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ON THE NON-EXISTENCE OF SOME INTERPOLATORY POLYNOMIALS

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ABSTRACT. Here we prove that if x_k, $k = 1,2,\ldots,n + 2$ are the zeros of $(1 - x^2)T_n(x)$ where $T_n(x)$ is the Tchebycheff polynomial of first kind of degree n, α_j, β_j, $j = 1,2,\ldots,n + 2$ and γ_j, $j = 2,3,\ldots,n + 1$ are any real numbers there does not exist a unique polynomial $Q_{3n+3}(x)$ of degree $\leq 3n + 3$ satisfying the conditions: $Q_{3n+3}(x_j) = \alpha_j$, $Q'_{3n+3}(x_j) = \beta_j$, $j = 1,2,\ldots,n + 2$ and $Q'''_{3n+3}(x_j) = \gamma_j$, $j = 2,3,\ldots,n + 1$. Similar result is also obtained by choosing the roots of $(1 - x^2)P_n(x)$ as the nodes of interpolation where $P_n(x)$ is the Legendre polynomial of degree n.

KEY WORDS AND PHRASES. Roots, interpolatory polynomials, non-existence, nodes.
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1. INTRODUCTION.

In [1] R.B. Saxena considered an interesting problem of (0,1,3) interpolation by taking the roots of $(1 - x^2)P_{n-2}(x)$, where $P_{n-2}(x)$ is the Legendre polynomial of degree $n - 2$, as the nodes of interpolation. By (0,1,3) interpolation, Saxena meant that for the collections $\{\alpha_j\}_1^n, \{\beta_j\}_2^{n-1}$, and $\{\gamma_j\}_1^n$ of real numbers and the zeros x_j of $(1 - x^2)P_{n-2}(x)$ arranged so that

$$-1 = x_n < x_{n-1} < \ldots < x_2 < x_1 = 1$$

a polynomial $R_n(x)$ of degree $\leq 3n - 3$ can be constructed so that

$$R_n(x_j) = \alpha_j;\ j = 1,2,\ldots,n,$$

$$R'_n(x_j) = \beta_j;\ j = 2,3,\ldots,n - 1,$$

and

$$R'''_n(x_j) = \gamma_j;\ j = 1,2,\ldots,n.$$

Saxena proved that such a polynomial exists uniquely if n is even and for n odd there does not exist a unique polynomial $R_n(x)$ satisfying the above conditions. Later Varma [2] obtained the following result in this direction:

THEOREM 1 (VARMA). Given a positive integer n and real numbers $\alpha_k(k = 1,2,\ldots, n + 2)$, $\beta_k,\gamma_k(k = 2,3,\ldots,n + 1)$ there is, in general no polynomial $F_{3n+1}(x)$ of degree $\leq 3n + 1$ such that $F_{3n+1}(x_k) = \alpha_k$; $k = 1,2,\ldots,n + 2$, $F'_{3n+1}(x_k) = \beta_k$;

$k = 2,3,\dots,n+1$ and $F'''_{3n+1}(x_k) = \gamma_k$; $k = 2,3,\dots,n+1$ provided $x_k's$ are the zeros of $(1-x^2)T_n(x)$ where $T_n(x)$ is Tchebycheff polynomial of first kind and if there exists such a polynomial then there is an infinity of them.

2. MAIN RESULTS.

In connection with the above results we shall prove the following.

THEOREM 2. For any positive integer n, with $1 = \xi_1 > \xi_2 > \dots > \xi_{n+1} > \xi_{n+2} = -1$ the zeros of $(1-x^2)P_n(x)$ where $P_n(x)$ is the Legendre polynomial of degree n, there is in general no polynomial $R_{3n+1}(x)$ of degree $\le 3n+1$ such that, for arbitrary real numbers $\{\alpha_j\}_1^{n+2}$, $\{\beta_j\}_2^{n+1}$ and $\{\gamma_j\}_2^{n+1}$ the conditions:

$$R_{3n+1}(\xi_j) = \alpha_j;\ j = 1,2,\dots,n+1,n+2, \tag{2.1}$$

$$R'_{3n+1}(\xi_j) = \beta_j;\ j = 2,3,\dots,n+1 \tag{2.2}$$

and

$$R'''_{3n+1}(\xi_j) = \gamma_j;\ j = 2,3,\dots,n+1 \tag{2.3}$$

are satisfied. If there does exist such a polynomial then there are infinitely many of them.

We also prove the following result for Tchebycheff nodes:

THEOREM 3. For any positive integer n, with $1 = x_1 > x_2 > \dots > x_n > x_{n+1} > x_{n+2} = -1$ the zeros of $\omega_n(x) = (1-x^2)T_n(x)$, there is in general no polynomial $Q_{3n+3}(x)$ of degree $\le 3n+3$ such that for arbitrary real numbers $\{\alpha_j\}_1^{n+2}$, $\{\beta_j\}_1^{n+2}$ and $\{\gamma_j\}_2^{n+1}$ the conditions:

$$Q_{3n+3}(x_j) = \alpha_j;\ j = 1,2,\dots,n+1,n+2, \tag{2.4}$$

$$Q'_{3n+3}(x_j) = \beta_j;\ j = 1,2,\dots,n+1,n+2 \tag{2.5}$$

and

$$Q'''_{3n+3}(x_j) = \gamma_j;\ j = 2,3,\dots,n+1 \tag{2.6}$$

are satisfied. If there does exist such a polynomial then there are infinitely many of them.

REMARK 1. The comparison of our Theorem 2 with the above mentioned result of Saxena shows that if we do not prescribe the third derivative at ± 1 then there does not exist a unique polynomial regardless whether n is even or odd. In an earlier work [3] we have shown that along with the conditions (2.1), (2.2) and (2.3) if we also prescribe the first derivative at ± 1 a unique polynomial of degree $\le 3n+3$ still does not exist. It is also evident from Theorem 3 that even if we prescribe the first derivative at ± 1 a unique polynomial of degree $\le 3n+3$ does not exist although the nodes of interpolation are different from that of [3].

REMARK 2. We shall give here the proof of Theorem 3 only. The proof of Theorem 2 can be obtained along the same lines.

PROOF OF THEOREM 3. We will show that if all of

$$\alpha_j = 0;\ j = 1,2,\dots,n+1,n+2, \tag{2.7}$$

$$\beta_j = 0;\ j = 1,2,\dots,n+1,n+2,$$

$$\gamma_j = 0;\ j = 2,3,\dots,n+1$$

then there exists a polynomial $Q_{3n+3}(x)$ of degree $\leq 3n + 3$ which is not identically zero, but satisfies (2.4), (2.5) and (2.6). The desired result then follows immediately from the theory of linear equations. From the definition of $\omega_n(x)$ and conditions (2.4), (2.5) and (2.6), together with the requirements (2.7), it is clear that the desired polynomial must be of the form

$$Q_{3n+3}(x) = (1 - x^2)^2 T_n^{\,2}(x) \hbar_{n-1}(x) \tag{2.8}$$

where $\hbar_{n-1}(x)$ is an unknown polynomial of degree $\leq n - 1$. Since we have also required $Q'''_{3n+3}(x_j) = 0$; for $j = 2,3,\dots,n + 1$, simple calculation provides

$$(1 - x^2)\hbar'_{n-1}(x) - 3x\hbar_{n-1}(x) = cT_n(x) \tag{2.9}$$

for unknown real constant c. Letting $x = \cos\theta$ and

$$\hbar_{n-1}(x) = \sum_{k=0}^{n-1} a_k \cos k\theta$$

we obtain

$$(1 - x^2)\hbar'_{n-1}(x) = \sum_{k=1}^{n-1} a_k k \sin k\theta \sin\theta.$$

Thus (2.9) becomes

$$c \cos n\theta = \sum_{k=0}^{n-1} a_k[k \sin k\theta \sin\theta - 3 \cos k\theta \cos\theta].$$

From this, we obtain on simplification

$$2\, c \cos n\theta = \sum_{k=0}^{n-1} a_k[(k - 3)\cos(k - 1)\theta - (k + 3)\cos(k + 1)\theta],$$

from which, by collecting the coefficients of $\cos k\theta$, for $k = 0,1,\dots,n$, we may write

$$\begin{aligned}
&-2a_1 - (6a_0 + a_2)\cos\theta - 4a_1\cos 2\theta \\
&+ \sum_{k=3}^{n-2} \{(k - 2)a_{k+1} - (k + 2)a_{k-1}\}\cos k\theta \\
&-(n + 1)a_{n-2}\cos(n - 1)\theta - (n + 2)a_{n-1}\cos n\theta \\
&= 2c \cos n\theta.
\end{aligned}$$

This, in turn, leads to the following system of equations

$$\begin{aligned}
&-2a_1 = 0 \\
&-(6a_0 + a_2) = 0, \\
&-4a_1 = 0, \\
&(k - 2)a_{k+1} - (k + 2)a_{k-1} = 0;\ k = 3,4,\dots,n - 2, \\
&-(n + 1)a_{n-2} = 0, \\
&-(n + 2)a_{n-1} = 2c.
\end{aligned}$$

If n is even, then

$$a_0 = a_2 = a_4 = \dots\dots = a_{n-2} = 0;\ a_1 = 0$$

but

$$a_{n-1-2j} = \frac{-2c}{n-2} \prod_{k=0}^{j} \left(\frac{n-2-2k}{n+2-2k}\right); \text{ for } j = 0,1,\dots,(n-4)/2$$

is not necessarily zero.

If n is odd, then

$$a_1 = a_3 = a_5 = \dots\dots = a_{n-2} = 0,$$

while

$$a_{2j} = \frac{-2c}{n-2} \prod_{k=j}^{(n-1)/2} \frac{2k-1}{2k+3};\ j = 1,2,\dots, \frac{(n-1)}{2}$$

with the special case

$$a_0 = -a_2/6$$

which are not necessarily zero. Hence regardless whether n is even or odd, in general, there does not exist a unique polynomial $Q_{3n+3}(x)$ of degree $\leq 3n+3$ satisfying (2.4), (2.5) and (2.6) and there are infinitely many if they exist.

This completes the proof of Theorem 3. For a complete history on lacunary interpolation we refer to a paper by J. Balázs [4].

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