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ON THE COMPUTATION OF THE CLASS NUMBERS OF SOME CUBIC FIELDS

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ABSTRACT. Class numbers are calculated for cubic fields of the form $x^3+12Ax-12 = 0$, $A > 0$, for $1 \le a \le 17$, and for some other values of A. These fields have a known unit, which under certain conditions is the fundamental unit, and are important in studying the Diophantine Equation $x^3 + y^3 + z^3 = 3$.

KEY WORDS AND PHRASES. Class numbers, cubic fields, Diophantine equation.
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1. INTRODUCTION AND SOME THEOREMS.

We consider the cubic fields defined by an equation of the form

$$f(x) = x^3 + 12Ax - 12 = 0, \tag{1.1}$$

where $A > 0$. The field defined by this equation is important because it is related to the Diophantine equation $x^3 + y^3 + z^3 = 3$ when $A = 9a^2$ [1]. Equation (1.1) is clearly irreducible, and as $f(x)$ is increasing, it defines a real cubic field K (with two complex conjugates) with exactly one fundamental unit. Let Θ be the real root of (1.1). We write $K = Q(\Theta)$. Note that $0 < \Theta < 1$. Also $\eta = \frac{\Theta^3}{12} = 1 - A\Theta$ defines a unit of K. As $0 < \eta < 1$, we have $\Theta < \frac{1}{A}$. The discriminant of $f(x)$ is $D = -2^4 \cdot 3^3 (16A^3 + 9)$. As $f(x)$ is an Eisenstein polynomial with respect to 3, we have $(3) = q^3$. Also as $\frac{6}{\Theta}$ satisfies $x^3 - 36Ax - 18 = 0$, we see that for the same reason $(2) = p^3$, and as $\frac{6}{\Theta} = 6A + \frac{\Theta^2}{2}$ we see that $\frac{\Theta^2}{2} \varepsilon\, 0_K$, the ring of integers of K. Thus the descriminant, D, of K, divides $-2^2 \cdot 3^3 (16A^3 + 9)$. We now state:

THEOREM 1. In K, the discriminant $D = \frac{-2^2 \cdot 3^3 (16A^3 +9)}{q^2}$ where q^2 is the largest square, prime to 3, dividing D. The unit η is never a cube, and if q=1 or q=5 then η is the fundamental unit except when $A = 1$. The class-number h, of K, is divisible by 3. The primes p_i dividing D (except for 2 and 3) ramify as

$(p_i) = p_i^2 q_i$. A basis for O_K is given by $\theta_o = 1$, $\theta_1 = \frac{\theta^2}{2}$, $\theta_2 = \frac{16A^2+3\theta+2A\theta^2}{3^i q}$,

$(\beta^i = (3,A))$.

As the proof is similar to the proof of the corresponding theorem in [1], we omit it, as well as the proof of the following two theorems, also in [1].

THEOREM 2. If the 3-component of the class-group of K is a direct product of cyclic groups of order 3, then

$$x^3 + 12Ax - 12 = 4z^3 \tag{1.2}$$

has no solutions.

Corollary: If $3 \nmid h$, then (1.2) has no solutions.

THEOREM 3. If $(h,2) = 1$, and $q = 1$, then solving $x^3 + 12Ax - 12 = y^2$ is equivalent to solving $-AG^4 - 2G^3H + 3H^4 = -1$ (This has no solutions (mod p) for small primes p, e.g. $A = 14$, $p=5$).

2. NUMERICAL COMPUTATIONS.

We note that $$\lim_{s\to 1+} \frac{\zeta_k(s)}{\zeta(s)} = \frac{4\pi \log \varepsilon \cdot h}{2\sqrt{2^2\cdot 3^3\ (16A^3+9)/q^2}} \tag{2.1}$$

where $\varepsilon > 1$ is the fundamental unit of K. As in [2], the left-hand side of (2.1) can be expressed as $f = \lim_{P\to\infty} f_P = \lim_{P\to\infty} \prod_{p=5}^{P} f(p)$ where

$$f(p) = \begin{cases} \frac{p}{p-1} & \text{if } p \text{ ramifies } ((p_i) = p_i^2 q_i) \\ \frac{p^2}{p^2+p+1} & \text{if } p \text{ remains inert} \\ \frac{p^2}{p^2-1} & \text{if } (p) = pq \\ \left(\frac{p}{p-1}\right)^2 & \text{if } p \text{ splits completely} \end{cases}$$

Hence (2.1) implies that approximately,

$$h = \frac{\sqrt{27\ (16A^3+9)}}{\pi \cdot q \cdot \log \varepsilon} f_P \tag{2.2}$$

for P sufficiently large.

For Table 1, the product in (2.2) was calculated for $P = P(2027)$, (at intervals of 50), where $P(i)$ indicates the i^{th} prime, and $1 \le A \le 36$:

TABLE 1

A	$-D/2^2\cdot 3^3$	h	A	$-D/2^2\cdot 3^3$	h
1	5^2	6	19	$7\cdot 15679$	39
2	137	3	20	$7\cdot 18287$	72
3	$3^2\cdot 7^2$	6	21	$3^2\cdot 5\cdot 37\cdot 89$	54
4	1033	6	22	$347\cdot 491$	36
5	$7^2\cdot 41$	3	23	194681	72
6	$3^2\cdot 5\cdot 7\cdot 11$	21	24	$3^2\cdot 7\cdot 3511$	54

A	$-D/2^2\cdot 3^3$	h	A	$-D/2^2\cdot 3^3$	h
7	$23\cdot 239$	9	25	$29\cdot 37\cdot 233$	72
8	$59\cdot 139$	18	26	$5^2\cdot 7\cdot 1607$	15
9	$3^2\cdot 1297$	12	27	$3^2\cdot 7\cdot 4999$	54
10	$7\cdot 2287$	27	28	$11\cdot 37\cdot 863$	78
11	$5\cdot 4261$	24	29	$359\cdot 1087$	48
12	$3^2\cdot 7\cdot 439$	24	30	$3^2\cdot 23\cdot 2087$	72
13	$7\cdot 5023$	48	31	$5\cdot 7\cdot 13619$	162
14	43913	21	32	$17\cdot 3084$	78
15	$3^2\cdot 17\cdot 353$	36	33	$7\cdot 82143$	114
16	$5\cdot 13109$	36	34	$7\cdot 89839$	87
17	$7\cdot 11\cdot 1021$	48	35	686009	75
18	$3^2\cdot 10369$	36	36	$3^2\cdot 5\cdot 53\cdot 313$	156

In all the cases above except when $A = 1$ or $A = 5$, $\eta = \frac{1}{\varepsilon}$ is the fundamental unit of K. When $A = 1,5$, $\eta = \varepsilon^{-2}$. K is a pure cubic field if and only if $A = 1$ or $A = 3$.

Also because of the equivalence of (1.2) with the Diophantine equation $x^3 + y^3 + z^3 = 3$ when $A = 9a^2$, the class-numbers of K were calculated using (2.2) for $1 \leq a \leq 17$ (Actually Cassels has shown that for solutions of (1.2) to exist in this case, one must have $3|a$ [3]). While most of the values obtained in this way were approximate, perhaps congruence conditions may be used to find them exactly, or perhaps they may be of use in regards to Brauer-Siegel Theorem, so we list them in Table 2. (The Brauer-Siegel Theorem applied here states $\log h \sim \frac{3}{2} \log A - \log q$).

TABLE 2

a	$-D/2^2\cdot 3^5$	I (P(I) is the I^{th} prime)	h
1	1297	8303	12
2	$5\cdot 53\cdot 313$	14903	156
3	$5\cdot 188957$	10803	216
4	5380417	4303	420
5	$3557\cdot 5693$	2201	789 (*)
6	$37\cdot 241\cdot 6781$	3003	1410
7	$5\cdot 30494621$	1002	3285
8	$5^3\cdot 17\cdot 29\cdot 37\cdot 149$	1002	873
9	$17\cdot 40514561$	1002	3549
10	$181\cdot 1361\cdot 5261$	1002	6999
11	$89\cdot 25797113$	212	6753
12	$5\cdot 23761\cdot 32573$	212	15999
13	$5\cdot 8821\cdot 141833$	212	21864

a	$-D/2^2 \cdot 3^5$	I (P(I) is the Ith prime)	h
14	$37 \cdot 263737261$	212	10062
15	$1193 \cdot 2381 \cdot 5197$	212	22653
16	$35801 \cdot 607337$	212	16764
17	$5^2 \cdot 1251291577$	212	4644

The second column gives the factorization of $-D/2^2 \cdot 3^5$.

(*) For $a \geq 5$, the values of h should be considered as estimates, but are probably accurate within ½%.

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