TIME AVERAGING FOR FUNCTIONAL DIFFERENTIAL EQUATIONS

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We present a result on the averaging for functional differential equations on finite time intervals. The result is formulated in both classical mathematics and nonstandard analysis; its proof uses some methods of nonstandard analysis.

1. Introduction

The idea of the method of averaging is to determine conditions in which solutions of an autonomous dynamical system can be used to approximate solutions of a more complicated time varying dynamical system. The method of averaging has become one of the most important tool ever developed for nonlinear time varying systems. Applications have been found in celestial mechanics, noise control, nonlinear oscillations, stability analysis, bifurcation theory and vibrational control, among many other fields. Although averaging of ordinary differential equations is considered a mature field—the reader may consult [1, 5, 8, 22, 24] for more references and information on the subject (see also [13, 14])—averaging of functional differential equations has only recently been developed (see [6, 7, 9, 11, 12, 15, 16, 19]).

This paper aims to present a result on the averaging for functional differential equations of the form

$$\dot{x}(t) = f\left(\frac{t}{\varepsilon}, x_t\right) \tag{1.1}$$

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on finite time intervals. The result is not new (see [10] and the references therein). However, by means of nonstandard analysis methods, we propose a new proof where all the analysis is achieved in $\mathbb{R}^d$ (it is not the case in [10]) which makes it more simple.

The paper is organized as follows. Section 2 contains the notation and conditions required to state and prove our main result as well as the main result itself. The proof of this result is given in Section 4.2. To avoid complicating the proof unnecessarily, several subsidiary lemmas have been placed in Section 4.1.

The main result is formulated in both classical mathematics and nonstandard analysis. Its proof makes use of Robinson's *nonstandard analysis* (NSA) [21]. We will work in the axiomatic form IST (for *internal set theory*) of nonstandard analysis, given by Nelson [20]. For that, Section 3.1 is devoted to a short description of IST. Then, in Section 3.2, we present the nonstandard translate (Theorem 3.6) in the language of IST of our main result (Theorem 2.2). We recall that IST is a *conservative extension* of ordinary mathematics. This means that any statement of ordinary mathematics which is a theorem of IST was already a theorem of ordinary mathematics, so there is no need to translate the proof.

2. Notation, conditions, and main result

Let $r \geq 0$ be a given constant. Throughout this paper $\mathcal{C}_0 = \mathcal{C}([-r,0],\mathbb{R}^d)$ will denote the Banach space of all continuous functions from $[-r,0]$ into $\mathbb{R}^d$ with the norm $\|\phi\| = \sup\{|\phi(\theta)| : -r \leq \theta \leq 0\}$, where $|\cdot|$ is a norm of $\mathbb{R}^d$. Let $t_0 \in \mathbb{R}$ and $T > t_0$. If $x(t)$ is a continuous function defined on $[t_0 - r, T]$ and $t \in [t_0, T]$, then $x_t \in \mathcal{C}_0$ is defined by $x_t(\theta) = x(t+\theta)$ for $\theta \in [-r,0]$.

The hypotheses, which are denoted by the letter H, are listed as follows.

(H1) The functional $f : \mathbb{R} \times \mathcal{C}_0 \to \mathbb{R}^d$ in (1.1) is continuous.

(H2) The functional f is Lipschitzian in $u \in \mathcal{C}_0$, that is, there exists some constant k such that

$$\left|f(\tau,u_1) - f(\tau,u_2)\right| \leq k\left\|u_1 - u_2\right\|, \quad \forall \tau \in \mathbb{R},\ \forall u_1, u_2 \in \mathcal{C}_0. \tag{2.1}$$

(H3) For all $u \in \mathcal{C}_0$, there exists a limit

$$f^0(u) := \lim_{T\to\infty} \frac{1}{T}\int_0^T f(\tau,u)\,d\tau. \tag{2.2}$$

For any $\phi \in \mathcal{C}_0$ and $t_0 \in \mathbb{R}$, the solution of the averaged equation

$$\dot{y}(t) = f^0(y_t) \tag{2.3}$$

(resp., the solution of (1.1)) such that $y_{t_0} = \phi$ (resp., $x_{t_0} = \phi$) is denoted by $y = y(\cdot;t_0,\phi)$ (resp., $x = x(\cdot;t_0,\phi)$) and J (resp., I) will denote its maximal interval of definition.

Remark 2.1. Existence and uniqueness of solutions of (2.3) will be justified a posteriori. Indeed, we will show in Lemma 4.1 below that the function f^0 is k-Lipschitz so that existence and uniqueness are guaranteed.

Under the above assumptions, we will state the main result of this paper which gives nearness of solutions of (1.1) and (2.3) on finite time intervals.

Theorem 2.2. *Let assumptions (H1), (H2), and (H3) hold. Let $\phi \in \mathcal{C}_0$ and $t_0 \in \mathbb{R}$. Let x be the solution of (1.1) and y the solution of (2.3) with $x_{t_0} = y_{t_0} = \phi$. Then for any $\delta > 0$ and $T > t_0$, $T \in J$, there exists $\varepsilon_0 = \varepsilon_0(\delta,T) > 0$ such that, for $\varepsilon \in (0,\varepsilon_0]$, x is defined at least on $[t_0,T]$ and $|x(t) - y(t)| < \delta$ on $t \in [t_0,T]$.*

3. Nonstandard main result

3.1. Internal set theory

In IST we adjoin to ordinary mathematics (say ZFC) a new undefined unary predicate *standard* (st). The axioms of IST are the usual axioms of ZFC plus three others which govern the use of the new predicate. Hence, *all theorems* of ZFC *remain valid* in IST. What is new in IST is an addition, not a change. We call a formula of IST *external* in the case where it involves the new predicate *st*; otherwise, we call it *internal*. Thus internal formulas are the formulas of ZFC. The theory IST is a *conservative extension* of ZFC, that is, every internal theorem of IST is a theorem of ZFC. Some of the theorems which are proved in IST are external and can be reformulated so that they become internal. Indeed, there is a *reduction algorithm* which reduces any external formula $F(x_1,\dots,x_n)$ of IST without other free variables than $x_1,\dots,x_n$, to an internal formula $F'(x_1,\dots,x_n)$ with the same free variables, such that $F \equiv F'$, that is, $F \Leftrightarrow F'$ for all standard values of the free variables. In other words, any result which may be formalized within IST by a formula $F(x_1,\dots,x_n)$ is equivalent to the classical property $F'(x_1,\dots,x_n)$, provided the parameters $x_1,\dots,x_n$ are restricted to standard values. Here is the reduction of the frequently occurring formula $\forall x\,(\forall^{\mathrm{st}} y\; A \Rightarrow \forall^{\mathrm{st}} z\; B)$ where A and B are internal formulas

$$\forall x\,\big(\forall^{\mathrm{st}} y\; A \Longrightarrow \forall^{\mathrm{st}} z\; B\big) \equiv \forall z\; \exists^{\mathrm{fin}} y'\; \forall x\,(\forall y \in y'\; A \Longrightarrow B). \tag{3.1}$$

The notations $\forall^{\mathrm{st}} X$ and $\exists^{\mathrm{fin}} X$ stand for $[\forall X,\ \mathrm{st}(X) \Rightarrow \cdots]$ and $[\exists X,\ X$ finite $\& \ldots]$, respectively.

A real number x is called *infinitesimal* when $|x| < a$ for all standard $a > 0$, *limited* when $|x| \leq a$ for some standard a, *appreciable* when it is limited and not infinitesimal, and *unlimited*, when it is not limited. We use the following notations: $x \simeq 0$ for x infinitesimal, $x \simeq +\infty$ for x unlimited positive, $x \gg 0$ for x noninfinitesimal positive. Thus we have

$$\begin{gathered} x \simeq 0 \Longleftrightarrow \forall^{\mathrm{st}} a > 0\; |x| < a, \\ x \gg 0 \Longleftrightarrow \exists^{\mathrm{st}} a > 0\; x \geq a, \\ x \text{ limited} \Longleftrightarrow \exists^{\mathrm{st}} a > 0\; |x| \leq a, \\ x \simeq +\infty \Longleftrightarrow \forall^{\mathrm{st}} a > 0\; x > a. \end{gathered} \tag{3.2}$$

Let (E,d) be a standard metric space. Two points x and y in E are called *infinitely close*, denoted by $x \simeq y$, when $d(x,y) \simeq 0$. If there exists in that space a standard x_0 such that $x \simeq x_0$, the element x is called *near-standard* in E and the standard point x_0 is called the *standard part* of x (it is unique) and is also denoted by ${}^{o}x$. A vector in $\mathbb{R}^d$ (d standard) is said to be *infinitesimal* (resp., *limited*) if its norm $|x|$ is infinitesimal (resp., limited), where $|\cdot|$ is a norm in $\mathbb{R}^d$.

We may not use external formulas to define subsets. The notations $\{x \in \mathbb{R} : x \text{ is limited}\}$ or $\{x \in \mathbb{R} : x \simeq 0\}$ are not allowed. Moreover, we can prove the following lemma.

Lemma 3.1. *There do not exist subsets $\mathcal{L}$ and $\mathcal{I}$ of $\mathbb{R}$ such that, for all $x \in \mathbb{R}$, x is in $\mathcal{L}$ if and only if x is limited, or x is in $\mathcal{I}$ if and only if x is infinitesimal.*

It happens sometimes in classical mathematics that a property is assumed, or proved, on a certain domain, and that afterwards it is noticed that the character of the property and the nature of the domain are incompatible. So actually the property must be valid on a larger domain. In the same manner, in nonstandard analysis, the result of Lemma 3.1 is frequently used to prove that the validity of a property exceeds the domain where it was established in direct way. Suppose that we have shown that A holds for every limited x, then we know that A holds for some unlimited x, for otherwise we could let $\mathcal{L} = \{x \in \mathbb{R} : A\}$. This statement is called the *Cauchy principle*. It has the following consequence.

Lemma 3.2 (Robinson's lemma). *Let g be a real function such that $g(t) \simeq 0$ for all limited $t \geq 0$, then there exists an unlimited positive number ω such that $g(t) \simeq 0$ for all $t \in [0,\omega]$.*

Proof. The set of all s such that for all $t \in [0,s]$ we have $|g(t)| < 1/s$ contains all limited $s \geq 1$. By the Cauchy principle it must contain some unlimited ω. □

We conclude this section with two other applications of the Cauchy principle which will be used later.

Lemma 3.3. *If $\mathcal{P}(\cdot)$ is an internal property such that $\mathcal{P}(\lambda)$ holds for all appreciable real numbers $\lambda > 0$, then there exists $0 < \lambda_0 \simeq 0$ such that $\mathcal{P}(\lambda_0)$ holds.*

Lemma 3.4. *Let $h : I \to \mathbb{R}$ be a function such that $h(t) \simeq 0$ for all $t \in I$. Then* $\sup\{h(t) : t \in I\} \simeq 0$.

Remark 3.5. The use of nonstandard analysis in perturbation theory of differential equations goes back to the seventies with the Reebian school (see [17, 18] and the references therein). It gave birth to the nonstandard perturbation theory of differential equations which has become today a well-established tool in asymptotic theory. For more informations on nonstandard analysis and its applications, the reader is referred to [2, 3, 4, 20, 21, 23].

3.2. Main result: nonstandard formulation

Hereafter we give the nonstandard formulation of Theorem 2.2. Then, by use of the reduction algorithm, we show that the reduction of Theorem 3.6 is Theorem 2.2.

Theorem 3.6. *Let $f : \mathbb{R} \times \mathcal{C}_0 \to \mathbb{R}^d$ be standard. Assume that assumptions (H1), (H2), and (H3) hold. Let $\phi \in \mathcal{C}_0$ and $t_0 \in \mathbb{R}$ be standard. Let x be the solution of (1.1) and y the solution of (2.3) with $x_{t_0} = y_{t_0} = \phi$. Let $\varepsilon > 0$ be infinitesimal. Then for any standard $T > t_0$, $T \in J$, x is defined at least on $[t_0,T]$ and $x(t) \simeq y(t)$ for all $t \in [t_0,T]$.*

The proof of Theorem 3.6 is postponed to Section 4. Theorem 3.6 is an external statement. As we have recalled, Nelson [20] proposed a reduction algorithm that reduces external theorems to equivalent internal forms. We show that the reduction of Theorem 3.6 is Theorem 2.2.

Reduction of Theorem 3.6. Without loss of generality, let $t_0 = 0$. Let $T > 0$, $T \in J$, and T standard. The characterization of the conclusion of

Theorem 3.6 is

$$\begin{aligned} \forall\varepsilon : \varepsilon \simeq 0 \Longrightarrow x \text{ is defined at least on } [0,T] \\ \text{and } x(t) \simeq y(t) \text{ for all } t \in [0,T]. \end{aligned} \tag{3.3}$$

Let B be the formula "If $\delta > 0$ then x is defined at least on $[0,T]$ and $|x(t) - y(t)| < \delta$ on $t \in [0,T]$." Using (3.2), formula (3.3) becomes

$$\forall\varepsilon\ (\forall^{\text{st}}\eta\ \varepsilon < \eta \Longrightarrow \forall^{\text{st}}\delta\ B). \tag{3.4}$$

In this formula T is standard and ε, η, and δ range over the strictly positive real numbers. By (3.1), formula (3.4) is equivalent to

$$\forall\delta\ \exists^{\text{fin}}\eta'\ \forall\varepsilon\ (\forall\eta \in \eta'\ \varepsilon < \eta \Longrightarrow B). \tag{3.5}$$

For η' a finite set, $\forall\eta \in \eta'$, $\varepsilon < \eta$ is the same as $\varepsilon < \varepsilon_0$ for $\varepsilon_0 = \min\eta'$, and so formula (3.5) is equivalent to

$$\forall\delta\ \exists\varepsilon_0\ \forall\varepsilon\ (\varepsilon < \varepsilon_0 \Longrightarrow B). \tag{3.6}$$

That is the statement of Theorem 2.2 holds for any standard $T > 0$, $T \in J$. By transfer, it holds for any $T > 0$, $T \in J$.

4. Proof of Theorem 3.6

4.1. Preliminary lemmas

In this subsection we state some results which are needed for our proof of Theorem 3.6. Let $f : \mathbb{R} \times \mathcal{C}_0 \to \mathbb{R}^d$ be standard. The external formulations of conditions (H1), (H2), and (H3) are, respectively,

(H1′) $\forall^{\text{st}}\tau \in \mathbb{R}\ \forall^{\text{st}}u \in \mathcal{C}_0\ \forall\tau' \in \mathbb{R}\ \forall u' \in \mathcal{C}_0$:

$$\tau' \simeq \tau,\ u' \simeq u \Longrightarrow f(\tau,u') \simeq f(\tau,u). \tag{4.1}$$

(H2′) There is a standard constant k such that

$$\left|f(\tau,u_1) - f(\tau,u_2)\right| \le k\left\|u_1 - u_2\right\|, \quad \forall^{\text{st}}\tau \in \mathbb{R},\ \forall^{\text{st}}u_1,u_2 \in \mathcal{C}_0 \tag{4.2}$$

(and by transfer the inequality holds for all $\tau \in \mathbb{R}$ and all $u_1,u_2 \in \mathcal{C}_0$).

(H3′) There is a standard functional $f^0 : \mathcal{C}_0 \to \mathbb{R}^d$ such that

$$\forall^{\text{st}}u \in \mathcal{C}_0,\ \forall T \simeq +\infty, \quad f^0(u) \simeq \frac{1}{T}\int_0^T f(\tau,u)\,d\tau. \tag{4.3}$$

We prove the following lemmas.

Lemma 4.1. *The functional f^0 is Lipschitz (with the same constant of Lipschitz as f) and satisfies*

$$f^0(u) \simeq \frac{1}{T}\int_0^T f(\tau,u)\,d\tau \tag{4.4}$$

for all $u \in \mathcal{C}_0$, u nearstandard, and all $T \simeq +\infty$.

Proof. First, let $u_1, u_2 \in \mathcal{C}_0$, with u_1 and u_2 standard. By means of conditions (H2) and (H3), we have

$$\left|f^0(u_1) - f^0(u_2)\right| \leq \lim_{T\to\infty}\frac{1}{T}\int_0^T \left|f(\tau,u_1) - f(\tau,u_2)\,d\tau\right| \leq k\|u_1 - u_2\|. \tag{4.5}$$

That is, f^0 is k-Lipschitz.

Next, let $u, {}^0u \in \mathcal{C}_0$ such that ${}^ou$ is standard and $u \simeq {}^0u$. By means of (4.5), conditions (H3′) and (H2′), respectively, for all $T \simeq +\infty$, we have

$$f^0(u) \simeq f^0({}^0u) \simeq \frac{1}{T}\int_0^T f(\tau,{}^0u)\,d\tau \simeq \frac{1}{T}\int_0^T f(\tau,u)\,d\tau. \tag{4.6}$$

□

Lemma 4.2. *There exists $\mu > 0$ such that whenever $t \geq 0$ is limited and $u \in \mathcal{C}_0$ is nearstandard there exists $\alpha > 0$ such that $\mu < \alpha \simeq 0$ and*

$$\frac{\varepsilon}{\alpha}\int_{t/\varepsilon}^{t/\varepsilon+\alpha/\varepsilon} f(\tau,u)\,d\tau \simeq f^0(u). \tag{4.7}$$

Proof. Let $t \geq 0$ be limited and let $u \in \mathcal{C}_0$ be nearstandard.

Case 1 (t is such that t/ε is limited). Let $S > 0$ be unlimited such that $\varepsilon S \simeq 0$. Then

$$\frac{1}{S}\int_{t/\varepsilon}^{t/\varepsilon+S} f(\tau,u)\,d\tau = \left(1 + \frac{t}{\varepsilon S}\right)\frac{1}{t/\varepsilon + S}\int_0^{t/\varepsilon+S} f(\tau,u)\,d\tau - \frac{1}{S}\int_0^{t/\varepsilon} f(\tau,u)\,d\tau. \tag{4.8}$$

By Lemma 4.1 we have

$$\frac{1}{t/\varepsilon + S}\int_0^{t/\varepsilon+S} f(\tau,u)\,d\tau \simeq f^0(u). \tag{4.9}$$

Since

$$\frac{1}{S}\int_0^{t/\varepsilon} f(\tau,u)\,d\tau \simeq 0, \qquad \frac{t}{\varepsilon S} \simeq 0 \tag{4.10}$$

we have

$$\frac{1}{S}\int_{t/\varepsilon}^{t/\varepsilon+S} f(\tau,u)d\tau \simeq f^0(u). \tag{4.11}$$

Then, it suffices to choose $\mu=\varepsilon$ and take $\alpha=\varepsilon S$.
Case 2 (t is such that t/ε is unlimited). Let $S>0$. We write

$$\frac{1}{S}\int_{t/\varepsilon}^{t/\varepsilon+S} f(\tau,u)d\tau = \frac{1}{t/\varepsilon+S}\int_0^{t/\varepsilon+S} f(\tau,u)d\tau + \frac{t}{\varepsilon S}\left(\frac{1}{t/\varepsilon+S}\int_0^{t/\varepsilon+S} f(\tau,u)d\tau - \frac{1}{t/\varepsilon}\int_0^{t/\varepsilon} f(\tau,u)d\tau\right). \tag{4.12}$$

By Lemma 4.1 we have

$$\frac{1}{t/\varepsilon+S}\int_0^{t/\varepsilon+S} f(\tau,u)d\tau \simeq f^0(u) \simeq \frac{1}{t/\varepsilon}\int_0^{t/\varepsilon} f(\tau,u)d\tau. \tag{4.13}$$

We denote

$$\eta(S) = \frac{t}{\varepsilon S}\left(\frac{1}{t/\varepsilon+S}\int_0^{t/\varepsilon+S} f(\tau,u)d\tau - \frac{1}{t/\varepsilon}\int_0^{t/\varepsilon} f(\tau,u)d\tau\right). \tag{4.14}$$

The quantity $\eta(S)$ is infinitesimal for all S such that $t/\varepsilon S$ is limited. By Lemma 3.2 this property holds for some S for which $t/\varepsilon S$ is unlimited. The real number S can be chosen so that $S>1$ and $t/\varepsilon S \simeq +\infty$. Since t is limited we have $\varepsilon S \simeq 0$. Then, it suffices to choose $\mu=\varepsilon$ and take $\alpha=\varepsilon S$. □

Lemma 4.3. *Let $\phi\in\mathcal{C}_0$ be standard. Let y be the solution of (2.3) on J with $y_0=\phi$, and let $T_1>0$ be standard such that $[0,T_1]\subset J$. Then there exist some positive integer N_0 and some infinitesimal partition $\{t_n : n=0,\dots,N_0+1\}$ of $[0,T_1]$ such that $t_0=0$, $t_{N_0}\le T_1<t_{N_0+1}$, $t_{n+1}=t_n+\alpha_n\simeq t_n$, and*

$$\frac{\varepsilon}{\alpha_n}\int_{t_n/\varepsilon}^{t_n/\varepsilon+\alpha_n/\varepsilon} f(\tau,y_{t_n})d\tau \simeq f^0(y_{t_n}). \tag{4.15}$$

Proof. It will be done in two steps.
Step 1. Let $t\in[0,T_1]$ and let us show that y_t is nearstandard.

As $y([-r,T_1])$ is a standard compact subset of $\mathbb{R}^d$, it suffices to show that y_t is S-continuous on $[-r,0]$ to deduce that it is nearstandard. Taking into account that $|f^0(0)|$ is standard, for $\theta,\theta' \in [-r,0]$, $\theta \le \theta'$, and $\theta \simeq \theta'$, we have

$$\begin{aligned} \left|y_t(\theta') - y_t(\theta)\right| &= \left|y(t+\theta') - y(t+\theta)\right| \\ &\le \int_{t+\theta}^{t+\theta'} \left|f^0(y_s)\right| ds \\ &\le \int_{t+\theta}^{t+\theta'} \left|f^0(y_s) - f^0(0)\right| ds + \int_{t+\theta}^{t+\theta'} \left|f^0(0)\right| ds \\ &\le k\int_{t+\theta}^{t+\theta'} \|y_s\| ds + (\theta' - \theta)\left|f^0(0)\right| \simeq 0. \end{aligned} \tag{4.16}$$

That is, y_t is S-continuous on $[-r,0]$.
Step 2. Let $\mu > 0$ be given as in Lemma 4.2, and define the set $A_\mu = \{\lambda \in \mathbb{R} \ / \ \forall t \in [0,T_1] \ \exists \alpha \in \mathbb{R} : \mathcal{P}_\mu(t,\alpha,\lambda)\}$ where

$$\mathcal{P}_\mu(t,\alpha,\lambda) \equiv \mu < \alpha < \lambda, \qquad \left|\frac{\varepsilon}{\alpha}\int_{t/\varepsilon}^{t/\varepsilon+\alpha/\varepsilon} f(\tau,y_t)d\tau - f^0(y_t)\right| < \lambda. \tag{4.17}$$

By Lemma 4.2 the set A_μ contains all the standard real numbers $\lambda > 0$. By Lemma 3.3 there exists $\lambda_0 \simeq 0$ in A_μ, that is, there exists $0 < \lambda_0 \simeq 0$ such that for all $t \in [0,T_1]$ there exists $\alpha \in \mathbb{R}$ such that $\mathcal{P}_\mu(t,\alpha,\lambda_0)$ holds. By the axiom of choice there exists a function $c : [0,T_1] \to \mathbb{R}$ such that $c(t) = \alpha$, that is, $\mathcal{P}_\mu(t,c(t),\lambda_0)$ holds for all $t \in [0,T_1]$. Since $c(t) > \mu$ for all $t \in [0,T_1]$, the conclusion of the lemma is immediate. □

Lemma 4.4. *Let $\phi \in \mathcal{C}_0$ be standard. Let y be the solution of (2.3) on J with $y_0 = \phi$, and let $T_1 > 0$ be standard such that $[0,T_1] \subset J$. Then for all $t \in [0,T_1]$,*

$$\int_0^t f\left(\frac{\tau}{\varepsilon}, y_\tau\right) d\tau \simeq \int_0^t f^0(y_\tau)d\tau. \tag{4.18}$$

Proof. Let $f_1(\tau,u) := f(\tau,u) - f^0(u)$, for $\tau \in \mathbb{R}$ and $u \in \mathcal{C}_0$. The functional f_1 is Lipschitzian in $u \in \mathcal{C}_0$, that is, there exists some standard constant k_1 ($k_1 = 2k$, where k is the Lipschitz constant of f) such that

$$\left|f_1(\tau,u_1) - f_1(\tau,u_2)\right| \le k_1\|u_1 - u_2\|, \quad \forall \tau \in \mathbb{R},\ \forall u_1,u_2 \in \mathcal{C}_0. \tag{4.19}$$

Next, by Lemma 4.3 there exists $\{t_n : n = 0,\dots,N_0+1\}$ such that $t_0 = 0$, $t_{N_0} \le T_1 < t_{N_0+1}$, $t_{n+1} = t_n + \alpha_n \simeq t_n$, and

$$\frac{\varepsilon}{\alpha_n}\int_{t_n/\varepsilon}^{t_n/\varepsilon+\alpha_n/\varepsilon} f_1(\tau,y_{t_n})d\tau \simeq 0. \tag{4.20}$$

Let $t \in [0,T_1]$ and let N be a positive integer such that $t_N \le t < t_{N+1}$. We have

$$\begin{aligned}
\left|\int_{t_N}^{t} f_1\left(\frac{\tau}{\varepsilon},y_\tau\right)d\tau\right| &\le \left|\int_{t_N}^{t} f_1\left(\frac{\tau}{\varepsilon},y_\tau\right)d\tau - \int_{t_N}^{t} f_1\left(\frac{\tau}{\varepsilon},0\right)d\tau\right| \\
&\quad + \left|\int_{t_N}^{t} f_1\left(\frac{\tau}{\varepsilon},0\right)d\tau\right| \\
&\le \int_{t_N}^{t}\left|f_1\left(\frac{\tau}{\varepsilon},y_\tau\right) - f_1\left(\frac{\tau}{\varepsilon},0\right)\right|d\tau + \left|\int_{t_N}^{t} f_1\left(\frac{\tau}{\varepsilon},0\right)d\tau\right| \\
&\le k_1\int_{t_N}^{t}\|y_\tau\|d\tau + \left|\int_{t_N}^{t} f_1\left(\frac{\tau}{\varepsilon},0\right)d\tau\right|.
\end{aligned} \tag{4.21}$$

As $y([-r,T_1])$ is a standard compact subset of $\mathbb{R}^d$, it follows that

$$\int_{t_N}^{t}\|y_\tau\|d\tau \simeq 0. \tag{4.22}$$

We now estimate the second term in the right-hand side of (4.21). For this, consider all the cases.
Case 1 (Both t_N/ε and t/ε are limited). In this case, it is clear that

$$\left|\int_{t_N}^{t} f_1\left(\frac{\tau}{\varepsilon},0\right)d\tau\right| = \varepsilon\left|\int_{t_N/\varepsilon}^{t/\varepsilon} f_1(s,0)ds\right| \simeq 0. \tag{4.23}$$

Case 2 (Both t_N/ε and t/ε are unlimited). By means of condition (H3′), we have

$$\begin{aligned}
\left|\int_{t_N}^{t} f_1\left(\frac{\tau}{\varepsilon},0\right)d\tau\right| &= \varepsilon\left|\int_{t_N/\varepsilon}^{t/\varepsilon} f_1(s,0)ds\right| \\
&\le t_N\left|\frac{1}{t_N/\varepsilon}\int_0^{t_N/\varepsilon} f_1(s,0)ds\right| + t\left|\frac{1}{t/\varepsilon}\int_0^{t/\varepsilon} f_1(s,0)ds\right| \simeq 0.
\end{aligned} \tag{4.24}$$

Case 3 (t_N/ε is limited and t/ε is unlimited). This case is a combination of Cases 1 and 2. We write

$$\begin{aligned}\left|\int_{t_N}^{t} f_1\left(\frac{\tau}{\varepsilon},0\right)d\tau\right| &= \varepsilon\left|\int_{t_N/\varepsilon}^{t/\varepsilon} f_1(s,0)ds\right| \\ &\le \varepsilon\left|\int_{0}^{t_N/\varepsilon} f_1(s,0)ds\right| + t\left|\frac{1}{t/\varepsilon}\int_{0}^{t/\varepsilon} f_1(s,0)ds\right| \\ &\simeq 0.\end{aligned} \tag{4.25}$$

Thus, we have

$$\left|\int_{t_N}^{t} f_1\left(\frac{\tau}{\varepsilon},0\right)d\tau\right| \simeq 0. \tag{4.26}$$

Therefore, from (4.21) and by means of (4.22) and (4.26), we obtain that

$$\left|\int_{t_N}^{t} f_1\left(\frac{\tau}{\varepsilon},y_\tau\right)d\tau\right| \simeq 0, \tag{4.27}$$

so that

$$\begin{aligned}&\int_0^t f\left(\frac{\tau}{\varepsilon},y_\tau\right)d\tau - \int_0^t f^0(y_\tau)d\tau \\ &\quad= \int_0^t f_1\left(\frac{\tau}{\varepsilon},y_\tau\right)d\tau \\ &\quad\simeq \sum_{n=0}^{N-1}\int_{t_n}^{t_{n+1}} f_1\left(\frac{\tau}{\varepsilon},y_\tau\right)d\tau \\ &\quad= \sum_{n=0}^{N-1}\int_{t_n}^{t_{n+1}} \left(f_1\left(\frac{\tau}{\varepsilon},y_\tau\right) - f_1\left(\frac{\tau}{\varepsilon},y_{t_n}\right)\right)d\tau \\ &\qquad+ \sum_{n=0}^{N-1}\int_{t_n}^{t_{n+1}} f_1\left(\frac{\tau}{\varepsilon},y_{t_n}\right)d\tau.\end{aligned} \tag{4.28}$$

Let $\tau \in [t_n,t_{n+1}]$, $n = 0,\ldots,N$, and let $\theta \in [-r,0]$. We have

$$\begin{aligned}\left|y_\tau(\theta) - y_{t_n}(\theta)\right| &= \left|y(\tau+\theta) - y(t_n+\theta)\right| \\ &\le \int_{t_n+\theta}^{\tau+\theta} \left|f^0(y_s)\right|ds\end{aligned}$$

$$\begin{aligned}&\le \int_{t_n+\theta}^{\tau+\theta} \left|f^0(y_s) - f^0(0)\right| ds + \int_{t_n+\theta}^{\tau+\theta} \left|f^0(0)\right| ds\\ &\le k \int_{t_n+\theta}^{\tau+\theta} \|y_s\| ds + (\tau - t_n)\left|f^0(0)\right|.\end{aligned} \tag{4.29}$$

As $y([-r,T_1])$ is a standard compact subset of $\mathbb{R}^d$ and $|f^0(0)|$ is standard, from (4.29) we deduce that

$$\left|y_\tau(\theta) - y_{t_n}(\theta)\right| \simeq 0. \tag{4.30}$$

That is, $y_\tau \simeq y_{t_n}$ for $\tau \in [t_n, t_{n+1}]$, $n = 0, \dots, N$. By Lemma 3.4, we have

$$\sup\left\{\|y_\tau - y_{t_n}\| : \tau \in \left[t_n, t_{n+1}\right],\ 0 \le n \le N-1\right\} \simeq 0 \tag{4.31}$$

and so is

$$k_1 \cdot \sup\left\{\|y_\tau - y_{t_n}\| : \tau \in \left[t_n, t_{n+1}\right],\ 0 \le n \le N-1\right\} \cdot t_N, \tag{4.32}$$

so that

$$\begin{aligned}&\left|\sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} f_1\left(\frac{\tau}{\varepsilon}, y_\tau\right) - f_1\left(\frac{\tau}{\varepsilon}, y_{t_n}\right) d\tau\right|\\ &\quad \le \sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} \left|f_1\left(\frac{\tau}{\varepsilon}, y_\tau\right) - f_1\left(\frac{\tau}{\varepsilon}, y_{t_n}\right)\right| d\tau\\ &\quad \le k_1 \sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} \|y_\tau - y_{t_n}\| d\tau\\ &\quad \le k_1 \cdot \sup\left\{\|y_\tau - y_{t_n}\| : \tau \in \left[t_n, t_{n+1}\right],\ 0 \le n \le N-1\right\} \cdot t_N\\ &\quad \simeq 0.\end{aligned} \tag{4.33}$$

Therefore, from (4.28) and by means of (4.20), it follows that

$$\begin{aligned}
&\int_0^t f\left(\frac{\tau}{\varepsilon}, y_\tau\right) d\tau - \int_0^t f^0(y_\tau)\, d\tau \\
&\quad \simeq \sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} f_1\left(\frac{\tau}{\varepsilon}, y_{t_n}\right) d\tau \\
&\quad = \sum_{n=0}^{N-1} \int_{t_n}^{t_n+\alpha_n} f_1\left(\frac{\tau}{\varepsilon}, y_{t_n}\right) d\tau \\
&\quad = \varepsilon \sum_{n=0}^{N-1} \int_{t_n/\varepsilon}^{t_n/\varepsilon+\alpha_n/\varepsilon} f_1(\tau, y_{t_n})\, d\tau \\
&\quad = \sum_{n=0}^{N-1} \alpha_n \left(\frac{\varepsilon}{\alpha_n} \int_{t_n/\varepsilon}^{t_n/\varepsilon+\alpha_n/\varepsilon} f_1(\tau, y_{t_n})\, d\tau \right) \\
&\quad = \sum_{n=0}^{N-1} \alpha_n \beta_n \\
&\quad \simeq 0
\end{aligned} \tag{4.34}$$

since $|\sum_{n=0}^{N-1} \alpha_n \cdot \beta_n| \leq \overline{\beta} \sum_{n=0}^{N-1} \alpha_n = \overline{\beta} \sum_{n=0}^{N-1} (t_{n+1} - t_n) = \overline{\beta} \cdot t_N$, where $\overline{\beta} = \max\{|\beta_n| : 0 \leq n \leq N-1\}$. By Lemma 3.4, $\overline{\beta}$ is infinitesimal and so is $\overline{\beta} \cdot t_N$. This completes the proof of Lemma 4.4. □

Lemma 4.5. *Let $\phi \in \mathcal{C}_0$ be standard. Let x be the solution of (1.1) on I, and y the solution of (2.3) on J, with $x_0 = y_0 = \phi$. Let $T_1 > 0$ be standard such that $[0, T_1] \subset I \cap J$. Then $x(t) \simeq y(t)$ for all $t \in [0, T_1]$.*

Proof. For $t \in [0, T_1]$, we have

$$x(t) = \phi(0) + \int_0^t f\left(\frac{\tau}{\varepsilon}, x_\tau\right) d\tau, \tag{4.35}$$

$$y(t) = \phi(0) + \int_0^t f^0(y_\tau)\, d\tau. \tag{4.36}$$

Subtraction of (4.35) and (4.36) gives

$$\begin{aligned}
|x(t) - y(t)| \leq &\left| \int_0^t \left(f\left(\frac{\tau}{\varepsilon}, x_\tau\right) - f\left(\frac{\tau}{\varepsilon}, y_\tau\right)\right) d\tau \right| \\
&+ \left| \int_0^t \left(f\left(\frac{\tau}{\varepsilon}, y_\tau\right) - f^0(y_\tau)\right) d\tau \right|
\end{aligned}$$

$$
\begin{aligned}
&\le \int_0^t \left| f\left(\frac{\tau}{\varepsilon}, x_\tau\right) - f\left(\frac{\tau}{\varepsilon}, y_\tau\right) \right| d\tau \\
&\quad + \left| \int_0^t \left(f\left(\frac{\tau}{\varepsilon}, y_\tau\right) - f^0(y_\tau) \right) d\tau \right| \\
&\le k \int_0^t \|x_\tau - y_\tau\| d\tau + \left| \int_0^t \left(f\left(\frac{\tau}{\varepsilon}, y_\tau\right) - f^0(y_\tau) \right) d\tau \right|.
\end{aligned}
\tag{4.37}
$$

Since, for $\tau \in [0,t]$, $\|x_\tau - y_\tau\| \le \sup_{s\in[0,\tau]} |x(s) - y(s)|$, it follows from (4.37) that

$$
\begin{aligned}
|x(t) - y(t)| \le k \int_0^t \sup_{s\in[0,\tau]} |x(s) - y(s)| d\tau \\
+ \left| \int_0^t \left(f\left(\frac{\tau}{\varepsilon}, y_\tau\right) - f^0(y_\tau) \right) d\tau \right|.
\end{aligned}
\tag{4.38}
$$

The first term on the right-hand side of (4.38) is increasing and therefore

$$
\begin{aligned}
\sup_{\tau\in[0,t]} |x(\tau) - y(\tau)| \le k \int_0^t \sup_{s\in[0,\tau]} |x(s) - y(s)| d\tau \\
+ \sup_{\tau\in[0,t]} \left| \int_0^\tau \left(f\left(\frac{s}{\varepsilon}, y_s\right) - f^0(y_s) \right) ds \right|.
\end{aligned}
\tag{4.39}
$$

By Gronwall's lemma, this implies that

$$
\sup_{\tau\in[0,t]} |x(\tau) - y(\tau)| \le e^{kt} \sup_{\tau\in[0,t]} \left| \int_0^\tau \left(f\left(\frac{s}{\varepsilon}, y_s\right) - f^0(y_s) \right) ds \right| \tag{4.40}
$$

and by means of Lemmas 4.4 and 3.4, respectively, we conclude that $x(t) \simeq y(t)$, which finishes the proof. □

4.2. Proof of Theorem 3.6

For notation simplicity, we let $t_0 = 0$. Let $T > 0$ be standard in J. Let K be a standard tubular neighborhood of diameter ρ around $\Gamma = y([0,T])$. Let I be the maximal interval of definition of x. Define the set $A = \{T_1 \in I \cap [0,T] \ / \ x([0,T_1]) \subset K\}$. The set A is nonempty ($0 \in A$) and bounded above by T. Let T_0 be a lower upper bound of A. There is $T_1 \in A$ such that $T_0 - \varepsilon^2 < T_1 \le T_0$. By continuation, there is T_2, T_2 appreciable, such that x remains defined on $[0, T_1 + \varepsilon T_2]$. Likewise, by continuation, y remains defined in particular on the same interval. By Lemma 4.5, we have $x(t) \simeq y(t)$ for $t \in [0, T_1 + \varepsilon T_2]$. Suppose $T_1 + \varepsilon T_2 \le T$. Then, $[0, T_1 + \varepsilon T_2] \subset I$ and

$x([0,T_1+\varepsilon T_2]) \subset K$ imply that $T_1+\varepsilon T_2 \in A$, which is a contradiction with $T_1+\varepsilon T_2 > T_0$. Thus $T_1+\varepsilon T_2 > T$, that is, we have $x(t) \simeq y(t)$ for all $t \in [0,T] \subset [0,T_1+\varepsilon T_2]$.

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