BOUNDARY VALUE PROBLEM WITH INTEGRAL CONDITIONS FOR A LINEAR THIRD-ORDER EQUATION

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Received 6 March 2003 and in revised form 29 July 2003

We prove the existence and uniqueness of a strong solution for a linear third-order equation with integral boundary conditions. The proof uses energy inequalities and the density of the range of the generated operator.

1. Introduction

In the rectangle $\Omega = [0,1] \times [0,T]$, we consider the equation

$$\pounds u = \frac{\partial^3 u}{\partial t^3} + \frac{\partial}{\partial x}\left(a(x,t)\frac{\partial u}{\partial x}\right) = f(x,t), \tag{1.1a}$$

with the initial conditions

$$u(x,0) = 0, \quad \frac{\partial u}{\partial t}(x,0) = 0, \quad x \in (0,1), \tag{1.1b}$$

the final condition

$$\frac{\partial^2 u}{\partial t^2}(x,T) = 0, \quad x \in (0,1), \tag{1.1c}$$

the Dirichlet condition

$$u(0,t) = 0, \quad \forall t \in (0,T), \tag{1.1d}$$

Journal of Applied Mathematics 2003:11 (2003) 553–567
2000 Mathematics Subject Classification: 35B45, 35G10
URL: http://dx.doi.org/10.1155/S1110757X03303031

and the integral condition

$$\int_0^1 u(x,t)\,dx = 0, \quad \forall t \in (0,T). \tag{1.1e}$$

In addition, we assume that the function $a(x,t)$ is bounded with

$$0 < a_0 \le a(x,t) \le a_1, \tag{1.2}$$

and has bounded partial derivatives such that

$$\begin{gathered} c'_k \le \frac{\partial^k a}{\partial t^k}(x,t) \le c_k, \quad \forall x \in (0,1),\ t \in (0,T),\ k = \overline{1,3},\ \text{with } c'_1 \ge 0, \\ \left|\frac{\partial a}{\partial x}(x,t)\right| \le b_1, \quad \text{for } (x,t) \in \Omega. \end{gathered} \tag{1.3}$$

Various problems arising in heat conduction [4, 6, 14, 15], chemical engineering [9], underground water flow [13], thermoelasticity [21], and plasmaphysics [19] can be reduced to the nonlocal problems with integral boundary conditions. This type of boundary value problems has been investigated in [1, 2, 3, 5, 6, 7, 9, 14, 15, 16, 20, 23] for parabolic equations, in [18, 22] for hyperbolic equations, and in [10, 11, 12] for mixed-type equations. The basic tool in [4, 10, 11, 12, 16, 23] is the energy inequality method which, of course, requires appropriate multipliers and functional spaces. In this paper, we extend this method to the study of a linear third-order partial differential equation. This type of problems is encountered in the study of thermal conductivity [17] and microscale heat transfer [8].

2. Preliminaries

In this paper, we prove the existence and uniqueness of a strong solution of problem (1.1). For this, we consider the solution of problem (1.1) as a solution of the operator equation $Lu = \mathcal{F}$, where L is the operator with domain of definition $D(L)$ consisting of functions $u \in E$ such that $\sqrt{1-x}(\partial^{k+1}u/\partial t^k \partial x)(x,t) \in L^2(\Omega)$, $k = \overline{0,3}$ and u satisfies conditions (1.1d) and (1.1e). The operator L is considered from E to F, where E is the Banach space of the functions u, $u \in L^2(\Omega)$, with the finite norm

$$\begin{aligned} \|u\|_E^2 = & \int_\Omega \frac{(1-x)^2}{2}\left\{\left|\frac{\partial^3 u}{\partial t^3}\right|^2 + \left|\frac{\partial^2 u}{\partial x^2}\right|^2\right\} dx\,dt \\ & + \int_\Omega \left(\frac{(1-x)^2}{2}\left|\frac{\partial u}{\partial x}\right|^2 + |u|^2\right) dx\,dt, \end{aligned} \tag{2.1}$$

and F is the Hilbert space of the functions $\mathcal{F} = (f,0,0,0)$, $f \in L^2(\Omega)$, with the finite norm

$$\|\mathcal{F}\|_F^2 = \int_\Omega (1-x)^2 |f|^2 dx\,dt. \tag{2.2}$$

Then we establish an energy inequality

$$\|u\|_E \le k\|Lu\|_F, \quad \forall u \in D(L), \tag{2.3}$$

and we show that the operator L has the closure $\overline{L}$.

Definition 2.1. A solution of the operator equation $\overline{L}u = \mathcal{F}$ is called a strong solution of problem (1.1).

Inequality (2.3) can be extended to $u \in D(\overline{L})$, that is,

$$\|u\|_E \le k\left\|\overline{L}u\right\|_F, \quad \forall u \in D\left(\overline{L}\right). \tag{2.4}$$

From this inequality, we obtain the uniqueness of a strong solution, if it exists, and the equality of the sets $R(\overline{L})$ and $\overline{R(L)}$. Thus, to prove the existence of a strong solution of problem (1.1) for any $\mathcal{F} \in F$, it remains to prove that the set $R(L)$ is dense in F.

3. An energy inequality and its applications

Theorem 3.1. *For any function $u \in D(L)$, there exists the a priori estimate*

$$\|u\|_E \le k\|Lu\|_F, \tag{3.1}$$

where

$$k^2 = \frac{17\exp(ct)\left[5 + 4(b_1)^2/(c_3' - 3cc_2 + 3c^2c_1' - c^3a_1 - b_1^2)\right] + 1}{\min\left(1, a_0^2, c_3' - 3cc_2 + 3c^2c_1' - c^3a_1 - b_1^2\right)}, \tag{3.2}$$

with the constant c satisfying

$$\begin{gathered}
\sup_{(x,t)\in\Omega}\left(\frac{1}{a}\frac{\partial a}{\partial t}\right) \le c < \inf_{(x,t)\in\Omega}\left(\frac{1}{a}\frac{\partial a}{\partial t} + 1\right), \\
c_3' - 3cc_2 + 3c^2c_1' - c^3a_1 - (b_1)^2 > 0, \\
c_2 - 2cc_1' + c^2a_1 - c_1' + ca_1 < 0.
\end{gathered} \tag{3.3}$$

Proof. Let

$$Mu = (1-x)^2 \frac{\partial^3 u}{\partial t^3} + 2(1-x) J_x \frac{\partial^3 u}{\partial t^3}, \tag{3.4}$$

where

$$J_x u = \int_0^x u(\zeta, t)\, d\zeta. \tag{3.5}$$

We consider the quadratic form

$$\Phi(u,u) = \operatorname{Re} \int_\Omega \exp(-ct) \pounds u \overline{Mu}\, dx\, dt, \tag{3.6}$$

with the constant c satisfying (3.3), obtained by multiplying (1.1a) by $\exp(-ct)\overline{Mu}$, integrating over Ω, and taking the real part. Substituting the expression of Mu in (3.6), we obtain

$$\begin{aligned}
&\operatorname{Re} \int_\Omega \exp(-ct) \pounds u \overline{Mu}\, dx\, dt \\
&\quad = \operatorname{Re} \int_\Omega \exp(-ct)(1-x)^2 \left| \frac{\partial^3 u}{\partial t^3} \right|^2 dx\, dt \\
&\qquad + 2 \operatorname{Re} \int_\Omega \exp(-ct)(1-x) \frac{\partial^3 u}{\partial t^3} J_x \overline{\frac{\partial^3 u}{\partial t^3}}\, dx\, dt \\
&\qquad + \operatorname{Re} \int_\Omega \exp(-ct) \frac{\partial}{\partial x}\left(a(x,t) \frac{\partial u}{\partial x} \right) \overline{Mu}\, dx\, dt.
\end{aligned} \tag{3.7}$$

Integrating the last two terms on the right-hand side by parts with respect to x in (3.7) and using the Dirichlet condition (1.1d), we obtain

$$2 \operatorname{Re} \int_0^1 (1-x) \exp(-ct) \frac{\partial^3 u}{\partial t^3} J_x \frac{\partial^3 \overline{u}}{\partial t^3}\, dx = \int_0^1 \exp(-ct) \left| J_x \frac{\partial^3 u}{\partial t^3} \right|^2 dx, \tag{3.8}$$

$$\begin{aligned}
&\operatorname{Re} \int_\Omega \exp(-ct) \frac{\partial}{\partial x}\left(a \frac{\partial u}{\partial x} \right) \overline{Mu}\, dx\, dt \\
&\quad = -\operatorname{Re} \int_\Omega \exp(-ct)(1-x)^2 a \frac{\partial u}{\partial x} \frac{\partial^4 \overline{u}}{\partial t^3 \partial x}\, dx\, dt \\
&\qquad - 2 \operatorname{Re} \int_\Omega \exp(-ct) \frac{\partial a}{\partial x} u J_x \frac{\partial^3 \overline{u}}{\partial t^3}\, dx\, dt \\
&\qquad - 2 \operatorname{Re} \int_\Omega \exp(-ct) a u \frac{\partial^3 \overline{u}}{\partial t^3}\, dx\, dt.
\end{aligned} \tag{3.9}$$

Integrating each term by parts in (3.9) with respect to t and using the initial and final conditions (1.1b) and (1.1c), we get

$$\begin{aligned}
&\operatorname{Re}\int_{\Omega}\exp(-ct)\frac{\partial}{\partial x}\left(a\frac{\partial u}{\partial x}\right)\overline{Mu}\,dx\,dt\\
&\quad=-2\operatorname{Re}\int_{\Omega}\exp(-ct)\frac{\partial a}{\partial x}uJ_x\frac{\partial^3\overline{u}}{\partial t^3}\,dx\,dt\\
&\qquad+\int_{\Omega}\exp(-ct)\left(\frac{\partial^3 a}{\partial t^3}-3c\frac{\partial^2 a}{\partial t^2}+3c^2\frac{\partial a}{\partial t}-c^3a\right)\\
&\qquad\quad\times\left[\frac{(1-x)^2}{2}\left|\frac{\partial u}{\partial x}\right|^2+|u|^2\right]dx\,dt\\
&\qquad-3\int_{\Omega}\exp(-ct)\left(\frac{\partial a}{\partial t}-ca\right)\left[\frac{(1-x)^2}{2}\left|\frac{\partial^2 u}{\partial t\partial x}\right|^2+\left|\frac{\partial u}{\partial t}\right|^2\right]dx\,dt\\
&\qquad+\int_0^1\exp(-ct)a\left[\frac{(1-x)^2}{2}\left|\frac{\partial^2 u}{\partial t\partial x}\right|^2+\left|\frac{\partial u}{\partial t}\right|^2\right]dx\bigg|_{T=t}\\
&\qquad-\int_0^1\exp(-ct)\left(\frac{\partial^2 a}{\partial t^2}-2c\frac{\partial a}{\partial t}+c^2a\right)\left[\frac{(1-x)^2}{2}\left|\frac{\partial u}{\partial x}\right|^2+|u|^2\right]dx\bigg|_{t=T}\\
&\qquad+\operatorname{Re}\int_0^1\exp(-ct)\left(\frac{\partial a}{\partial t}-ca\right)\left\{(1-x)^2\frac{\partial^2\overline{u}}{\partial t\partial x}\frac{\partial u}{\partial x}+2u\frac{\partial\overline{u}}{\partial t}\right\}\bigg|_{T=t}dx.
\end{aligned}\tag{3.10}$$

Substituting (3.8) and (3.10) in (3.7) and using conditions (1.2), (1.3), and (3.3), we obtain

$$\begin{aligned}
&\int_{\Omega}\exp(-ct)(1-x)^2\left|\frac{\partial^3 u}{\partial t^3}\right|^2dx\,dt\\
&\qquad+\int_{\Omega}\exp(-ct)\{c_3'-3cc_2+3c^2c_1'-c^3a_1-b_1^2\}\\
&\qquad\quad\times\left[\frac{(1-x)^2}{2}\left|\frac{\partial u}{\partial x}\right|^2+|u|^2\right]dx\,dt\\
&\quad\le\operatorname{Re}\int_{\Omega}\exp(-ct)\pounds uM\overline{u}\,dx\,dt.
\end{aligned}\tag{3.11}$$

Again, substituting the expression of Mu in (3.11) and using elementary inequality, we get

$$
\begin{aligned}
&\int_\Omega \exp(-ct)\frac{(1-x)^2}{2}\left|\frac{\partial^3 u}{\partial t^3}\right|^2 dx\,dt\\
&\qquad+\int_\Omega \exp(-ct)\left\{c_3'-3cc_2+3c^2c_1'-c^3a_1-b_1^2\right\}\\
&\qquad\qquad\times\left[\frac{(1-x)^2}{2}\left|\frac{\partial u}{\partial x}\right|^2+|u|^2\right]dx\,dt\\
&\quad\le 17\int_\Omega \exp(-ct)(1-x)^2|f|^2 dx\,dt.
\end{aligned}
\tag{3.12}
$$

By virtue of (1.1a), we have

$$
\begin{aligned}
&\int_\Omega a_0\left|\frac{\partial^2 u}{\partial x^2}\right|^2\frac{(1-x)^2}{2}dx\,dt\\
&\quad\le\int_\Omega (1-x)^2|f|^2 dx\,dt+\int_\Omega 2(1-x)^2\left|\frac{\partial^3 u}{\partial t^3}\right|^2 dx\,dt\\
&\qquad+4\int_\Omega b_1^2\left\{\frac{(1-x)^2}{2}\left|\frac{\partial u}{\partial x}\right|^2+|u|^2\right\}dx\,dt.
\end{aligned}
\tag{3.13}
$$

This last inequality combined with (3.12) yields

$$
\begin{aligned}
&\int_\Omega \frac{(1-x)^2}{2}\left|\frac{\partial^3 u}{\partial t^3}\right|^2 dx\,dt\\
&\qquad+\int_\Omega \left(c_3'-3cc_2+3c^2c_1'-c^3a_1-b_1^2\right)\left\{\frac{(1-x)^2}{2}\left|\frac{\partial u}{\partial x}\right|^2+|u|^2\right\}dx\,dt\\
&\qquad+\int_\Omega a_0^2\frac{(1-x)^2}{2}\left|\frac{\partial^2 u}{\partial x^2}\right|^2 dx\,dt\\
&\quad\le\left\{17\exp(cT)\left[5+\frac{4b_1^2}{c_3'-3cc_2+3c^2c_1'-c^3a_1-b_1^2}\right]+1\right\}\\
&\qquad\times\int_\Omega (1-x)^2|f|^2 dx\,dt.
\end{aligned}
\tag{3.14}
$$

Thus, this inequality implies

$$
\begin{aligned}
&\int_\Omega \frac{(1-x)^2}{2}\left\{\left|\frac{\partial^3 u}{\partial t^3}\right|^2+\left|\frac{\partial^2 u}{\partial x^2}\right|^2\right\}dx\,dt+\int_\Omega \frac{(1-x)^2}{2}\left|\frac{\partial u}{\partial x}\right|^2+|u|^2 dx\,dt\\
&\qquad\le k^2\int_\Omega (1-x)^2|f|^2 dx\,dt,
\end{aligned}
\tag{3.15}
$$

where

$$k^2 = \frac{17\exp(cT)\left[5+4b_1^2/\left(c_3'-3cc_2+3c^2c_1'-c^3a_1-b_1^2\right)\right]+1}{\min\left(1,a_0^2,c_3'-3cc_2+3c^2c_1'-c^3a_1-b_1^2\right)}. \tag{3.16}$$

Then,

$$\|u\|_E \le k\|Lu\|_F, \quad \forall u \in D(L). \tag{3.17}$$

Thus, we obtain the desired inequality. □

LEMMA 3.2. *The operator L from E to F admits a closure.*

Proof. Suppose that $(u_n) \in D(L)$ is a sequence such that

$$u_n \longrightarrow 0 \quad \text{in } E, \qquad Lu_n \longrightarrow \mathcal{F} \quad \text{in } F. \tag{3.18}$$

We need to show that $\mathcal{F} = 0$. We introduce the operator

$$£_0 v = -(1-x)^2\frac{\partial^3 v}{\partial t^3} + \frac{\partial}{\partial x}\left\{a(x,t)\frac{\partial}{\partial x}\left[(1-x)^2 v\right]\right\}, \tag{3.19}$$

with domain $D(£_0)$ consisting of functions $v \in W_2^{2,3}(\Omega)$ satisfying

$$v|_{t=0} = 0, \qquad \left.\frac{\partial v}{\partial t}\right|_{t=0} = 0, \qquad \left.\frac{\partial^2 v}{\partial t^2}\right|_{t=0} = 0, \qquad v|_{x=0} = 0, \qquad \left.\frac{\partial v}{\partial x}\right|_{x=0} = 0. \tag{3.20}$$

We note that $D(£_0)$ is dense in the Hilbert space obtained by completing $L^2(\Omega)$ with respect to the norm

$$\int_\Omega (1-x)^2|v|^2\,dx\,dt = \|v\|^2. \tag{3.21}$$

Since

$$\begin{aligned}\int_\Omega (1-x)^2 f\overline{v}\,dx\,dt &= \lim_{n\to+\infty}\int_\Omega (1-x)^2 £u_n\overline{v}\,dx\,dt \\ &= \lim_{n\to+\infty}\int_\Omega u_n £_0\overline{v}\,dx\,dt = 0,\end{aligned} \tag{3.22}$$

for any function $v \in D(£_0)$, it follows that $f = 0$. □

Theorem 3.1 is valid for a strong solution, then we have the inequality

$$\|u\|_E \leq k\|\overline{L}u\|_F, \quad \forall u \in D(\overline{L}). \tag{3.23}$$

Hence we obtain the following corollary.

Corollary 3.3. *A strong solution of problem (1.1) is unique if it exists, and depends continuously on $\mathcal{F}$.*

Corollary 3.4. *The range $R(\overline{L})$ of the operator $\overline{L}$ is closed in F, and $R(\overline{L}) = \overline{R(L)}$.*

4. Solvability of problem (1.1)

To prove the solvability of problem (1.1), it is sufficient to show that $R(L)$ is dense in F. The proof is based on the following lemma.

Lemma 4.1. *Suppose that $a(x,t)$ and its derivatives $\partial^4 a/\partial t^3\partial x$ and $\partial^2 a/\partial t\partial x$ are bounded. Let $D_0(L) = \{u \in D(L) : u(x,0) = 0,\ (\partial u/\partial t)(x,0) = 0,\ (\partial^2 u/\partial t^2)(x,T) = 0\}$. If, for $u \in D_0(L)$ and for some functions $w \in L^2(\Omega)$,*

$$\int_\Omega (1-x)\pounds u\overline{w}\,dx\,dt = 0, \tag{4.1}$$

then $w = 0$.

Proof. Equality (4.1) can be written as follows:

$$\int_\Omega (1-x)\overline{w}\frac{\partial^3 u}{\partial t^3}dx\,dt = -\int_\Omega \frac{\partial}{\partial x}\left(a(1-x)\frac{\partial u}{\partial x}\right)\left\{\overline{w} - \int_0^x \frac{\overline{w}}{1-\zeta}d\zeta\right\}dx\,dt. \tag{4.2}$$

For a given $w(x,t)$, we introduce the function $v(x,t)$ such that

$$v(x,t) = w(x,t) - \int_0^x \frac{w(\zeta,t)}{1-\zeta}d\zeta. \tag{4.3}$$

From (4.3), we conclude that $\int_0^1 v(x,t)dx = 0$, and thus, we have

$$\int_\Omega \frac{\partial^3 u}{\partial t^3}\overline{Nv}\,dx\,dt = -\int_\Omega A(t)u\overline{v}\,dx\,dt, \tag{4.4}$$

where $A(t)u = (\partial/\partial x)(a(1-x)(\partial u/\partial x))$ and $Nv = (1-x)v + Jv$.

Following [23], we introduce the smoothing operators

$$J_\varepsilon^{-1} = \left(I - \epsilon\left(\frac{\partial^3}{\partial t^3}\right)\right)^{-1}, \qquad (J_\varepsilon^{-1})^* = \left(I + \epsilon\left(\frac{\partial^3}{\partial t^3}\right)\right)^{-1}, \tag{4.5}$$

with respect to t, which provide the solutions of the respective problems

$$\begin{aligned} &g_\epsilon - \epsilon\frac{\partial^3 g_\varepsilon}{\partial t^3} = g, \qquad g_\epsilon(0) = 0, \qquad \frac{\partial g_\epsilon}{\partial t}(0) = 0, \qquad \frac{\partial^2 g_\epsilon}{\partial t^2}(T) = 0, \\ &g_\epsilon^* + \epsilon\frac{\partial^3 g_\epsilon^*}{\partial t^3} = g, \qquad g_\epsilon^*(0) = 0, \qquad \frac{\partial g_\epsilon^*}{\partial t}(T) = 0, \qquad \frac{\partial^2 g_\epsilon^*}{\partial t^2}(T) = 0. \end{aligned} \tag{4.6}$$

We also have the following properties: for any $g \in L^2(0,T)$, the functions $J_\epsilon^{-1}(g)$, $(J_\epsilon^{-1})^* g \in W_2^3(0,T)$. If $g \in D(L)$, then $J_\epsilon^{-1}(g) \in D(L)$ and we have

$$\begin{aligned} \lim \left\| (J_\epsilon^{-1})^* g - g \right\|_{L^2[0,T]} &= 0 \quad \text{for } \epsilon \longrightarrow 0, \\ \lim \left\| (J_\epsilon^{-1}) g - g \right\|_{L^2[0,T]} &= 0 \quad \text{for } \epsilon \longrightarrow 0. \end{aligned} \tag{4.7}$$

Substituting the function u in (4.4) by the smoothing function u_ε and using the relation

$$A(t)u_\varepsilon = J_\varepsilon^{-1} A u - \epsilon J_\varepsilon^{-1} \beta_\epsilon(t) u_\varepsilon, \tag{4.8}$$

where

$$\beta_\epsilon(t)u_\varepsilon = 3\frac{\partial^2 A(t)}{\partial t^2}\frac{\partial u_\varepsilon}{\partial t} + 3\frac{\partial A(t)}{\partial t}\frac{\partial^2 u_\varepsilon}{\partial^2 t} + \frac{\partial^3 A(t)}{\partial t^3} u_\varepsilon, \tag{4.9}$$

we obtain

$$-\int_\Omega u N \frac{\partial^3 \overline{v_\varepsilon}^*}{\partial^3 t} dx\, dt = \int_\Omega A(t) u \overline{v_\varepsilon^*} dx\, dt - \epsilon \int_\Omega \beta_\epsilon(t) u_\varepsilon \overline{v_\varepsilon^*} dx\, dt. \tag{4.10}$$

Passing to the limit, the equality in the relation (4.10) remains true for all functions $u \in L^2(\Omega)$ such that $(1-x)(\partial u/\partial x)$, $(\partial/\partial x)((1-x)(\partial u/\partial x)) \in L^2(\Omega)$, and satisfying condition (1.1d).

The operator $A(t)$ has a continuous inverse in $L^2(0,1)$ defined by

$$A^{-1}(t)g=-\int_0^x \frac{1}{1-\zeta}\frac{1}{a(\zeta,t)}\int_0^\zeta g(\eta,t)\,d\eta\,d\zeta + C(t)\int_0^x \frac{1}{1-\zeta}\frac{1}{a(\zeta,t)}\,d\zeta, \tag{4.11}$$

where

$$C(t)=\frac{\int_0^1 \left(d\zeta/a(\zeta,t)\right)\int_0^\zeta g(\eta,t)\,d\eta}{\int_0^1 \left(d\zeta/a(\zeta,t)\right)}. \tag{4.12}$$

Then, we have $\int_0^1 A^{-1}(t)g\,dx=0$, hence the function $u_\varepsilon=(J_\varepsilon)^{-1}u$ can be represented in the form

$$u_\varepsilon=\left(J_\varepsilon\right)^{-1}A^{-1}(t)A(t)u. \tag{4.13}$$

Then

$$\begin{aligned} B_\varepsilon(t)g={}&\frac{\partial^4 a}{\partial t^3\partial x}J_\varepsilon^{-1}\left[\frac{1}{a(x,t)}\left(\int_0^x g(\eta,t)\,d\eta-C(t)\right)\right]\\ &+\frac{\partial^3 a}{\partial t^3}J_\varepsilon^{-1}\left[\frac{g}{a}-\frac{a_x}{a^2(x,t)}\left(\int_0^x g(\eta,t)\,d\eta-C(t)\right)\right]\\ &+3\frac{\partial}{\partial t}\frac{\partial^2 a}{\partial t^2\partial x}\frac{\partial}{\partial t}J_\varepsilon^{-1}\frac{1}{a(x,t)}\left(\int_0^x g(\eta,t)\,d\eta-C(t)\right)\\ &+\frac{\partial a}{\partial t}\frac{\partial}{\partial t}J_\varepsilon^{-1}\frac{g}{a}-\frac{a_x}{a^2(x,t)}\left(\int_0^x g(\eta,t)\,d\eta-C(t)\right). \end{aligned} \tag{4.14}$$

The adjoint of $B_\varepsilon(t)$ has the form

$$\begin{aligned} B_\varepsilon^*(t)={}&\frac{1}{a}\left(J_\varepsilon^{-1}\right)^*\left[\frac{\partial^3 a}{\partial t^3}\overline{h}\right]+\frac{3}{a}\left(J_\varepsilon^{-1}\right)^*\frac{\partial}{\partial t}\left(\frac{\partial a}{\partial t}\frac{\partial \overline{h}}{\partial t}\right)\\ &+\left(G_\varepsilon h\right)(x)-\frac{\int_0^x \left(1/a(\eta,t)\right)d\eta}{\int_0^1 \left(1/a(x,t)\right)dx}\left(G_\varepsilon h\right)(1), \end{aligned} \tag{4.15}$$

where

$$(G_\epsilon h)(x) = \int_0^x \Bigg(-\frac{3}{a(\zeta,t)} (J_\epsilon^{-1})^* \frac{\partial}{\partial t}\left(\frac{\partial^2}{\partial t \partial \zeta}\frac{\partial h}{\partial t}\right) + 3\frac{\partial a}{\partial \zeta}\frac{1}{a^2(\zeta,t)}(J_\epsilon^{-1})^*\frac{\partial}{\partial t}\left(\frac{\partial a}{\partial t}\frac{\partial h}{\partial t}\right) - \frac{1}{a(\zeta,t)}(J_\varepsilon^{-1})^*\left(\frac{\partial^4 a}{\partial t^3 \partial \zeta} h\right) + \frac{\partial a}{\partial \zeta}\frac{1}{a^2(\zeta,t)}(J_\varepsilon^{-1})^*\left(\frac{\partial^3 a}{\partial t^3} h\right)\Bigg) d\zeta. \tag{4.16}$$

Consequently, equality (4.10) becomes

$$-\int_\Omega uN\frac{\partial^3 \overline{v_\varepsilon^*}}{\partial t^3} dx\, dt = \int_\Omega A(t) u \overline{h_\varepsilon} dx\, dt, \tag{4.17}$$

where $h_\varepsilon = v_\varepsilon^* - \varepsilon B_\varepsilon^* v_\varepsilon^*$.

The left-hand side of (4.17) is a continuous linear functional of u. Hence the function h_ε has the derivatives $(1-x)(\partial h_\varepsilon/\partial x)$, $(\partial/\partial x)((1-x)(\partial h_\varepsilon/\partial x)) \in L^2(\Omega)$ and the following conditions are satisfied: $h_\varepsilon|_{x=0} = 0$, $h_\varepsilon|_{x=1} = 0$, and $(1-x)(\partial h_\varepsilon/\partial x)|_{x=1} = 0$.

From the equality

$$(1-x)\frac{\partial h_\varepsilon}{\partial x} = \left[I - \varepsilon\frac{1}{a}(J_\varepsilon^{-1})^*\frac{\partial^3 a}{\partial t^3}\right](1-x)\frac{\partial v_\varepsilon^*}{\partial x} - 3\varepsilon\frac{1}{a}(J_\varepsilon^{-1})^*\frac{\partial}{\partial t}\left(\frac{\partial a}{\partial t}\frac{\partial}{\partial t}(1-x)\frac{\partial v_\varepsilon^*}{\partial x}\right), \tag{4.18}$$

and since the operator $(J_\varepsilon^{-1})^*$ is bounded in $L^2(\Omega)$, for sufficiently small ε, we have $\|\varepsilon(1/a)(J_\varepsilon^{-1})^*(\partial^3 a/\partial t^3)\| < 1$. Hence the operator $I - \varepsilon(1/a)(J_\varepsilon^{-1})^*(\partial^3 a/\partial t^3)$ has a bounded inverse in $L^2(\Omega)$. We conclude that $(1-x)(\partial v_\varepsilon^*/\partial x) \in L^2(\Omega)$.

Similarly, we conclude that $(\partial/\partial x)((1-x)(\partial v_\varepsilon^*/\partial x))$ exists and belongs to $L^2(\Omega)$, and the following conditions are satisfied:

$$v_\varepsilon^*|_{x=0} = 0, \qquad v_\varepsilon^*|_{x=1} = 0, \qquad (1-x)\frac{\partial v_\varepsilon^*}{\partial x}\bigg|_{x=1} = 0. \tag{4.19}$$

Substituting $u = \int_0^t \int_0^\eta \int_\zeta^T \exp(c\tau) v_\varepsilon^*(\tau)\, d\tau\, d\zeta\, d\eta$ in (4.4), where the constant c satisfies (3.3), we obtain

$$\int_\Omega \exp(ct) v_\varepsilon^* N\overline{v}\, dx\, dt = -\int_\Omega A(t) u\overline{v}\, dx\, dt. \tag{4.20}$$

Using the properties of smoothing operators, we have

$$\int_\Omega \exp(ct)v_\varepsilon^* N\overline{v}\,dx\,dt = -\int_\Omega A(t)u\overline{v_\varepsilon^*}dx\,dt - \varepsilon\int_\Omega A(t)u\frac{\partial^3\overline{v_\varepsilon^*}}{\partial t^3}dx\,dt, \tag{4.21}$$

and from

$$\begin{aligned}
\varepsilon\,\mathrm{Re}\int_\Omega A(t)u\frac{\partial^3\overline{v_\varepsilon^*}}{\partial t^3}dx\,dt &= \varepsilon\int_\Omega (1-x)a\frac{\partial u}{\partial x}\frac{\partial}{\partial x}\frac{\partial^3\overline{v_\varepsilon^*}}{\partial t^3}dx\,dt\\
&= -\varepsilon\,\mathrm{Re}\int_\Omega (1-x)\frac{\partial a}{\partial t}\frac{\partial u}{\partial x}\frac{\partial^2}{\partial t^2}\frac{\partial\overline{v_\varepsilon^*}}{\partial x}dx\,dt\\
&\quad + \varepsilon\,\mathrm{Re}\int_\Omega (1-x)\frac{\partial a}{\partial t}\frac{\partial^2 u}{\partial t\partial x}\frac{\partial}{\partial t}\frac{\partial\overline{v_\varepsilon^*}}{\partial x}dx\,dt\\
&\quad + \varepsilon\int_\Omega a\exp(-ct)(1-x)\left|\frac{\partial\overline{v_\varepsilon^*}}{\partial x}\right|^2 dx\,dt\\
&\quad + \varepsilon\,\mathrm{Re}\int_\Omega (1-x)\frac{\partial a}{\partial t}\frac{\partial^2 u}{\partial t\partial x}\frac{\partial\overline{v_\varepsilon^*}}{\partial x}dx\,dt,
\end{aligned} \tag{4.22}$$

we have

$$\begin{aligned}
&\varepsilon\,\mathrm{Re}\int_\Omega A(t)u\frac{\partial^3\overline{v_\varepsilon^*}}{\partial t^3}dx\,dt\\
&\quad\geq \varepsilon\int_\Omega a\exp(+ct)(1-x)\left|\frac{\partial\overline{v_\varepsilon^*}}{\partial x}\right|^2 dx\,dt\\
&\qquad - \varepsilon\int_\Omega (1-x)\frac{1}{4a}\left(\frac{\partial a}{\partial t}\right)^2\exp(-ct)\left|\frac{\partial^3 u}{\partial t^2\partial x}\right|^2 dx\,dt\\
&\qquad - \varepsilon\int_\Omega a\exp(+ct)(1-x)\left|\frac{\partial\overline{v_\varepsilon^*}}{\partial x}\right|^2 dx\,dt\\
&\qquad - \varepsilon\int_\Omega \frac{1-x}{2}\left(\frac{\partial a}{\partial t}\right)^2\exp(-ct)\left|\frac{\partial u}{\partial x}\right|^2 dx\,dt\\
&\qquad - \varepsilon\int_\Omega \exp(+ct)\frac{1-x}{2}\left|\frac{\partial^3\overline{v_\varepsilon^*}}{\partial t^2\partial x}\right|^2 dx\,dt\\
&\qquad - \varepsilon\int_\Omega \exp(+ct)\frac{1}{2}\left|\frac{\partial^2\overline{v_\varepsilon^*}}{\partial t\partial x}\right|^2 dx\,dt\\
&\qquad - \varepsilon\int_\Omega \frac{1-x}{2}\left(\frac{\partial a}{\partial t}\right)^2\exp(-ct)\left|\frac{\partial^2 u}{\partial x\partial t}\right|^2 dx\,dt.
\end{aligned} \tag{4.23}$$

Integrating the first term on the right-hand side by parts in (4.21), we obtain

$$
\begin{aligned}
\operatorname{Re}&\int_\Omega A(t)u\overline{v_\varepsilon^*}\,dx\,dt\\
&\geq -\frac{3}{2}\int_\Omega (1-x)\exp(-ct)\left(\frac{\partial a}{\partial t}-ca\right)\left|\frac{\partial^2\overline{u}}{\partial t\partial x}\right|^2 dx\,dt\\
&\quad+\frac{1}{2}\int_0^1 (1-x)\exp(-ct)\left(a-\left|\frac{\partial a}{\partial t}-ca\right|\right)\left|\frac{\partial^2\overline{u}}{\partial t\partial x}\right|^2 dx\Big|_{t=T}\\
&\quad-\frac{1}{2}\int_0^1 (1-x)\exp(-ct)\left\{\frac{\partial^2 a}{\partial t^2}-2c\frac{\partial a}{\partial t}+c^2a+\left|\frac{\partial a}{\partial t}-ca\right|\right\}\left|\frac{\partial u}{\partial x}\right|^2\Big|_{t=T} dx\\
&\quad+\frac{1}{2}\int_\Omega (1-x)\exp(-ct)\left\{\frac{\partial^3 a}{\partial t^3}-3c\frac{\partial^2 a}{\partial t^2}+3c^2\frac{\partial a}{\partial t}-c^3a\right\}\left|\frac{\partial u}{\partial x}\right|^2 dx\,dt.
\end{aligned}
\tag{4.24}
$$

Combining (4.23) and (4.24), we get

$$
\begin{aligned}
\operatorname{Re}&\int_\Omega \exp(ct)v_\varepsilon^* N\overline{v}\,dx\,dt\\
&\leq \frac{3}{2}\int_\Omega (1-x)\exp(-ct)(c_1-ca_0)\left|\frac{\partial^2\overline{u}}{\partial t\partial x}\right|^2 dx\,dt\\
&\quad-\frac{1}{2}\int_0^1 (1-x)\exp(-ct)\{a_0-c_1'-ca_1\}\left|\frac{\partial^2 u}{\partial t\partial x}\right|^2 dx\Big|_{t=T}\\
&\quad+\frac{1}{2}\int_0^1 (1-x)\exp(-ct)\{c_2-2c_1'c-c^2a_1-c_1'+ca_1\}\left|\frac{\partial u}{\partial x}\right|^2\Big|_{t=T} dx\\
&\quad-\frac{1}{2}\int_\Omega (1-x)\exp(-ct)\{c_3'-3c_2c+3c^2c_1'-c^3a_1\}\left|\frac{\partial u}{\partial x}\right|^2 dx\,dt\\
&\quad+\varepsilon\Bigg(\int_\Omega (1-x)\exp(-ct)\frac{c_1^2}{4a_0}\left|\frac{\partial^3\overline{u}}{\partial t^2\partial x}\right|^2 dx\,dt\\
&\qquad+\int_\Omega (1-x)\exp(-ct)\frac{c_1^2}{2}\left|\frac{\partial u}{\partial x}\right|^2 dx\,dt\\
&\qquad+\int_\Omega \frac{1-x}{2}\exp(ct)\left|\frac{\partial^3 v_\varepsilon^*}{\partial t^2\partial x}\right|^2 dx\,dt\\
&\qquad+\int_\Omega (1-x)\exp(-ct)\frac{c_1^2}{2}\left|\frac{\partial^2\overline{u}}{\partial t\partial x}\right|^2 dx\,dt\\
&\qquad+\int_\Omega \frac{1-x}{2}\exp(ct)\left|\frac{\partial^2 v_\varepsilon^*}{\partial t\partial x}\right|^2 dx\,dt\Bigg).
\end{aligned}
\tag{4.25}
$$

Using conditions (3.3) and inequalities (4.23) and (4.24), we obtain

$$\operatorname{Re}\int_{\Omega}\exp(ct)vN\overline{v}\,dx\,dt\leq 0,\quad \text{as } \varepsilon\longrightarrow 0. \tag{4.26}$$

Since $\operatorname{Re}\int_{\Omega}\exp(ct)v\overline{J_x v}\,dx\,dt=0$, then $v=0$ a.e.

Finally, from the equality $(1-x)v+J_x v=(1-x)w$, we conclude $w=0$. □

Theorem 4.2. *The range $R(\overline{L})$ of $\overline{L}$ coincides with F.*

Proof. Since F is Hilbert space, then $R(\overline{L})=F$ if and only if the relation

$$\int_{\Omega}(1-x)^2 \pounds u\overline{f}\,dx\,dt=0, \tag{4.27}$$

for arbitrary $u\in D_0(L)$ and $\mathcal{F}\in F$, implies that $f=0$.

Taking $u\in D_0(L)$ in (4.27) and using Lemma 4.1, we obtain that $w=(1-x)f=0$, then $f=0$. □

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