CONDITIONAL VALUE-AT-RISK BOUNDS FOR COMPOUND POISSON RISKS AND A NORMAL APPROXIMATION

WERNER HÜRLIMANN

A considerable number of equivalent formulas defining conditional value-at-risk and expected shortfall are gathered together. Then we present a simple method to bound the conditional value-at-risk of compound Poisson loss distributions under incomplete information about its severity distribution, which is assumed to have a known finite range, mean, and variance. This important class of nonnormal loss distributions finds applications in actuarial science, where it is able to model the aggregate claims of an insurance-risk business.

1. Introduction

Value-at-risk, or VaR for short, which is defined as the α-quantile of a loss distribution for some prescribed confidence level $\alpha \in (0,1)$, is a popular measure of risk used to assess capital requirements in the insurance and finance industry. However, VaR suffers from various shortcomings pointed out in recent studies. For example, numerical instability and difficulties occur for nonnormal loss distributions, especially in the presence of "fat tails" and/or empirical discreteness. Furthermore, VaR is not a coherent measure of risk in the sense of Artzner et al. [6, 7], and it does not take into account the severity of an incurred adverse loss event.

A simple alternative measure of risk with some significant advantages over VaR is *conditional value-at-risk* or *expected shortfall*, abbreviated CVaR and ES, respectively, which is intuitively grasped as "the average of the $100(1 - \alpha)\%$ worst losses." This measure of risk is able to quantify dangers beyond VaR and it is coherent. Moreover, it provides a numerical efficient and stable tool in optimization problems under uncertainty. Some

recent studies presenting these advantages and further desirable properties include Acerbi [1], Acerbi et al. [2], Acerbi and Tasche [3, 4], Bertsimas et al. [8], Hürlimann [16, 17], Kusuoka [21], Pflug [26], Rockafellar and Uryasev [28, 29], Testuri and Uryasev [31], Wirch and Hardy [34], Yamai and Yoshiba [35, 36, 37].

The present paper gathers together a considerable number of equivalent formulas defining CVaR and ES, which are scattered through the recent literature on the subject. Beside this, it provides a simple method to bound the CVaR of compound Poisson loss distributions under incomplete information about its severity distribution. The latter is assumed to have a known finite range and a given mean and variance. This important class of nonnormal loss distributions finds applications in actuarial practice, where it is able to model the aggregate claims of an insurance risk business.

In Section 2, CVaR and ES are defined and a lot of their equivalent formulas are summarized. Furthermore, it is recalled that this measure of risk preserves the stop-loss order or, equivalently, the increasing convex order. Then, in Section 3, we show how to compute CVaR bounds for compound Poisson distributions knowing only the finite range, mean, and variance of the severity distribution. Finally, Section 4 contains a numerical illustration. It compares the average value of the obtained bounds with a normal approximation. The approximation turns out to be useful for large values of the Poisson parameter, where the bounds are difficult to evaluate numerically due to the underflow and overflow technical problem inherent in any computer-based quantitative evaluation.

2. Equivalent definitions and the stop-loss order-preserving property

Let (Ω, A, P) be a probability space such that Ω is the sample space, A is the σ-field of events, and P is the probability measure. For a measurable real-valued random variable X on this probability space, that is, a map $X : \Omega \to R$, the probability distribution of X is defined and denoted by $F_X(x) = P(X \leq x)$.

In the present paper, X represents a loss random variable such that for $\omega \in \Omega$, the real number $X(\omega)$ is the realization of a loss-and-profit function with $X(\omega) \geq 0$ for a loss and $X(\omega) < 0$ for a profit. Given X, consider the VaR to the confidence level α, defined as the *lower α-quantile*,

$$\mathrm{VaR}_\alpha[X] = Q_X^\hbar(\alpha) = \inf\{x : F_X(x) \geq \alpha\}, \qquad (2.1)$$

and the *upper conditional value-at-risk* (CVaR$^+$) to the confidence level α, defined by Rockafellar and Uryasev [29] as the mean excess loss

above VaR,

$$\text{CVaR}_\alpha^+[X] = E\big[X \mid X > \text{VaR}_\alpha[X]\big]. \qquad (2.2)$$

The VaR quantity represents the maximum possible loss, which is not exceeded with the probability α (in practice, $\alpha = 95\%$, 99%, 99.75%). The CVaR$^+$ quantity is the conditional-expected loss given that the loss strictly exceeds its VaR. Next, consider the *α-tail transform* X^α of X with the distribution function

$$F_{X^\alpha}(x) = \begin{cases} 0, & x < \text{VaR}_\alpha[X], \\ \dfrac{F_X(x) - \alpha}{1 - \alpha}, & x \geq \text{VaR}_\alpha[X]. \end{cases} \qquad (2.3)$$

Rockafellar and Uryasev [29] define CVaR to the confidence level α as expected value of the α-tail transform, that is, by

$$\text{CVaR}_\alpha[X] = E\big[X^\alpha\big]. \qquad (2.4)$$

The obtained measure is a coherent risk measure in the sense of Artzner et al. [6, 7] and coincides with CVaR$^+$ only under technical conditions, for example, in the case of continuous distributions (see Remarks 2.2 and Corollary 2.3). However, the reader should be warned that in many of the cited papers, the notion of CVaR is defined as the noncoherent measure (2.2) and is nevertheless claimed to be coherent. For instance, Pflug [26] proves the coherence of CVaR using (2.14), but then extends the proof to the noncoherent expression (2.2). Rockafellar and Uryasev [28] rely on Pflug's result and bear the same mistake. Bertsimas et al. [8] define CVaR as (2.2), but then identify it with (2.5). In fact, all literature before spring 2001 defines CVaR as (2.2) and claims erroneously that it is coherent. It is only after the appearance of Acerbi et al. [2, 4] that Rockafellar and Uryasev [29] propose a clear distinction between the notions of CVaR$^+$ and CVaR.

Alternatively, the ES to the confidence level α is defined as

$$\text{ES}_\alpha[X] = \frac{1}{\varepsilon} \cdot \int_\alpha^1 \text{VaR}_u[X]\,du \qquad (2.5)$$

and represents the average of the $100\varepsilon\%$ worst losses, where $\varepsilon = 1 - \alpha$ denotes the loss probability. The CVaR and ES quantities coincide and satisfy a lot of equivalent formulas. The alternative expressions are based on several transforms associated with X, which are of common use in the fields of reliability, actuarial science, finance, and economics.

The following standard definitions and notations are used throughout. The *survival function* associated with the probability distribution of X is denoted by $\overline{F}_X(x) = 1 - F_X(x)$. For $u \in (0,1)$, the *upper u-quantile* is the quantity $Q_X^u(u) = \inf\{x : F_X(x) > u\}$, and an arbitrary *u-quantile* $Q_X(u)$ denotes an element of the interval $[Q_X^\hbar(u), Q_X^u(u)]$. The *stop-loss transform* of X is the real-valued function defined by $\pi_X(x) = E[(X - x)_+] = \int_x^\infty \overline{F}_X(t)dt$, where $x_+ = x$ if $x \geq 0$ and $x_+ = 0$, otherwise. The *mean excess function* of X is the real-valued function defined by $m_X(x) = E[X - x \mid X > x] = \pi_X(x)/\overline{F}_X(x)$. Under the *right-spread transform* of X, we mean the real-valued function defined by $S_X(u) = \pi_X[Q_X(u)]$, $u \in [0,1]$ (e.g., Fernandez-Ponce et al. [11] and Shaked and Shanthikumar [30]). The *Lorenz transform* of X is defined by $L_X(u) = \int_0^u Q_X(t)dt$, $u \in [0,1]$, while the *dual Lorenz transform* is $L_X^*(u) = \int_u^1 Q_X(t)dt$. A standard reference for the Lorenz transform is Arnold [5], while its dual has been considered by Heilmann [12]. Another important probability transform is the *Hardy-Littlewood transform* defined by $\mathrm{HL}_X(u) = L_X^*(u)/(1-u)$ if $u \in [0,1)$ and $\mathrm{HL}_X(1) = Q_X(1)$, which has been considered in many papers (e.g., Kertz and Rösler [18, 19, 20], Hürlimann [15], and references therein). We know that it identifies with the quantile function $Q_{X^{\mathrm{HL}}}(u) = \mathrm{HL}_X(u)$, $u \in [0,1]$, of a random variable X^{HL} associated with X, which is called here *Hardy-Littlewood random variable* and which turns out to be the least majorant with respect to the stochastic dominance of first order among all random variables Y preceding X in the increasing convex order (e.g., Meilijson and Nádas [22]). For an increasing concave function $g : [0,1] \to [0,1]$ such that $g(0) = 0$, $g(1) = 1$, we consider, in actuarial science, the *distortion transform* of X defined by $D_g[X] = \int_{-\infty}^0 (g[\overline{F}_X(t)] - 1)dt + \int_0^\infty g[\overline{F}_X(t)]dt$ (e.g., Wang et al. [33] and references herein). Finally, the *total-time-on-test* or *TTT-transform* of X is the real-valued function defined by $T_X(u) = \int_0^{Q_X(u)} \overline{F}_X(t)dt$, $u \in (0,1)$, and is widely used in reliability. We note that many properties of the transforms L_X, T_X, and their relationships have been discussed in Pham and Turkkan [27].

PROPOSITION 2.1. *Let X be a real-valued random variable defined on the probability space (Ω, A, P). Then $\mathrm{CVaR}_\alpha[X] = \mathrm{ES}_\alpha[X]$, and these quantities can be represented by the following equivalent formulas:*

$$\frac{F_X(\mathrm{VaR}_\alpha[X]) - \alpha}{1 - \alpha} \cdot \mathrm{VaR}_\alpha[X] + \frac{1 - F_X(\mathrm{VaR}_\alpha[X])}{1 - \alpha} \cdot \mathrm{CVaR}_\alpha^+[X], \quad (2.6)$$

$$Q_X(\alpha) + \frac{1}{\varepsilon} \cdot S_X(\alpha), \quad (2.7)$$

$$\frac{1}{\varepsilon} \cdot \{E[X] - L_X(\alpha)\}, \quad (2.8)$$

$$\frac{1}{\varepsilon} \cdot L_X^*(\alpha), \tag{2.9}$$

$$\mathrm{HL}_X(\alpha), \tag{2.10}$$

$$Q_{X^{\mathrm{HL}}}(\alpha), \tag{2.11}$$

$$D_{g_\varepsilon}[X], \qquad g_\varepsilon(x) = \min\left\{\frac{x}{\varepsilon}, 1\right\}, \tag{2.12}$$

$$\frac{1}{\varepsilon} \cdot \left\{ E\left[X \cdot 1_{\{X > Q_X(\alpha)\}}\right] + Q_X(\alpha) \cdot \left(\varepsilon - \overline{F}_X[Q_X(\alpha)]\right) \right\}, \tag{2.13}$$

$$\min_\zeta \left\{ \zeta + \frac{1}{\varepsilon} \cdot \pi_X(\zeta) \right\}. \tag{2.14}$$

Proof. We first show that $\mathrm{CVaR}_\alpha[X] = \mathrm{ES}_\alpha[X]$. Rearranging and making a change of variables, we obtain

$$
\begin{aligned}
\mathrm{ES}_\alpha[X] &= Q_X^\hbar(\alpha) + \frac{1}{\varepsilon} \cdot \int_\alpha^1 \left(Q_X^\hbar(u) - Q_X^\hbar(\alpha)\right) du \\
&= Q_X^\hbar(\alpha) + \frac{1}{\varepsilon} \cdot \int_{Q_X^\hbar(\alpha)}^\infty \left(x - Q_X^\hbar(\alpha)\right) dF_X(x) \\
&= Q_X^\hbar(\alpha) + \frac{1}{\varepsilon} \cdot \pi_X\left[Q_X^\hbar(\alpha)\right].
\end{aligned}
\tag{2.15}
$$

On the other hand, by definition of CVaR, we have

$$\mathrm{CVaR}_\alpha[X] = E\left[X^\alpha\right] = \int_0^\infty \overline{F}_{X^\alpha}(x)dx - \int_{-\infty}^0 F_{X^\alpha}(x)dx. \tag{2.16}$$

Using (2.3) and distinguishing between the two cases $\mathrm{VaR}_\alpha[X] \geq 0$ and $\mathrm{VaR}_\alpha[X] < 0$, we, without difficulty, obtain that

$$\mathrm{CVaR}_\alpha[X] = \mathrm{VaR}_\alpha[X] + \frac{1}{\varepsilon} \cdot \pi_X\left[\mathrm{VaR}_\alpha[X]\right] = \mathrm{ES}_\alpha[X]. \tag{2.17}$$

The weighted average formula (2.6) is Proposition 6 in Rockafellar and Uryasev [29]. Since the integral in (2.5) does not depend on the choice of the α-quantile, we, similarly to the above, obtain that

$$\mathrm{HL}_X(\alpha) = \frac{1}{\varepsilon} \cdot \int_\alpha^1 Q_X(u)du = \mathrm{ES}_\alpha[X] = Q_X(\alpha) + \frac{1}{\varepsilon} \cdot \pi_X\left[Q_X(\alpha)\right], \quad (2.18)$$

which yields (2.7) and (2.10). Now, (2.9) is immediate by the definition of the Hardy-Littlewood transform, and (2.8) follows immediately through rearrangement, noting that $E[X] = \int_0^1 Q_X(u)du$. The relationship (2.11) is clear by the definition of Hardy-Littlewood random variable.

Formula (2.12) is an easy exercise. Formula (2.13), which expressed in terms of the worth or gain random variable $-X$ is (3.11) in Acerbi and Tasche [4], is obtained as follows. We have

$$
\begin{aligned}
E\big[X \cdot 1_{\{X > Q_X(\alpha)\}}\big] &= E\big[(X - Q_X(\alpha)) \cdot 1_{\{X > Q_X(\alpha)\}}\big] \\
&\quad + E\big[Q_X(\alpha) \cdot 1_{\{X > Q_X(\alpha)\}}\big] \\
&= E\big[(X - Q_X(\alpha))_+\big] + Q_X(\alpha) \cdot \overline{F}_X\big[Q_X(\alpha)\big],
\end{aligned}
\tag{2.19}
$$

which, inserted in (2.13), immediately yields (2.7). Finally, the minimization formula (2.14) is found in Rockafellar and Uryasev [29].   □

Remarks 2.2. Up to (2.6), (2.13) and (2.14) and under the assumption of continuous distributions, these equivalent expressions for CVaR are also derived in Hürlimann [17]. The many available alternative formulations for CVaR and ES suggest that, besides most recent ones, several proofs of the coherence of this measure are known. In particular, the distortion transform representation (2.12) can be traced back to Denneberg [9, 10], Wang [32], Wang et al. [33], and Hürlimann [15], which contain proofs of the coherence of this measure.

Besides the identification of CVaR$^+$ with CVaR in the case of continuous distributions, we note that a huge number of further equivalent formulas could be found. Only two attractive possibilities are mentioned.

COROLLARY 2.3. *Under the assumption of continuous distributions,* CVaR$_\alpha^+[X]$ = CVaR$_\alpha[X]$ *and these quantities are equivalent to the following formulas:*

$$
Q_X(\alpha) + m_X\big[Q_X(\alpha)\big],
\tag{2.20}
$$

$$
\frac{1}{\varepsilon} \cdot E[X] - \int_0^{1-\varepsilon} \frac{T_X(x)}{(1-x)^2}\,dx.
\tag{2.21}
$$

Proof. If the distribution function is continuous, we have that $F_X(\mathrm{VaR}_\alpha[X]) = \alpha$ and (2.6) coincides with (2.2). Formula (2.20) follows from (2.7), noting that

$$
m_X\big[Q_X(\alpha)\big] = \frac{\pi_X\big[Q_X(\alpha)\big]}{\overline{F}_X\big[Q_X(\alpha)\big]} = \frac{1}{\varepsilon} \cdot S_X(\alpha).
\tag{2.22}
$$

Finally, (2.21) follows from a result due to Pham and Turkkan [27, Theorem 2 and formula (5)]. We have $T_X(u) = L_X(u) + (1 - u) \cdot Q_X(u)$.

Since $Q_X(u)$ is continuous, we have $L'_X(u) = Q_X(u)$, which yields a linear differential equation in $L_X(u)$. Its solution is

$$L_X(u) = (1 - u) \cdot \int_0^u \frac{T_X(x)}{(1 - x)^2} dx. \tag{2.23}$$

Inserted in (2.8), we obtain (2.21). $\square$

In the special situation of discrete arithmetic loss distributions defined on the nonnegative integers, which will be used to evaluate our CVaR bounds in Section 3, numerical evaluation proceeds as follows. Let $f_k = \Pr(X = k)$ denote the probability that the nonnegative loss takes the value k, where $k = 0, 1, 2, \dots$, and assume that the finite mean $\mu_X = E[X]$ is known. Determine the unique index k_α such that

$$\sum_{k=1}^{k_\alpha - 1} f_k < \alpha \leq \sum_{k=1}^{k_\alpha} f_k. \tag{2.24}$$

Then we have

$$\mathrm{VaR}_\alpha[X] = Q_X^h(\alpha) = k_\alpha, \tag{2.25}$$

and we obtain from (2.7) that

$$\mathrm{CVaR}_\alpha[X] = Q_X(\alpha) + \frac{1}{\varepsilon} \cdot \left\{ \mu_X - Q_X(\alpha) + E\left[(Q_X(\alpha) - X)_+ \right] \right\}$$
$$= \frac{1}{\varepsilon} \cdot \left\{ \mu_X - \alpha \cdot k_\alpha + \sum_{k=0}^{k_\alpha} (k_\alpha - k) \cdot f_k \right\}. \tag{2.26}$$

In particular, the loss probabilities must only be evaluated up to the index k_α satisfying inequality (2.24).

It is important to observe that the CVaR functional is preserved under the stop-loss order or equivalently the increasing convex order. This fact is a main ingredient underlying the construction of CVaR bounds in Section 3. The next result is a slight generalization of Theorem 1.1 in Hürlimann [16], which is valid there for continuous distributions only. Recall that a loss X precedes another one Y in the stop-loss order, written $X \leq_{\mathrm{sl}} Y$ if $\pi_X(x) \leq \pi_Y(x)$ for all x.

PROPOSITION 2.4. *Let X and Y be two real-valued random variables defined on the probability space (Ω, A, P). Then $X \leq_{\mathrm{sl}} Y$ if and only if $\mathrm{CVaR}_\alpha[X] \leq \mathrm{CVaR}_\alpha[Y]$ for all $\alpha \in [0, 1]$.*

Proof. By (2.11) we have $\mathrm{CVaR}_\alpha[X] = Q_{X^{\mathrm{HL}}}(\alpha)$ and $\mathrm{CVaR}_\alpha[Y] = Q_{Y^{\mathrm{HL}}}(\alpha)$. The result follows from the fact that $X \leq_{\mathrm{sl}} Y$ if and only if $X^{\mathrm{HL}} \leq_{\mathrm{sl}} Y^{\mathrm{HL}}$, where $\leq_{\mathrm{sl}}$ denotes the stochastic dominance of first order (e.g., Kertz and Rösler [20, Lemma 1.8], or Hürlimann [15, Theorem 2.3]).   □

3. CVaR bounds for compound Poisson risks

An important risk management issue of an insurance company is the construction of more or less accurate bounds on risk measures like CVaR or ES for compound random sums $S = X_1 + \cdots + X_N$, where the claim number N is Poisson (λ), the claim sizes X_i are independent and identically distributed as X, and X_i is independent from N. By incomplete information about the claim size, say X belongs to the set $D = D([0,b]; \mu, \sigma)$ of all nonnegative random variables with maximum claim size b, known mean μ, and standard deviation σ, simple bounds are obtained as follows.

Following Hürlimann [14, Section 3], consider the stop-loss-ordered extreme random variables $X_{\min}$ and $X_{\max}$ for the set D such that

$$X_{\min} \leq_{\mathrm{sl}} X \leq_{\mathrm{sl}} X_{\max}, \quad \forall X \in D. \tag{3.1}$$

Then replace X by $X_{\min}$ and $X_{\max}$ in the compound Poisson random sums to get random sums $S_{\min}$ and $S_{\max}$ such that

$$S_{\min} \leq_{\mathrm{sl}} S \leq_{\mathrm{sl}} S_{\max}, \quad \forall X \in D. \tag{3.2}$$

Since CVaR is preserved under stop-loss order by Proposition 2.4, we obtain the bounds

$$\mathrm{CVaR}_\alpha\left[S_{\min}\right] \leq \mathrm{CVaR}_\alpha[S] \leq \mathrm{CVaR}_\alpha\left[S_{\max}\right], \quad \forall \alpha \in [0,1]. \tag{3.3}$$

For computational reasons, it is more advantageous to evaluate bounds based on finite atomic claim sizes. Now, the minimum $X_{\min}$ is already 2-atomic while the maximum $X_{\max}$ has a probability distribution of mixed discrete and continuous type. The latter can be replaced through mass dispersion by a 4-atomic stop-loss larger discrete approximation $X_{\max} \leq_{\mathrm{sl}} X_{\max}^d$ such that $S_{\max} \leq_{\mathrm{sl}} S_{\max}^d$ and $\mathrm{CVaR}_\alpha[S_{\max}] \leq \mathrm{CVaR}_\alpha[S_{\max}^d]$ for all $\alpha \in [0,1]$. Recall the structure of the finite atomic random variables $X_{\min}$ and $X_{\max}^d$. Let $v = (\sigma/\mu)^2$, the relative variance of the claim size, $v_0 = (b - \mu)/\mu$, the maximum relative variance for the set D, and $v_r = v/v_0$, a relative variance ratio. The discrete supports and probabilities of these

random variables are described for $X_{\min}$ by

$$\{x_1, x_2\} = \{(1 - v_r)\mu, (1 + v)\mu\}, \qquad \{p_1, p_2\} = \left\{\frac{v_0}{1 + v_0}, \frac{1}{1 + v_0}\right\}, \quad (3.4)$$

and for $X_{\max}^d$ by

$$\{x_0, x_1, x_2, x_3\} = \left\{0, \frac{1}{2}(1 + v)\mu, \left[1 + \frac{1}{2}(v_0 - v_r)\right]\mu, (1 + v_0)\mu\right\},$$

$$\{p_0, p_1, p_2, p_3\} = \left\{\frac{v}{1 + v}, \frac{v_0 - v}{(1 + v)(1 + v_0)}, \frac{v_0 - v}{(v_r + v_0)(1 + v_0)}, \frac{v_r}{v_r + v_0}\right\}.$$

$$(3.5)$$

In practice, we choose the parameters and fix a unit of money in such a way that the atoms x_i are nonnegative integers. Recall that the probabilities f_k, $k = 0, 1, 2, \ldots$, of a compound Poisson (λ) distribution with nonnegative integer claim sizes $x_0 = 0 < x_1 < \cdots < x_m$, and the corresponding probabilities $p_0, p_1, \ldots, p_m$ are best numerically evaluated using the following Adelson-Panjer recursive algorithm (e.g., Panjer [23], Hürlimann [13]):

$$f_0 = e^{-\lambda(1 - p_0)}, \qquad f_k = \frac{\lambda}{k} \sum_{j=1}^{m} \delta(k - x_j) x_j p_j f_{k - x_j}, \quad k = 1, 2, 3, \ldots, \quad (3.6)$$

where $\delta(x) = 1$ if $x \geq 0$ and $\delta(x) = 0$ else. Finally, to obtain $\mathrm{CVaR}_\alpha[S]$, we use formulas (2.24) and (2.26).

Since computers represent only a finite number of digits, it remains to discuss the technical problems of round-off errors and underflow/overflow. Regarding round-off errors, it has been shown by Panjer and Wang [24] that the recursive formula (3.6) is strongly stable such that this algorithm works well. However, for large values of λ, underflow/overflow occurs. In this situation, some methods have been proposed in Panjer and Willmot [25]. In Section 4, we use exponential scaling/descaling as follows. Let $\mu_S = \lambda\mu$, $\sigma_S^2 = \lambda(\mu^2 + \sigma^2)$ be the mean and variance of S. Choose appropriately $M = \mu_S - t \cdot \sigma_S$ for some t ($t = 19, 25.5$ in our example in Section 4 for $\lambda = 2000, 3000$), and let $r = \lambda(1 - p_0)/M$, $m_0 = [M]$, the greatest integer less than M. Exponential scaling and recursion yields

$$h_0 = 1, \qquad h_k = \frac{\lambda}{k} \sum_{j=1}^{m} \delta(k - x_j) x_j p_j e^{-r x_j} h_{k - x_j}, \quad k = 1, 2, \ldots, m_0. \quad (3.7)$$

Then apply exponential descaling setting $f_k = h_k e^{r(k - M)}$, $k = 0, \ldots, m_0$, and continue the evaluation of f_k for $k > m_0$ with the recursion (3.6).

4. Bounds on the insurance economic-risk capital

We are interested in the evaluation of economic-risk capital of an insurance portfolio whose compound Poisson aggregate claims S at a future date are covered by a risk premium $P > \mu_S$. The future random loss of the portfolio can be decomposed as follows:

$$S - P = (\mu_S - P) + (S - \mu_S). \tag{4.1}$$

The first component, which is the negative of the insurance margin, represents the future expected insurance gain and belongs to the stakeholders of the insurance company. To protect this expected gain, we require some economic-risk capital to cover the insurance loss $L = S - \mu_S$ (signed deviation from the mean aggregate claims). Using CVaR as risk measure, the future value of this economic-risk capital is equal to

$$\text{CVaR}_\alpha[L] = \text{CVaR}_\alpha[S] - \mu_S, \tag{4.2}$$

where α is some prescribed confidence level. Note that the equality in (4.2) follows from the translation invariant property of CVaR, which is one of the axioms required to define a coherent risk measure.

The following numerical illustration is based on the approximate figures of a real-life portfolio of grouped life insurance contracts from the early 1980s. For some unit of money, our choice for the claim-size parameters is $\mu = 12$, $\sigma^2 = 360$, and $b = 48$; hence $v = 5/2$, $v_0 = 3$, and $v_r = 5/6$. According to (3.4) and (3.5), the discrete supports and probabilities are given for $X_{\min}$ by

$$\{x_1, x_2\} = \{2, 42\}, \qquad \{p_1, p_2\} = \{0.75, 0.25\}, \tag{4.3}$$

and for $X_{\min}^d$ by

$$\{x_0, x_1, x_2, x_3\} = \{0, 21, 25, 48\},$$
$$\{p_0, p_1, p_2, p_3\} = \{0.71429, 0.03571, 0.03261, 0.21739\}. \tag{4.4}$$

Table 4.1 displays the values $\text{CVaR}_{\min} = (1/\mu_S) \cdot \text{CVaR}_\alpha[L_{\min}]$ and $\text{CVaR}_{\max} = (1/\mu_S) \cdot \text{CVaR}_\alpha[L_{\max}^d]$ with $L_{\min} = S_{\min} - \mu_S$, $L_{\max}^d = S_{\max}^d - \mu_S$, which represent bounds on the insurance economic-risk capital per unit of mean aggregate claims for $\alpha = 95\%, 99\%, 99.75\%$ by varying the expected number of claims λ. The *average rate*

$$\text{CVaR}^A = \frac{1}{2}\left(\text{CVaR}_{\min} + \text{CVaR}_{\max}\right) \tag{4.5}$$

TABLE 4.1. CVaR bounds and normal approximation as percentages of μ_S.

α	λ	CVaR$_{\min}$	CVaR$_{\max}$	CVaR^A	CVaR^N	D^N
95%	100	38.123	41.944	40.033	38.590	−1.443
	200	26.571	29.232	27.901	27.287	−0.614
	300	21.554	23.711	22.632	22.280	−0.352
	400	18.593	20.453	19.523	19.295	−0.228
	500	16.585	18.244	17.414	17.258	−0.156
	1000	11.648	12.812	12.230	12.203	−0.027
	2000	8.197	9.015	8.606	8.629	0.023
	3000	6.678	7.345	7.011	7.046	0.035
99%	100	50.251	55.297	52.774	49.862	−2.912
	200	34.837	38.331	36.584	35.257	−1.327
	300	28.189	31.013	29.601	28.788	−0.813
	400	24.279	26.711	25.495	24.931	−0.564
	500	21.634	23.800	22.717	22.299	−0.418
	1000	15.154	16.669	15.912	15.768	−0.144
	2000	10.643	11.706	11.174	11.149	−0.025
	3000	8.663	9.529	9.096	9.103	0.007
99.75%	100	59.333	65.315	62.342	58.077	−4.265
	200	40.987	45.103	43.045	41.067	−1.978
	300	33.109	36.430	34.770	33.531	−1.239
	400	28.488	31.343	29.916	29.039	−0.877
	500	25.366	27.908	26.637	25.973	−0.664
	1000	17.735	19.510	18.622	18.366	−0.256
	2000	12.439	13.682	13.060	12.986	−0.074
	3000	10.119	11.130	10.625	10.603	−0.022

is compared with the *normal approximation rate*

$$\text{CVaR}^N = \frac{1}{\varepsilon}\varphi\left[\Phi^{-1}(\alpha)\right] \cdot \frac{\sigma_S}{\mu_S},$$

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-(1/2)t^2} dt, \qquad \varphi(x) = \Phi'(x), \tag{4.6}$$

which is obtained by approximating S by a normal random variable S^N with a mean μ_S and a standard deviation σ_S. The approximation error is measured here by the *signed normal deviation rate*

$$D^N = \text{CVaR}^N - \text{CVaR}^A. \tag{4.7}$$

The following observations are noted. By fixed confidence level α, the normal approximation underestimates the average rate up to some fixed, rather large, expected number of claims λ, and then overestimates it. The underestimation increases by increasing the confidence level α. Since

computational difficulties with the exponential scaling/descaling method of Section 3 arise for values of λ beyond 3000, the normal approximation appears useful in this range provided insurers agree to set insurance economic-risk capital rates at the proposed average rate (4.5).

Acknowledgment

References

[1]   C. Acerbi, *Spectral measures of risk: a coherent representation of subjective risk aversion*, Journal of Banking and Finance **26** (2002), 1505–1518.

[2]   C. Acerbi, C. Nordio, and C. Sirtori, *Expected shortfall as a tool for financial-risk management*, in preparation.

[3]   C. Acerbi and D. Tasche, *Expected shortfall: a natural coherent alternative to value-at-risk*, Economic Notes by Banca Monte dei Paschi di Siena SpA vol. 31 no. 2-2002, 379–388.

[4]   ———, *On the coherence of expected shortfall*, Journal of Banking and Finance **26** (2002), 1487–1503.

[5]   B. C. Arnold, *Majorization and the Lorenz Order: a Brief Introduction*, Lecture Notes in Statistics, vol. 43, Springer-Verlag, Berlin, 1987.

[6]   P. Artzner, F. Delbaen, J.-M. Eber, and D. Heath, *Thinking coherently*, Risk Magazine **10** (1997), no. 11, 68–71.

[7]   ———, *Coherent measures of risk*, Math. Finance **9** (1999), no. 3, 203–228.

[8]   D. Bertsimas, G. J. Lauprete, and A. Samarov, *Shortfall as a risk measure: properties, optimization and applications*, Sloan School of Management, MIT, Massachusetts, 2000, to appear.

[9]   D. Denneberg, *Premium calculation: why standard deviation should be replaced by absolute deviation*, Astin Bull. **20** (1990), 181–190.

[10] ———, *Non-Additive Measure and Integral*, Theory and Decision Library. Series B: Mathematical and Statistical Methods, vol. 27, Kluwer Academic Publishers, Dordrecht, 1994.

[11] J. M. Fernandez-Ponce, S. C. Kochar, and J. Muñoz-Perez, *Partial orderings of distributions based on right-spread functions*, J. Appl. Probab. **35** (1998), no. 1, 221–228.

[12] W.-R. Heilmann, *Transformations of claim distributions*, Mitt. Verein. Schweiz. Versicherungsmath. (1985), no. 1, 57–69.

[13] W. Hürlimann, *An elementary proof of the Adelson-Panjer recursion formula*, Insurance Math. Econom. **7** (1988), no. 1, 39–40.

[14] ———, *Improved analytical bounds for some risk quantities*, Astin Bull. **26** (1996), 185–199.

[15] ———, *On stop-loss order and the distortion pricing principle*, Astin Bull. **28** (1998), no. 1, 119–134.

[16] ———, *Analytical evaluation of economic risk capital for portfolios of Gamma risks*, Astin Bull. **31** (2001), 107–122.

[17] _____ , *A unified approach to conditional value-at-risk, with application to RAROC and RARAROC*, manuscript, 2001.

[18] R. P. Kertz and U. Rösler, *Martingales with given maxima and terminal distributions*, Israel J. Math. **69** (1990), no. 2, 173–192.

[19] _____ , *Hyperbolic-concave functions and Hardy-Littlewood maximal functions*, Stochastic Inequalities (Seattle, WA, 1991), IMS Lecture Notes Monogr. Ser., vol. 22, Institute of Mathematical Statistics, California, 1992, pp. 196–210.

[20] _____ , *Stochastic and convex orders and lattices of probability measures, with a martingale interpretation*, Israel J. Math. **77** (1992), no. 1-2, 129–164.

[21] S. Kusuoka, *On law invariant coherent risk measures*, Advances in Mathematical Economics, Vol. 3 (S. Kusuoka, ed.), Springer, Tokyo, 2001, pp. 83–95.

[22] I. Meilijson and A. Nádas, *Convex majorization with an application to the length of critical paths*, J. Appl. Probab. **16** (1979), no. 3, 671–677.

[23] H. H. Panjer, *Recursive evaluation of a family of compound distributions*, Astin Bull. **12** (1981), no. 1, 22–26.

[24] H. H. Panjer and S. Wang, *On the stability of recursive formulas*, Astin Bull. **23** (1993), 227–258.

[25] H. H. Panjer and G. E. Willmot, *Computational aspects of recursive evaluation of compound distributions*, Insurance Math. Econom. **5** (1986), 113–116.

[26] G. Ch. Pflug, *Some remarks on the value-at-risk and the conditional value-at-risk*, Probabilistic Constrained Optimization (S. Uryasev, ed.), Nonconvex Optim. Appl., vol. 49, Kluwer Academic Publishers, Dordrecht, 2000, pp. 272–281.

[27] T. G. Pham and N. Turkkan, *The Lorenz and the scaled total-time-on-test transform curves: a unified approach*, IEEE Trans. Reliab. **43** (1994), no. 1, 76–84.

[28] R. T. Rockafellar and S. Uryasev, *Optimization of conditional value-at-risk*, The Journal of Risk **2** (2000), no. 3, 21–41.

[29] _____ , *Conditional value-at-risk for general loss distributions*, Journal of Banking and Finance **26** (2002), 1443–1471.

[30] M. Shaked and J. G. Shanthikumar, *Two variability orders*, Probab. Engrg. Inform. Sci. **12** (1998), no. 1, 1–23.

[31] C. E. Testuri and S. Uryasev, *On relation between expected regret and conditional value-at-risk*, 2000, in preparation.

[32] S. S. Wang, *Premium calculation by transforming the layer premium density*, Astin Bull. **26** (1996), 71–92.

[33] S. S. Wang, V. R. Young, and H. H. Panjer, *Axiomatic characterization of insurance prices*, Insurance Math. Econom. **21** (1997), no. 2, 173–183.

[34] J. L. Wirch and M. R. Hardy, *A synthesis of risk measures for capital adequacy*, Insurance Math. Econom. **25** (1999), 337–347.

[35] Y. Yamai and T. Yoshiba, *Comparative analyses of expected shortfall and value-at-risk: their estimation error, decomposition, and optimization*, in preparation.

[36] _____ , *On the validity of value-at-risk: comparative analyses with expected shortfall*, in preparation.

[37] T. Yoshiba and Y. Yamai, *Comparative analyses of expected shortfall and value-at-risk (2): expected utility maximization and tail risk*, in preparation.

Werner Hürlimann: Schönholzweg 24, CH-8409 Winterthur, Switzerland
E-mail address: whurlimann@bluewin.ch