\omega^2 + g(\bar{r},t) = 0. \tag{1.7}$$

With the wavevectors regarded as momenta, Equation (1.7) may be considered a Hamiltonian

$$H = \bar{p}\cdot\bar{p} - f(\bar{r},t)\omega^2 + g(\bar{r},t). \tag{1.8}$$

Equation (1.7) is a first order non-linear partial differential equation for the phase $S(\bar{r},t)$ and may be solved by introducing Hamilton's equations

$$\frac{d\bar{r}}{d\gamma} = \nabla_p H \qquad \frac{d\bar{p}}{d\gamma} = -\nabla_r H \tag{1.9}$$

$$\frac{dt}{d\gamma} = \frac{-\partial H}{\partial\omega} \qquad \frac{d\omega}{d\gamma} = \frac{\partial H}{\partial t} \quad . \tag{1.10}$$

The solution of Equations (1.9) and (1.10) are the space-time ray trajectories (map)

$$\bar{r} = \bar{r}(\gamma,\bar{\sigma}) \qquad \bar{p} = \bar{p}(\gamma,\bar{\sigma}) \tag{1.11}$$

$$t = t(\gamma,\bar{\sigma}) \qquad \omega = \omega(\gamma,\bar{\sigma}), \tag{1.12}$$

where γ is the ray-path parameter and $\bar{\sigma}$ a parametrized initial condition [2]. (In this parametrization, time and frequency (t and ω) appear only implicitly in the space and wave vector ($\bar{r}$ and $\bar{p}$, respectively) coordinates, i.e., through γ and $\bar{\sigma}$. In some parametrizations [3,4], time is chosen as the ray parameter.) Integrating along the trajectories obtains the phase

$$S(\bar{r},t) = \int_{\bar{r}_o,t_o}^{\bar{r},t} \bar{p}\cdot d\bar{r} - \omega d\gamma + S(\bar{r}_o,t_o) \ . \tag{1.13}$$

Once the phase is known, the amplitudes can be determined from the coefficients of the $i\lambda$ and $(i\lambda)^0$ terms in Equation (1.5). Usually, these terms are re-grouped, using Equations (1.6), into a first-order (transport) equation

$$[\nabla\cdot\bar{p} + f(\bar{r},t)\frac{\partial\omega}{\partial t} + 2\ \bar{p}\cdot\nabla + 2\omega f(\bar{r},t)\frac{\partial}{\partial t}]A_k = -\ (\nabla^2 - f(\bar{r},t)\frac{\partial^2}{\partial t^2})A_{k-1},\ k\geq 1. \tag{1.14}$$

If the amplitude A_o is specified at the source, $\bar{r}_o$, at some initial time, t_o, and with some (from Equation (1.7)) propagation frequency ω_o, then A_o at any space-time field point may be determined from

$$A_o(\bar{r},t) = A_o(\bar{r}_o,t_o)\left[\frac{f(\bar{r}_o,t_o)\ J_{t_o}(\bar{r}_o,\bar{\mu}_o)}{f(\bar{r},t)\ \ J_t(\bar{r},\bar{\mu})}\right]^{1/2} \ , \tag{1.15}$$

where, following Lewis [5], J_t is the Jacobian of the ray transformation $\bar{\mu} \to \bar{r}$ at each time t, i.e.,

$$J_t(\bar{r},\bar{\mu}) = \frac{\partial(\bar{r})}{\partial(\bar{\mu})} \ , \tag{1.16}$$

where $\bar{\mu} = (\gamma,\bar{\sigma})$. With $A_o(\bar{r},t)$ known, the other A_k's may be obtained recursively.

This algorithm suffices to determine the asymptotic solution at most field points. At caustic points, points where the spatial and temporal inhomogeneities of the media effect a focusing of trajectories, the ray transformation from parameter space to coordinate space ($\bar{\mu} \to \bar{r}$) becomes singular, i.e., $J_t(\bar{r},\bar{\mu}) = 0$. As an example, we consider waves propagating from a point source at the origin $\bar{r} = (0,0)$ at $t = 0$ in a medium with $f(\bar{r},t) = 1$ and $g(\bar{r},t) = x + t - k^2$. Then the dispersion equation

$$\bar{p}\cdot\bar{p} - \omega^2 + x + t - k^2 = 0 \tag{1.17}$$

leads to the map

$$\begin{aligned} x &= -\gamma^2 + 2\rho\gamma\cos\theta & \qquad p_x &= -\gamma + \rho\cos\theta \\ y &= 2\rho\gamma\sin\theta & p_y &= \rho\sin\theta \\ t &= \gamma^2 + 2\gamma\Omega & \omega &= \gamma + \Omega \ . \end{aligned}$$

In these equations θ is an initial propagation angle, taken with respect to the positive x-axis, Ω is the initial frequency and $\rho = (\Omega^2 + k^2)^{1/2}$. For definiteness, at $t = 0$, let $\Omega = 3$, $k = 4$, $\rho = 5$. At $\bar{\mu} = (\gamma,\theta) = (5.77, 30°)$, i.e., the space-time point $(x,y,t) = (16.68,\ 28.85,\ 67.91)$, the map $\bar{\mu} \to \bar{r}$ is singular and the technique predicts unbounded amplitudes. (We note that at $\gamma = 5.77$, $\omega = 8.77$.)

Such difficulties can often be circumvented using the Lagrange manifold formalism of Maslov [7] and Arnold [8]. A modification of their technique has been applied to enable straightforward calculation of the field at caustic (turning) points associated with the vector Helmholtz equation [9] and with dispersive waves [10]. Here, drawing on Kratsov's treatment [11] of the Schrodinger equation with a potential that varies both spatially and temporally, we adapt the Lagrange manifold technique to determine a class of asymptotic solutions at caustic points characteristic of media with both spatial and temporal inhomogeneities. Also, we determine the transport equation at caustic points associated with the Schrodinger equation considered by Kratsov, complementing his off-caustic treatment.

2. FORMALISM

To begin the algorithm, we assume that near caustic (turning) points of the highest order

$$\nabla^2\psi(\bar{r},t) - f(\bar{r},t)\frac{\partial^2\psi(\bar{r},t)}{\partial t^2} - \lambda^2 g(\bar{r},t)\psi(\bar{r},t) = 0 \tag{2.1}$$

has an asymptotic solution of the form

$$\psi(\bar{r},t) - \int A(\bar{r},\bar{p},t,\lambda)\exp\{i\lambda(\bar{r}\cdot\bar{p} - \omega t - S(\bar{p}))\}d\bar{p} = O(\lambda^{-\infty}) , \tag{2.2}$$

where the amplitude $A(\bar{r},\bar{p},t,\lambda)$ and its derivatives are assumed bounded and $\bar{r}\cdot\bar{p} - \omega t - S(\bar{p})$ may be regarded as a phase, i.e.,

$$\phi(\bar{r},\bar{p},t,\omega) = \bar{r}\cdot\bar{p} - \omega t - S(\bar{p}) . \tag{2.3}$$

($S(\bar{p})$ will be seen to be the generating function of a canonical transformation; although neither time nor frequency appear explicitly in this transformation, both appear implicitly, analogous to the development above.) The technique proceeds by carrying the differentiation in Equation (2.1) across the integral in Equation (2.2) obtaining

$$\int d\bar{p}\, \exp\{i\lambda(\bar{r}\cdot\bar{p}-\omega t-S(\bar{p}))\}\{(i\lambda)^2(\bar{p}\cdot\bar{p}-f(\bar{r},t)\omega^2 + g(\bar{r},t))A +$$

$$i\lambda(2\bar{p}\cdot\nabla_r A+2\omega f(\bar{r},t)\frac{\partial A}{\partial t}) + (i\lambda)^0(\nabla_r^2 A-f(\bar{r},t)\frac{\partial^2 A}{\partial t^2})\} = O(\lambda^{-\infty}). \tag{2.4}$$

The coefficient of the $(i\lambda)^2$ term is Maslov's Hamiltonian

$$H = \bar{p}\cdot\bar{p} - f(\bar{r},t)\omega^2 + g(\bar{r},t)$$

cf. Equation (1.8). Then by invoking the stationary phase condition $[\nabla_p\phi = 0]$, we determine the Lagrange manifold

$$\bar{r} = \nabla_p S(\bar{p}) \tag{2.5}$$

and Maslov's Hamiltonian becomes an eikonal equation on the Lagrange manifold

$$\bar{p}\cdot\bar{p} - f(\nabla_p S,t)\omega^2 + g(\nabla_p S,t) = 0 \tag{2.6}$$

cf. Equation (1.7). To obtain the phase we once again use Hamilton's equations (Equations (1.9) and (1.10)) to obtain the trajectories

$$\bar{r} = \bar{r}(\gamma,\bar{\sigma}) \qquad \bar{p} = \bar{p}(\gamma,\bar{\sigma})$$

$$t = t(\gamma,\bar{\sigma}) \qquad \omega = \omega(\gamma,\bar{\sigma}).$$

At any space-time point where the map from $\bar{\mu} = (\gamma,\bar{\sigma})$ to coordinate space is singular, i.e., Equation (1.16) is zero, the map $\bar{\mu} \to \bar{p}$ is first inverted to obtain

$$\gamma = \gamma(\bar{p}) \qquad\qquad \bar{\sigma} = \bar{\sigma}(\bar{p}) \ .$$

Then substituting into the coordinate space map determines the Lagrange manifold explicitly

$$\bar{r} = \bar{r}(\gamma(\bar{p}),\bar{\sigma}(\bar{p})) = \nabla_p S(\bar{p}). \tag{2.7}$$

Finally, by integrating along the trajectories we obtain

$$S(\bar{p}) = \int_{\bar{p}_o}^{\bar{p}} \bar{r}\cdot d\bar{p} \tag{2.8}$$

analogous to Equation (1.13), and thus the phase

$$\phi(\bar{r},\bar{p},t,\omega) = \bar{r}\cdot\bar{p} - \omega t - S(\bar{p}). \tag{2.9}$$

In the parametrization specified by the Lagrange manifold, caustic points are those space-time points at which

$$\det\left\{\frac{\partial^2\phi}{\partial p_i \partial p_j}\right\} = \det\left\{\frac{\partial^2 S}{\partial p_i \partial p_j}\right\} = 0 \tag{2.10}$$

Each triplet $(\bar{p})$ that satisfies Equation (2.10) corresponds to a point on the caustic in configuration space obtained by substituting into the Lagrange manifold. The locus of these points specifies the caustic in configuration space. The level-equivalence between this parametrization and the classical approach, i.e., that regular points are carried to regular points and caustic points are carried to caustic points, is illustrated below.

We note that, from Equation (2.7), the Lagrange manifold may be regarded as a coordinate transformation from $\bar{p} \to \bar{r}$ (with generating function $S(\bar{p})$) analogous to the coordinate transformation from $\bar{\mu} \to \bar{r}$ (specified by Hamilton's equations). Just as time t appears implicitly in the map $\bar{\mu} \to \bar{r}$, cf. Equation (1.16), frequency ω, conjugate variable of t, is implicit in the Lagrange manifold. That is, although ω appears to be a free parameter, its value is specified at any $(\gamma,\bar{\sigma})$ by the Hamiltonian map (as is the value of t), even at the caustic point.

To obtain the transport equation for the field amplitudes, we proceed as with caustics associated with dispersive waves [10]. Briefly, Taylor expanding the Hamiltonian near the Lagrange manifold obtains

$$\bar{p}\cdot\bar{p}-f(\bar{r},t)\omega^2+g(\bar{r},t) = \bar{p}\cdot\bar{p}-f(\nabla_p S,t)\omega^2+g(\nabla_p S,t)+(\bar{r}-\nabla_p S)\cdot\bar{D} = (\bar{r}-\nabla_p S)\cdot\bar{D} \ ,$$

where

$$\bar{D} = \int_0^1 \nabla_r H(\xi(\bar{r}-\nabla_p S)+\nabla_p S,\bar{p},t,\omega)d\xi \ . \tag{2.11}$$

Substituting into Equation (2.4) leads to

$$\int d\bar{p}\ \exp\{i\lambda\phi\}\{i\lambda[-(\nabla_p A)\cdot\bar{D}-A(\nabla\cdot\bar{D})+2\bar{p}\cdot\nabla_r A+2\omega f(\bar{r},t)\frac{\partial A}{\partial t}]+\frac{1}{i\lambda}(\nabla_r^2 A-f(\bar{r},t)\frac{\partial^2 A}{\partial t^2})\} = O(\lambda^{-\infty}) \ .$$

Then by introducing the flow

$$\frac{d\bar{r}}{d\gamma} = 2\bar{p} \qquad \frac{d\bar{p}}{d\gamma} = -\bar{D}$$
$$\frac{dt}{d\gamma} = 2\omega f(\bar{r},t) \qquad \frac{d\omega}{d\gamma} = \frac{\partial H}{\partial t} \tag{2.12}$$

and requiring that

$$-(\nabla_p A)\cdot\bar{D} - A(\nabla\cdot\bar{D}) + 2\bar{p}\cdot\nabla_r A + 2\omega f(\bar{r},t)\frac{\partial A}{\partial t} + \frac{1}{i\lambda}(\nabla_r^2 A - f(\bar{r},t)\frac{\partial^2 A}{\partial t^2}) = 0 \tag{2.13}$$

in a neighborhood of the Lagrange manifold, we obtain the transport equation

$$\frac{dA_k}{d\gamma} - A_k \nabla_p \cdot D + \frac{1}{i\lambda}\left\{\nabla_r^2 - f(\bar{r},t)\frac{\partial^2}{\partial t^2}\right\} A_{k-1} = 0 \tag{2.14}$$

for the evolution of the amplitudes A_k, where

$$A(\bar{r},\bar{p},t,\lambda) \simeq \sum_{k=0} A_k(\bar{r},\bar{p},t)(i\lambda)^{-k} \quad .$$

The asymptotic evaluation of the field integrals

$$\int A_k(\bar{r},\bar{p},t)\exp\{i\lambda\phi(\bar{r},\bar{p},t,\omega)\}d\bar{p} \tag{2.15}$$

at the caustic point proceeds by transforming the phase to a canonical form, followed by a modified stationary phase technique. The procedures for determining the appropriate canonical form, constructing the coordinate transformations carrying the phase to the canonical form and actually evaluating the integrals have been detailed elsewhere [9,10,12]. For brevity, we do not repeat them here.

3. EXAMPLE

Returning to the example above, i.e., a point source of radiation located at the origin $\bar{r} = (0,0)$, propagating in a medium with $f(\bar{r},t) = 1$ and $g(\bar{r},t) = x+t-k^2$, the equation we consider is

$$\nabla_r^2\psi(\bar{r},t) - \frac{\partial^2\psi(\bar{r},t)}{\partial t^2} - \lambda^2(x+t-k^2)\psi(\bar{r},t) = 0 \ . \tag{3.1}$$

We assume an asymptotic solution of the form

$$\psi(\bar{r},t) - \int A(\bar{r},\bar{p},t,\lambda)\exp\{i\lambda(\bar{r}\cdot\bar{p} - \omega t - S(\bar{p}))\}d\bar{p} = O(\lambda^{-\infty}) \ . \tag{3.2}$$

Proceeding through the algorithm, Maslov's Hamiltonian

$$H = \bar{p}\cdot\bar{p} - \omega^2 + x + t - k^2 \tag{3.3}$$

leads to the same map as above

$$x = -\gamma^2 + 2\rho\gamma\cos\theta \qquad p_x = -\gamma + \rho\cos\theta$$
$$y = 2\rho\gamma\sin\theta \qquad p_y = \rho\sin\theta$$
$$t = \gamma^2 + 2\gamma\Omega \qquad \omega = \gamma + \Omega.$$

Then by invoking the stationary phase condition Maslov's Hamiltonian becomes an eikonal equation

$$\bar{p}\cdot\bar{p} - \omega^2 + x + t - k^2 = 0 \tag{3.4}$$

on the Lagrange manifold

$$\bar{r} = \nabla_p S(\bar{p}) \quad ,$$

which we determine explicitly by inverting the $\bar{\mu} \to \bar{p}$ map and substituting into the map from $\bar{\mu} \to \bar{r}$,

$$\begin{aligned} x &= k^2 + \Omega^2 - p_x{}^2 - p_y{}^2 \\ y &= 2p_y(\rho - p_y{}^2)^{1/2} - 2p_x p_y \quad , \end{aligned} \tag{3.5}$$

and leads to the phase

$$\phi(\bar{r},\bar{p},t,\omega) = \bar{r}\cdot\bar{p} - \omega t - (k^2 + \Omega^2 + p_y{}^2)p_x - \frac{p_x{}^3}{3} - \frac{2}{3}(\rho^2 - \rho_y{}^2)^{3/2} \ . \tag{3.6}$$

When Ω=3, k=4, ρ=5, at $(\gamma,\theta)=(5.77,30^0)$, i.e., the field point (x,y,t)=(16.68, 28.85, 67.91), the classical map becomes singular. Corresponding to this field point, $(p_x,p_y,\omega)=(-1.44,2.5,8.77)$. For these values of (p_x,p_y,ω), the Hessian determinant of the phase, Equation (2.10), is zero, demonstrating the level-equivalence of the classical map and the transformation specified by the Lagrange manifold. For completeness, let $A(\bar{r},\bar{p},t,\lambda)=1$ at the emitter. Then the field at (x,y,t)=(16.68, 28.85, 67.91) is represented by

$$\psi(16.68,28.85,67.91) \int A(16.68,28.85,\bar{\beta},67.91,\lambda)\exp\{i\lambda(1.898\pi-\beta_1^2-\beta_2^3)\}d\bar{\beta}. \tag{3.7}$$

The asymptotic evaluation of the above integral proceeds from a modification of the classical stationary phase technique [9,10,11]. The first two terms in the expansion are

$$\begin{aligned}\psi(16.68,28.85,67.91) \simeq -\ .746\lambda^{-5/6}\exp\{i\lambda(1.65\pi)\}\Gamma(\tfrac{1}{3})\cos(\tfrac{\pi}{6}) \ + \\ .148\lambda^{-7/6}\exp\{i\lambda(1.65\pi)\}\Gamma(\tfrac{2}{3})\sin(\tfrac{\pi}{3}) \ .\end{aligned} \tag{3.8}$$

4. VECTOR FIELDS

The same algorithm applies to vector (electric, $\bar{E}$, or magnetic, H) field propagation. For example, using the electric field, $\bar{E}$, we consider the equation

$$\nabla_r^2\ \bar{E}(\bar{r},t) - f(\bar{r},t)\frac{\partial^2\ \bar{E}(\bar{r},t)}{\partial t^2} - \lambda^2 g(\bar{r},t)\bar{E} = 0 \quad , \tag{4.1}$$

where $\bar{E}\ (\bar{r},t)$ is a column vector. Assuming an asymptotic solution of the form

$$\bar{E}(\bar{r},t) - \int \bar{E}(\bar{r},\bar{p},t,\lambda)\exp\{i\lambda(\bar{r}\cdot\bar{p}-\omega t-S(\bar{p}\ ))\}d\bar{p} = O(\lambda^{-\infty}) \tag{4.2}$$

and proceeding through the algorithm, we find that requiring

$$-(\nabla_p\cdot\bar{D})\bar{E} + \bar{D}\nabla_p\cdot\bar{E} + 2\bar{p}(\nabla\cdot\bar{E}) + 2\omega f(\bar{r},t)\frac{\partial\bar{E}}{\partial t} + \frac{1}{i\lambda}[\nabla_r^2\bar{E} - f(\bar{r},t)\frac{\partial^2\bar{E}}{\partial t^2}] = 0 \tag{4.3}$$

where $\bar{D}$ is the operator from Equation (2.11), in a neighborhood of the Lagrange manifold and leads to the transport equation

$$\frac{d\overline{E}}{d\gamma} - (\nabla_p \cdot \overline{D})\overline{E} + \frac{1}{i\lambda}[\nabla^2\overline{E} - f(\overline{r},t)\frac{\partial^2 E}{\partial t^2}] = 0 \quad , \tag{4.4}$$

if we introduce the flow from Equations (2.12).

The zeroth-order approximation to the time-averaged Poynting vector (power density) $\overline{S}$ on the caustic

$$\overline{S} = R_e(\overline{E} \times \overline{H}^*) = \left(\frac{\varepsilon_o}{\mu_o}\right)^{1/2} E_0^2\overline{p} \ ,$$

where ε_o and μ_o are the permittivity and permeability, respectively, of vacuum, also proceeds from the zeroth-order term of Equation (4.3), i.e.,

$$-(\nabla_p \cdot \overline{D})\overline{E}_o + \overline{D}\nabla_p \cdot \overline{E}_o + 2p_o(\nabla \cdot \overline{E}_o) + 2\omega f(\overline{r},t)\frac{\partial \overline{E}_o}{\partial t} = 0 \ . \tag{4.5}$$

Following the same procedure used in the consideration of dispersive waves [10], scalar multiplication of Equation (4.5) by $\overline{E}_o^*$ and similarly multiplying the complex conjugate of Equation (4.5) by $\overline{E}_o$ and introducing the flow from Equations (2.12) leads to

$$\frac{dE_o^2}{d\gamma} - 2E_o^2(\nabla_p \cdot \overline{D}) = 0 \tag{4.6}$$

and

$$E_o^2(\gamma) = E_o^2(\gamma=0)\exp\{2\int(\nabla_p \cdot \overline{D})d\gamma\} \quad , \tag{4.7}$$

paralleling the result for dispersive waves. Another interesting correspondence concerns the polarization

$$\overline{P} = \frac{\overline{E}_o}{(\overline{E}_o \cdot \overline{E}_o^*)^{1/2}} \ . \tag{4.8}$$

Differentiating Equation (4.8)

$$\frac{d\overline{P}}{d\gamma} = \frac{1}{E_o}\frac{d\overline{E}_o}{d\gamma} - \frac{\overline{E}_o}{E_o}\frac{dE_o}{d\gamma} \quad , \tag{4.9}$$

then combining Equation (4.6) with the first term of Equation (4.5) and noting that the remaining terms

$$-\overline{D}(\nabla_p \cdot E_o) + 2\overline{p}(\nabla \cdot \overline{E}_o) + 2\omega f(\overline{r},t)\frac{\partial \overline{E}_o}{\partial t} = \frac{d\overline{E}_o}{d\gamma}$$

leads to

$$-\overline{E}_o\{\frac{1}{E_o}\frac{dE_o}{d\gamma} + \frac{dE_o}{d\gamma}\} = 0 \ . \tag{4.10}$$

Dividing by $\overline{E}_o$ and comparing with Equation (4.9) obtains the result that on the flow in Equations (2.12)

$$\frac{d\overline{P}}{d\gamma} = 0 \quad ,$$

i.e., the polarization is a constant, cf. [10].

5. THE SCHRODINGER EQUATION

Because the above approach applies so directly to determine asymptotic solutions at caustic points for the Schrodinger equation with a potential that varies both in space and time,

$$i\hbar \frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2\psi + V(\bar{r},t)\psi \quad , \tag{5.1}$$

where $\hbar$ is the reduced Planck's constant ($h/2\pi$) and m is the mass, we merely sketch those aspects of the algorithm that most complement Kratsov's off-caustic treatment [11]. Away from caustics, Kratsov assumes an asymptotic solution of the form

$$\psi(x,t) = \sum_{k=o} \exp(i\phi(x,t)/\hbar)A_k(x,t)(i\hbar)^k \quad . \tag{5.2}$$

Then following the classical procedure outlined above leads to a Hamilton-Jacobi equation for the phase

$$\frac{\partial\phi}{\partial t} + \frac{\bar{p}\cdot\bar{p}}{2m} + V(\bar{r},t) = 0 \tag{5.3}$$

and a transport equation for the amplitude $A_o(x,t)$

$$\frac{\partial A_o}{\partial t} + \frac{1}{2m}(2\nabla A_o\cdot\nabla\phi + A_o\nabla^2\phi) = 0 \quad , \tag{5.4}$$

whose solution Kratsov details. Analogously, near caustics, we assume a solution of the form

$$\psi(\bar{r},t) - \int A(\bar{r},\bar{p},t,i\hbar)\exp\{i(\bar{r}\cdot\bar{p}-\omega t-S(\bar{p}))/\hbar\}d\bar{p} = O((i\hbar)^\infty) \quad . \tag{5.5}$$

Proceeding through the algorithm leads to the Hamiltonian

$$H = \frac{\bar{p}\cdot\bar{p}}{2m} - \omega + V(x,t) \tag{5.6}$$

and transport equation

$$\frac{\partial A_k}{\partial t} + \frac{1}{2m}(2\bar{p}\cdot\nabla A_k - \nabla A_k\cdot\bar{D} - A_k\nabla\cdot\bar{D} + \nabla_r{}^2A_{k-1}) = 0 \quad , \tag{5.7}$$

where $k \geq 0$, $A_{-1} = 0$ and

$$\bar{D} = \int_0^1 \nabla_r H(\xi(\bar{r}-\nabla_p S) + \nabla_p S,\bar{p},t,\omega)d\xi \quad .$$

Equation (5.7) may be re-grouped into a first order ordinary differential equation using Equation (2.12) flow, after which the procedure detailed above applies directly. (A more extensive treatment of this Schrodinger equation using the Lagrange manifold approach has recently been presented by Bernstein [13].)

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