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ITERATIONS CONVERGING TO DISTINCT SOLUTIONS OF SOME NONLINEAR OPERATOR EQUATIONS IN BANACH SPACE

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ABSTRACT. We examine the solvability of multilinear equations of the form

$$\underbrace{M_k(x,x,\ldots,x)}_{\text{-k times-}} = y, \ k = 2,3,\ldots$$

where M_k is a k-linear operator on a Banach space X and $y \in X$ is fixed.

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1. INTRODUCTION.

We study the quadratic equation

$$B(x,x) = y \tag{1.1}$$

in a Banach space X, where B is a bounded symmetric bilinear operator on X and y is fixed in X [2], [3], [7], [9], [10]. We consider two cases.

CASE 1. Let $y = 0$ and set $x = \bar{x} - h$ for some $\bar{x}$ such that the linear operator $2B(\bar{x})$ is invertible then (1.1) becomes

$$\bar{B}(h,h) = h - \bar{y} \tag{1.2}$$

where $\bar{B} = (2B(\bar{x}))^{-1}B$, $\bar{y} = (2B(\bar{x}))^{-1}B(\bar{x},\bar{x})$ and $h \in X$ is to be determined.

We introduce the iteration

$$h_{n+1} = (\bar{B}(h_n))^{-1}(h_n - \bar{y}) \quad \text{for some} \quad h_0 \in X \tag{1.3}$$

to find a solution h of (1.2) such that $h \neq \bar{x}$.

It turns out under certain assumptions that iteration (1.3) converges to an $h \in X$ such that $h \neq \bar{x}$, therefore $x = \bar{x} - h$ is a nonzero solution of (1.1).

CASE 2. Let $y \neq 0$, we then introduce the iteration

$$x_{n+1} = B(x_n)^{-1}(y) \quad \text{for some} \quad x_0 \in X \tag{1.4}$$

to find solutions of (1.1).

The results obtained here can be generalized to include multilinear equations of the form

$$M_k(\underbrace{x,x,\ldots,x}_{k\text{ times}}) = y$$

where M_k is a k-linear operator on X and y is fixed in X [10].

We now state the following lemma. The proof can be found in [10].

2. EXISTENCE THEORY.

LEMMA 1. Let L_1 and L_2 be bounded linear operators in a Banach space X, where L_1 is invertible, and $\|L_1^{-1}\| \cdot \|L_2\| < 1$. Then $(L_1 + L_2)^{-1}$ exists, amd

$$\|(L_1 + L_2)^{-1}\| \leq \frac{\|L_1^{-1}\|}{1-\|L_2\| \cdot \|L_1^{-1}\|}.$$

LEMMA 2. Let $z \neq 0$ be fixed in X. Assume that the linear operator $\overline{B}(z)$ is invertible then $\overline{B}(x)$ is also invertible for all $x \in U(z,r) = \{x \in X \mid \|x-z\| < r\}$, where $r \in (0,r_0)$ and $r_0 = [\|\overline{B}\| \cdot \|\overline{B}(z)^{-1}\|]^{-1}$.

PROOF. We have

$$\begin{aligned}\|\overline{B}(x-z)\| \cdot \|\overline{B}(z)^{-1}\| &\leq \|\overline{B}\| \cdot \|x-z\| \cdot \|\overline{B}(z)^{-1}\| \\ &\leq \|\overline{B}\| \cdot \|\overline{B}(z)^{-1}\| \cdot r \\ &< 1\end{aligned}$$

for $r \in (0,r_0)$. The result now follows from Lemma 1 for $L_1 = \overline{B}(z)$, $L_2 = \overline{B}(x-z)$ and $x \in U(z,r)$.

DEFINITION 1. Assume that the linear operator $\overline{B}(z)$ is invertible.

Define the operators P,T on $U(z,r)$ for some $r > 0$ by

$$P(x) = \overline{B}(x,x) + \overline{y} - x, \quad T(x) = (\overline{B}(x))^{-1}(x-\overline{y})$$

and the real polynomials $f(r)$, $g(r)$ on R by

$$\begin{aligned}f(r) &= a'r^2 + b'r + c', \quad g(r) = ar^2 + br + c, \\ a' &= (\|\overline{B}\| \cdot \|\overline{B}(z)^{-1}\|)^2, \\ b' &= -2\|\overline{B}\| \cdot \|\overline{B}(z)^{-1}\|, \\ c' &= 1 - \|\overline{B}(z)^{-1}\| - \|\overline{B}\| \cdot \|\overline{B}(z)^{-1}\|^2 \cdot \|z-\overline{y}\|, \\ a &= \|\overline{B}\| \, \|\overline{B}(z)^{-1}\|, \\ b &= \|\overline{B}(z)^{-1}(I-\overline{B}(z))\| - 1,\end{aligned}$$

and

$$c = \|\overline{B}(z)^{-1}P(z)\|.$$

THEOREM 1. Let $z \in X$ be such that $\overline{B}(z)$ is invertible and that the following are true:

a) $c' > 0$;

b) $b < 0$, $b^2 - 4ac > 0$, and

c) there exists $r > 0$ such that $f(r) > 0$ and $g(r) \leq 0$

then the iteration

$$h_{n+1} = \overline{B}(h_n)^{-1}(h_n-\overline{y}), \quad n = 0,1,2,\ldots$$

is well defined and it converges to a unique solution h of (1.2) in $\bar{U}(z,r)$ for any $h_0 \in \bar{U}(z,r)$.

PROOF. T is well defined by Lemma 2.

CLAIM 1. T maps $\bar{U}(z,r)$ into $\bar{U}(z,r)$.

If $x \in \bar{U}(z,r)$ then

$$T(x) - z = \bar{B}(x)^{-1}(x-\bar{y}) - z$$
$$= \bar{B}(x)^{-1}[(I - \bar{B}(z))(x-z) - P(z)]$$

so

$$\|T(x) - z\| \le r$$

if

$$\frac{1}{1-\|\bar{B}\| \cdot \|\bar{B}(z)^{-1}\|r}[\|\bar{B}(z)^{-1}(I - \bar{B}(z))\|r + \|\bar{B}(z)^{-1}P(z)\|] \le r$$

(using Lemma 1 for $L_1 = \bar{B}(z)$ and $L_2 = \bar{B}(x-z)$) or $g(r) \le 0$ which is true by hypothesis.

CLAIM 2. T is a contraction operator on $\bar{U}(z,r)$.

If $w,v \in \bar{U}(z,r)$ then

$$\|T(w) - T(v)\|$$
$$= \|\bar{B}(w)^{-1}(w-\bar{y}) - \bar{B}(v)^{-1}(v-\bar{y})\|$$
$$= \|\bar{B}(w)^{-1}[I - \bar{B}(\bar{B}(v)^{-1}(v-\bar{y})](w-v)\|$$
$$= \|\bar{B}(w)^{-1}[I - \bar{B}(\bar{B}(v)^{-1}(v-z)) + \bar{B}(\bar{B}(v)^{-1}(z-\bar{y}))](w-v)\|$$
$$\le \frac{1}{1 - \|\bar{B}\| \cdot \|\bar{B}(z)^{-1}\| \cdot r}\left[\|\bar{B}(z)^{-1}\| + \frac{\|\bar{B}\| \cdot \|\bar{B}(z)^{-1}\|^2 r + \|\bar{B}\|\|\bar{B}(z)^{-1}\|^2\|z-\bar{y}\|}{1 - \|\bar{B}\| \cdot \|\bar{B}(z)^{-1}\| \cdot r}\right] \cdot \|w-v\|$$
$$= q \cdot \|w-v\|.$$

So T is a contraction on $\bar{U}(z,r)$ if $0 < q < 1$, where

$$q = \frac{1}{1 - \|\bar{B}\| \cdot \|\bar{B}(z)^{-1}\| \cdot r}\left[\|\bar{B}(z)^{-1}\| + \frac{\|\bar{B}\| \cdot \|\bar{B}(z)^{-1}\|^2 r + \|\bar{B}\| \cdot \|\bar{B}(z)^{-1}\|^2\|z-\bar{y}\|}{1 - \|\bar{B}\| \cdot \|\bar{B}(z)^{-1}\| \cdot r}\right]$$

which is true since $f(r) > 0$.

THEOREM 2. Assume that there exist $r > 0$, $z, \bar{x} \in X$ satisfying the hypotheses of Theorem 1 and

(a) $0 < \|\bar{x}\| < \dfrac{-1 + \sqrt{1 + 4\|\bar{B}\|\|\bar{y}\|}}{2\|\bar{B}\|}$;

(b) $r + \|z\| < \dfrac{\|y\|}{1 + \|\bar{B}\| \cdot \|\bar{x}\|}$

then if $\|\bar{x}\| < h_0 \le r + \|z\|$, the solution h if (1.2) is such that

$$\|\bar{x}\| < \|h\| \le r + \|z\|.$$

Moreover, $x = \bar{x} - h$ is a nonzero solution of (1.1).

PROOF. By Theorem 1 $h \in \bar{U}(z,r)$ therefore

$$\|h\| \le r + \|z\|.$$

Assume that $\|h_k\| > \|\bar{x}\|$ for $k = 0,1,2,\ldots,n$. By iteration (1.3) we have

$$\bar{B}(h_{n+1},h_n) = h_n - \bar{y}$$

or

$$\|\bar{B}\|\|h_{n+1}\| \cdot \|h_n\| \geq \|h_n - \bar{y}\| \geq \|\bar{y}\| - \|h_n\|$$

so

$$\|h_{n+1}\| \geq \frac{\|\bar{y}\| - \|h_n\|}{\|\bar{B}\| \cdot \|h_n\|},$$

to show that

$$\|h_{n+1}\| > \|\bar{x}\| ,$$

it suffices to show

$$\frac{\|\bar{y}\| - \|h_n\|}{\|\bar{B}\|\|h_n\|} > \|\bar{x}\|$$

which is true by (b). For consistency we must have

$$\|\bar{x}\| < \frac{\|\bar{y}\|}{1 + \|\bar{B}\| \cdot \|\bar{x}\|}$$

which is true by (a). The result now follows by taking the limit as $n \to \infty$ in (2.1).

Finally note that since $\|h\| > \|\bar{x}\|$, $\bar{x} - h \neq 0$ therefore $x = \bar{x} - h$ is a non-zero solution of (1.1).

DEFINITION 2. Assume that the linear operator $B(z)$ is invertible for some $z \in X$. Define the operator $\bar{P}$ on $U(z,r)$ for some $r > 0$ by

$$\bar{P}(x) = B(x,x) - y, \quad y \neq 0$$

and the real polynomials $\bar{f}(r)$, $\bar{g}(r)$ on R by

$$\bar{f}(r) = s_1' r^2 + s_2' r + s_3', \quad \bar{g}(r) = s_1 r^2 + s_2 r + s_3,$$

where

$$\begin{aligned}
s_1' &= (\|B\| \cdot \|B(z)^{-1}\|)^2 \\
s_2' &= -2\|B\| \cdot \|B(z)^{-1}\| \\
s_3' &= 1 - \|B\| \cdot \|B(z)^{-1}\|^2 \\
s_1 &= \|B\| \cdot \|B(z)^{-1}\| \\
s_2 &= \|B\| \\
s_3 &= \|B(z)^{-1}\bar{P}(z)\|.
\end{aligned}$$

The proofs of the following theorems are omitted as similar to Theorems 1 and 2.

THEOREM 3. Let $z \in X$ be such that the linear operator $B(z)$ is invertible and that the following are true:

a) $s_3' > 0$;

b) $s_2 > 0$, $s_2^2 - 4s_1 s_2 > 0$, and

c) there exists $r > 0$ such that $\bar{f}(r) > 0$ and $\bar{g}(r) \leq 0$

then the iteration

$$x_{n+1} = B(x_n)^{-1}(y)$$

for some $x_0 \in X$ is well defined and it converges to a solution x of (1.1) which is unique in $\bar{U}(z,r)$ for any $x_0 \in \bar{U}(z,r)$.

THEOREM 4. Let z,r be such that the hypotheses of Theorem 3 are satisfied. Let $p < q$ be positive numbers such that

a) $pq\|B\| \leq \|y\|$;

b) $$\frac{\|B(z)^{-1}\|}{1 - \|B\| \cdot \|B(z)^{-1}\| r} \leq q \leq r + \|z\|$$

then if $p \leq \|x_0\| \leq q$ then the solution x of (1.1) is such that

$$p \leq \|x\| \leq q.$$

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