ON CONTROLLABILITY, PARAMETRIZATION, AND OUTPUT TRACKING OF A LINEARIZED BIOREACTOR MODEL

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The paper deals with a distributed parameter system related to the so-called fixed-bed bioreactor. The original nonlinear partial differential system is linearized around the steady state. We find that the linearized system is not exactly controllable but it is approximatively controllable when certain algebraic equations hold. We apply frequency-domain methods (transfer function analysis) to consider a related output tracking problem. The input-output system can be formulated as a translation invariant pseudodifferential equation. A simulation shows that the calculation scheme is stable. An idea to use frequency-domain methods and certain pseudodifferential operators for parametrization of control systems of more general systems is pointed out.

1. Introduction

Various boundary value control systems related to partial differential equations are used to model the propagation in science and technology. These systems are infinite dimensional in a sense that the corresponding state variables belong to the infinite-dimensional linear spaces such as the Lebesgue or more generally the Sobolev spaces. One has developed, for example, functional analytic (see, e.g., [1, 4, 7, 16, 26]) and algebraic (e.g., [9, 20, 23]) methods to study these systems. Two of the main issues are the *controllability* and (asymptotic) *output tracking* of the system. Applying the functional analytic methods, the infinite-dimensional case has diverse collection of controllability concepts such as exact, approximate, and null controllability. There is no general correlation between these controllability properties and the output tracking. This is mainly due to

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the fact that the state, output, and input spaces may be completely different.

One of the more novel structural property of the control system is the *parametrizability*. In some cases, it can be studied using also algebraic methods such as torsion-freeness of certain structural factor modules. Torsion elements correspond to uncontrollable modes of the system. Parametrizability is closely related also to the flatness of the system. The main practical advantage of parametrizable systems is the potential usefulness in the (output) tracking problems (cf. [9, 22]). One is able more easily to design realizable controllers applying suitable parametrization. In the connection of boundary value control problems, the application of pseudodifferential and boundary value operators (see [13, 14, 24, 25]) are useful in parametrization. One of the potential advantages is that the (generalized) inverses and adjoints can be analyzed and treated because they are usually in the same class of operators.

In this paper, we consider the controllability, parametrization, and output tracking properties of the so-called fixed-bed bioreactor model [6, 27, 30]. The reactor is tubular and the waste water flows continuously through it. It is filled with a material in which the micro-organisms are fixed. The unwanted constituents of the waste water are consumed by the organisms and they convert into less harmful substituents.

The spatially one-dimensional model of a fixed-bed bioreactor consists of a pair of nonlinear partial differential equations (see [6, 19, 27])

$$\frac{\partial v_1}{\partial t} = -k_d v_1 + \mu(v_1, v_2) v_1, \tag{1.1}$$

$$\frac{\partial v_2}{\partial t} = D\frac{\partial^2 v_2}{\partial x^2} - c(t)\frac{\partial v_2}{\partial x} - k_1 \mu(v_1, v_2) v_1, \tag{1.2}$$

where the spatial variable x belongs to the interval $G =]0,1[\subset \mathbb{R}$ and the evolving times $t \in [0, t_0[$, $t_0 \le \infty$. The commonly used boundary conditions for stirred tank reactors are of the form

$$\frac{\partial v_2}{\partial x}(0,t) = \frac{c(t)}{D}\left(v_2(0,t) - S_a(t)\right), \qquad \frac{\partial v_2}{\partial x}(1,t) = 0, \tag{1.3}$$

for $t \in]0, t_0[$. The states $v_1 = v_1(x,t)$ and $v_2 = v_2(x,t)$ are the concentrations of the biomass (fixed in the reactor) and the substrate (flowing through the reactor), respectively. The specific growth rate of the microorganisms (in biomass) is modelled by the nonlinear law

$$\mu(v_1, v_2) = \mu_m \frac{v_2}{k_2 v_1 + v_2}. \tag{1.4}$$

The input flow $c = c(x,t)$ is the control variable. The input substrate concentration $S_a = S_a(t)$ is a disturbance variable in the system. The relevant output function y (the measurable variable) is usually the substrate concentration at the end of the reactor, that is,

$$y(t) = v_2(1,t). \tag{1.5}$$

The initial conditions

$$v_1(x,0) = \overline{v}_1(x), \qquad v_2(x,0) = \overline{v}_2(x) \tag{1.6}$$

are typically chosen in such a way that $\overline{v}_1$ and $\overline{v}_2$ are the steady state solutions of (1.2) and (1.3) before the simulated step changes the input flow c and/or the initial concentration S_a of the substrate. In the steady state, c and S_a are independent of time and in that case they are denoted by $\overline{c}$ and $\overline{S}_a$. We assume that $c(0) = \overline{c}$ and $S_a(0) = \overline{S}_a$. The real world model is spatially three-dimensional for which some simulations are given in [30].

At first, we linearize systems (1.2) and (1.3) around the steady state $\overline{v} = (\overline{v}_1, \overline{v}_2)$. The appropriate changes of variables have been performed. We formulate the linearized problem abstractly in the corresponding Sobolev spaces. As a result, we obtain a linear control system

$$\begin{gathered} \frac{\partial W}{\partial t} = \mathbf{A}W + \mathbf{B}_1 u + \mathbf{B}_2 u', \\ Y = \mathbf{D}_1 W + \mathbf{D}_2 u, \\ W(\cdot,0) = u(0) = 0, \end{gathered} \tag{1.7}$$

where $W = (W_1, W_2)$, $u = \binom{C}{S}$, and Y are the new state variable, input variable, and output variable, respectively.

In controllability problems, S_a is assumed to be constant (which means that $S = 0$). We show that the system

$$\frac{\partial W}{\partial t} = \mathbf{A}W + \mathbf{B}_1 u + \mathbf{B}_2 u' \tag{1.8}$$

is not exactly controllable. For the approximate controllability, we give a characterization with the help of algebraic equations. In addition, we verify that the closed loop system

$$\begin{gathered} \frac{\partial W}{\partial t} = \mathbf{A}W + \mathbf{B}_1 u + \mathbf{B}_2 u', \\ Y = \mathbf{D}_1 W + \mathbf{D}_2 u \end{gathered} \tag{1.9}$$

is not exactly output trackable in the chosen spaces.

In this paper, our aim is to analyze only the linearized model. We are not straightforwardly able to transfer the corresponding controllability properties for the original nonlinear partial differential system without careful further analysis. One of the difficulties in the nonlinear analysis arises because of the nonlinearity $c(\partial v_2/\partial x)$. The nonlinearity $\mu(v_1, v_2)v_1$ is easier because it is globally Lipschitz continuous (in appropriate spaces). The other difficulties arise because of the boundary values contain t-dependent variables c, S_a and the boundary conditions are nonlinear. The existing literature contains numerous results for nonlinear partial differential control problems but to our knowledge these results cannot be routinely applied in our case. The study of the linearized model in itself is motivated for the following reasons. As well known, the results for the linearized model can be potentially applied in the analysis of the nonlinear model. In addition, the linearized system in itself models quite accurately the bioreactor around the steady state because the changes (disturbances and changes in output level) are quite small in practise.

There are some places in the text where certain generalizations are possible for the more general (linearized) systems containing the derivative of control. We remark, however, that to get algebraic criteria like in Theorem 3.2, we must likely apply eigenfunction analysis or other analyses. Explicit eigenfunction analysis is strongly dependent on the application. For these reasons, we restrict to our application although techniques give some inspiration for generalizations.

In Section 4.1, we consider some stability properties of the input-output system. Here we assume that S_a is not constant. The transfer function of the linearized problem is considered. As, in general, for infinite-dimensional situations, our transfer function is not rational and more novel frequency-domain analysis is required. In infinite-dimensional case, the successful transfer function categories are, for example, Callier-Desoer classes. The integrated state- and frequency-domain method of infinite-dimensional systems is a useful control theoretic approach today (see, e.g., [1, 4, 17]). We find that the transfer function $G(\lambda) = (G_1(\lambda)\ \ G_2(\lambda))$ belongs to the Callier-Desoer class $\hat{A}_-(0)$. This result implies the internal input-output stability of the system (see [4, pages 457–470]).

In Section 4.2, we show how the frequency space factorizations can be applied to get state space parametrization. This technique has its preimage in control theory of ordinary differential equation systems, where one is able to get flat outputs (or parametrizations) for certain MIMO systems (cf. [5, 8, 23]). Our methodology is based on the use of pseudodifferential operators.

Applying the transfer function analysis, we give a scheme of the output tracking for certain reference outputs. We find that the needed realizations can be calculated and analyzed by using translation invariant pseudodifferential operators (i.e., operators with spatially independent symbols). The used methodologies have potential generalizations to certain classes of boundary value control problems. Our approach is closely related to π-freeness (see [9]) which is an extension for partial differential systems of flatness of certain finite-dimensional systems. The corresponding algebra is consisting of pseudodifferential operators. Here we, however, omit algebraic considerations.

We design a stabilizing compensator which (asymptotically) tracks the given reference output. A simulation shows the functionality of the method in practise.

1.1. Basic notations

We give some preliminary notations. Let G be an open set in $\mathbb{R}^n$. The spaces $C^\infty(\overline{G})$ and $C^\infty(\overline{G}\times\Delta)$ are correspondingly the collections of smooth functions $\overline{G}\mapsto\mathbb{C}$ and $\overline{G}\times\Delta\mapsto\mathbb{C}$. The space $L_p(G)$, $p\in[1,\infty[$, is the Lebesgue space of pth-power integrable functions $f:G\mapsto\mathbb{C}$. The space $W^{l,p}(G)$, $l\in\mathbb{N}_0$, is the Sobolev space (see [31]) equipped with the usual norm $\|v\|_{W^{l,p}(G)}$. We denote $H^l(G)=W^{l,2}(G)$. Let Δ be an interval in $\mathbb{R}$. We define the subspace $H_0^l(\overline{\Delta})=\{v\in H^l(\mathbb{R})\mid \operatorname{supp} v\subset\overline{\Delta}\}$. The spaces $H_0^l(\overline{\Delta})$ can be defined for any $l\in\mathbb{R}$ [31]. The space $C^l(\Delta,X)$ consists of all l times continuously differentiable functions $f:\Delta\mapsto X$, when X is a normed space.

For $\mu\in\mathbb{R}$, we denote $\mathbb{C}_\mu^+=\{\lambda\in\mathbb{C}\mid\Re\lambda>\mu\}$. In addition, we denote $\mathbb{C}^-=\{\lambda\in\mathbb{C}\mid\Re\lambda<0\}$ and $\mathbb{C}^+=\{\lambda\in\mathbb{C}\mid\Re\lambda>0\}$.

The Laplace transform of an appropriate function $u:[0,\infty[\,\mapsto\mathbb{R}$ is denoted by

$$\hat{u}(\lambda)=\int_0^\infty u(t)e^{-\lambda t}dt,\quad \Re\lambda>0. \tag{1.10}$$

Alternatively, we denote $\hat{u}=Lu$. The inverse Laplace transform is

$$L^{-1}f(t)=\frac{1}{2\pi}\int_{\alpha-\infty i}^{\alpha+\infty i}\hat{f}(\alpha+i\xi)e^{(\alpha+i\xi)t}d\xi. \tag{1.11}$$

The integral is taken in the sense of principal value, if necessary. For any $f\in H_0^1([0,\infty[)$, it holds that $L^{-1}\hat{f}=f$. The Fourier transform of an appropriate function $u:\mathbb{R}\mapsto\mathbb{R}$ is denoted by

$$Fu(\xi)=\int_{-\infty}^\infty u(t)e^{-it\xi}dt. \tag{1.12}$$

2. Linearized control system

2.1. Linearization and abstract formulations

Consider the nonlinear model (1.2) and (1.3). In the following, we use the notations

$$P = \frac{\overline{c}}{D}, \qquad p = \frac{P}{2}, \qquad a = \frac{k_1(\mu_m - k_d)}{k_2\overline{c}}, \qquad q = \sqrt{\frac{P^2}{4} + Pa}. \tag{2.1}$$

The steady state solutions $\overline{v}_1$ and $\overline{v}_2$ can be explicitly solved [19].

Denote $U_1 = v_1 - \overline{v}_1$, $U_2 = v_2 - \overline{v}_2$, $v = (v_1, v_2)$, $\overline{v} = (\overline{v}_1, \overline{v}_2)$, $U = (U_1, U_2)$, and $C = c - \overline{c}$, $S = S_a - \overline{S}_a$. Applying the total derivatives, we obtain by simple computations the following linearized approximation:

$$\frac{\partial U_1}{\partial t} = -a_1 U_1 + a_2 U_2, \tag{2.2}$$

$$\frac{\partial U_2}{\partial t} = D\frac{\partial^2 U_2}{\partial x^2} - \overline{c}\frac{\partial U_2}{\partial x} - C(t)\frac{\partial \overline{u}_2}{\partial x}, -a_3 U_1 - a_4 U_2, \tag{2.3}$$

for $(x,t) \in G \times]0, t_0[$,

$$\frac{\partial U_2}{\partial x}(0,t) = \frac{C(t)}{D}\left(\overline{v}_2(0) - \overline{S}_a\right) + \frac{\overline{c}}{D}U_2(0,t) - \frac{\overline{c}}{D}S, \qquad \frac{\partial U_2}{\partial x}(1,t) = 0, \tag{2.4}$$

for $t \in]0, t_0[$, and

$$U_1(x,0) = 0, \qquad U_2(x,0) = 0, \tag{2.5}$$

for $x \in G$. Above, a_1, a_2, a_3, and a_4 are positive numbers defined by

$$\begin{aligned} a_1 &= \frac{k_d(\mu_m - k_d)}{\mu_m}, \qquad a_2 = \frac{(\mu_m - k_d)^2}{\mu_m k_2}, \\ a_3 &= -k_1(a_1 - k_d) = \frac{k_1 k_d^2}{\mu_m}, \qquad a_4 = k_1 a_2. \end{aligned} \tag{2.6}$$

The typical output function y related to problems (1.2) and (1.3) is $y(t) = v_2(1,t)$. Thus a relevant output function Y, associated with the linearized problem, is given by

$$Y(t) = U_2(1,t) = y(t) - \overline{v}_2(1). \tag{2.7}$$

Denote $s(t) = (C(t)/\overline{c})(\overline{v}_2(0) - \overline{S}_a) - S(t)$. Let

$$V_2 = U_2 + s(t), \qquad V_1 = U_1, \qquad V = (V_1, V_2). \tag{2.8}$$

For some technical simplifications, we finally substitute

$$W = (W_1, W_2) = (\kappa V_1, V_2), \tag{2.9}$$

where $\kappa = \sqrt{a_3/a_2}$. With this notation, systems (2.3), (2.4), and (2.5) becomes

$$\begin{aligned}
\frac{\partial W_1}{\partial t} &= -a_1 W_1 + \kappa a_2 W_2 - \kappa \frac{a_2}{\overline{c}}(\overline{v}_2(0) - \overline{S}_a)C(t) + \kappa a_2 S(t), \\
\frac{\partial W_2}{\partial t} &= D\frac{\partial^2 W_2}{\partial x^2} - \overline{c}\frac{\partial W_2}{\partial x} + \frac{1}{\overline{c}}(\overline{v}_2(0) - S_a)C'(t) - S'(t) \\
&\quad - \frac{\partial \overline{v}_2}{\partial x}C(t) + \frac{a_4}{\overline{c}}(\overline{v}_2(0) - S_a)C(t) - a_4 S(t) - \frac{a_3}{\kappa}W_1 - a_4 W_2,
\end{aligned} \tag{2.10}$$

for $(x,t) \in G \times]0, t_0[$, and

$$\frac{\partial W_2}{\partial x}(0,t) - \frac{\overline{c}}{D}W_2(0,t) = 0, \qquad \frac{\partial W_2}{\partial x}(1,t) = 0, \tag{2.11}$$

for $t \in]0, t_0[$,

$$W_1(x,0) = 0, \qquad W_2(x,0) = 0. \tag{2.12}$$

Systems (2.10), (2.11), and (2.12) can be written abstractly as follows. Denote $Q_1 = (1/\overline{c})(\overline{v}_2(0) - \overline{S}_a)$ and

$$\begin{aligned}
Q_2 &= Q_2(x) \\
&= \frac{\partial \overline{v}_2}{\partial x} = -\frac{PaS_a \sinh(q(1-x))e^{px}}{q\cosh q + (a+p)\sinh q} \\
&= \gamma\left(e^{q+(p-q)x} - e^{-q+(p+q)x}\right),
\end{aligned} \tag{2.13}$$

where $\gamma = -(PaS_a/2(q\cosh q + (a+p)\sinh q))$. Let $u = \binom{C}{S}$. Define operators $\mathbf{B}_1, \mathbf{B}_2 : \mathbb{R}^2 \mapsto L_2(G)^2$ by

$$\mathbf{B}_1 u = \begin{pmatrix} -\kappa a_2 Q_1 & \kappa a_2 \\ -Q_2 + a_4 Q_1 & -a_4 \end{pmatrix} u, \qquad \mathbf{B}_2 u = \begin{pmatrix} 0 & 0 \\ Q_1 & -1 \end{pmatrix} u. \tag{2.14}$$

Furthermore, define a linear operator $\mathbf{A} : L_2(G)^2 \mapsto L_2(G)^2$ by

$$D(\mathbf{A}) = L_2(G) \times \left\{ W_2 \in H^2(G) \mid \frac{\partial W_2}{\partial x}(0) - \frac{\overline{c}}{D} W_2(0) = 0,\ \frac{\partial W_2}{\partial x}(1) = 0 \right\},$$
$$\mathbf{A}W = \left(-a_1 W_1 + \kappa a_2 W_2, D\frac{\partial^2 W_2}{\partial x^2} - \overline{c}\frac{\partial W_2}{\partial x} - \frac{a_3}{\kappa} W_1 - a_4 W_2 \right). \tag{2.15}$$

Then the linearized problems (2.10), (2.11), and (2.12) can be written in the abstract form

$$\begin{aligned} \frac{\partial W}{\partial t} &= \mathbf{A}W + \mathbf{B}_1 u + \mathbf{B}_2 u', \\ W(\cdot, 0) &= 0. \end{aligned} \tag{2.16}$$

Here $W \in C([0,t_0[, L_2(G)^2) \cap C^1(]0,t_0[, L_2(G)^2)$ and $u \in H_0^1([0,t_0[)$.

Since the output $Y(t) = U_2(1,t) = W_2(1,t) - s(t)$, the associated input-output control system can be written as

$$\frac{\partial W}{\partial t} = \mathbf{A}W + \mathbf{B}_1 u + \mathbf{B}_2 u', \tag{2.17}$$
$$Y = \mathbf{D}_1 W + \mathbf{D}_2 u, \tag{2.18}$$

where $\mathbf{D}_1 : L_2(G)^2 \mapsto \mathbb{R}$ is the (unbounded) operator $\mathbf{D}_1 W = W_2(1) = (pr_2(W))(1)$ and where $\mathbf{D}_2 : \mathbb{R}^2 \mapsto \mathbb{R}$ is the operator $\mathbf{D}_2 u = -Q_1 C + S = (-Q_1 \ \ 1)\binom{C}{S}$. Due to the Sobolev lemma (see [31]),

$$\sup_{x \in \overline{G}} \left| \partial^\alpha v(x) \right| \le C \|v\|_{H^k(G)}, \quad v \in H^k(G), \tag{2.19}$$

for $k > l + 1/2$ and $\alpha \le l$, we find that $\mathbf{D}_1 W = W_2(1)$ is well defined for any $W \in D(\mathbf{A})$. Hence system (2.18) is sensible.

In the case where $S = 0$, the above system is

$$\begin{aligned} \frac{\partial W}{\partial t} &= \mathbf{A}W + \mathbf{B}_{11} C + \mathbf{B}_{21} C', \\ Y &= \mathbf{D}_1 W - Q_1 C, \end{aligned} \tag{2.20}$$

where

$$\mathbf{B}_{11} C = \begin{pmatrix} -\kappa a_2 Q_1 \\ -Q_2 + a_4 Q_1 \end{pmatrix} C, \qquad \mathbf{B}_{21} C = \begin{pmatrix} 0 \\ Q_1 \end{pmatrix} C. \tag{2.21}$$

The existence of solutions for the linearized problem

$$\begin{aligned} \frac{\partial W}{\partial t} &= \mathbf{A}W + \mathbf{B}_1 u + \mathbf{B}_2 u', \\ W(\cdot,0) &= W_0 \end{aligned} \tag{2.22}$$

can be studied by the semigroup theory (e.g., [11, 21, 28], cf. also [12] where the original nonlinear system is considered). For example, in the case where $C, S \in H^1([0,\infty[)$ and where the derivatives C', S' are locally Lipschitz continuous, the global ($t_0 = \infty$) classical solution exists.

2.2. *Semigroups generated by* **A**

The operator **A** satisfies the following boundedness and coercivity conditions.

Theorem 2.1. *For all $W, V \in H := L_2(G) \times H^1(G)$,*

$$\left|\langle \mathbf{A}W, V\rangle_{L_2(G)^2}\right| \le C\|W\|_H\|V\|_H, \tag{2.23}$$

$$\Re\langle -\mathbf{A}W, W\rangle_{L_2(G)^2} \ge c\|W\|_H^2, \tag{2.24}$$

where $c = \min\{a_1, a_4, D\}$.

Proof. (A) The boundedness is easily shown by noting that, due to the Sobolev imbedding theorem (2.19),

$$\left|W_2(0)\right| \le C\left\|W_2\right\|_{H^1(G)}. \tag{2.25}$$

(B) We shortly consider the coercivity. By the direct computation, we see that, for all $W \in D(\mathbf{A})$,

$$\begin{aligned} \langle \mathbf{A}W, W\rangle_{L_2(G)^2} = &-a_1\int_G W_1\overline{W}_1\,dx - a_4\int_G W_2\overline{W}_2\,dx \\ &+ \kappa a_2\int_G W_2\overline{W}_1\,dx - \frac{a_3}{\kappa}\int_G W_1\overline{W}_2\,dx - \overline{c}W_2(0)\overline{W}_2(0) \\ &- D\int_G \frac{\partial W_2}{\partial x}\overline{\frac{\partial W_2}{\partial x}}\,dx - \overline{c}\int_G \frac{\partial W_2}{\partial x}\overline{W}_2\,dx, \end{aligned} \tag{2.26}$$

where we used the fact that, for $W \in D(\mathbf{A})$,

$$D\frac{\partial W_2}{\partial x}(0)\overline{W}_2(0) = \overline{c}W_2(0)\overline{W}_2(0). \tag{2.27}$$

Hence noting that $2\Re\int_G(\partial W_2/\partial x)\overline{W_2}dx = |W_2(1)|^2 - |W_2(0)|^2$, we find that, for $W \in D(\mathbf{A})$,

$$\Re\langle -\mathbf{A}W, W\rangle_{L^2(G)^2} = a_1\int_G |W_1|^2 dx + a_4\int_G |W_2|^2 dx \\ + D\int_G \left|\frac{\partial W_2}{\partial x}\right|^2 dx + \frac{\overline{c}}{2}\left(|W_2(1)|^2 + |W_2(0)|^2\right), \tag{2.28}$$

where we used the relation $\kappa a_2 = a_3/\kappa$. The estimate (2.28) immediately implies that, for $W \in D(\mathbf{A})$,

$$\Re\langle -\mathbf{A}W, W\rangle_{L_2(G)^2} \geq \min\{a_1, a_4, D\}\|W\|^2_{L_2(G)\times H^1(G)}. \tag{2.29}$$

This completes the proof. □

The adjoint operator $\mathbf{A}^* : L_2(G)^2 \mapsto L_2(G)^2$ of $\mathbf{A}$ is given by

$$D(\mathbf{A}^*) = L_2(G) \times \left\{W_2 \in H^2(G) \mid \frac{\partial W_2}{\partial x}(1) + \frac{\overline{c}}{D}W_2(1) = 0,\ \frac{\partial W_2}{\partial x}(0) = 0\right\}, \\ \mathbf{A}^*W = \left(-a_1W_1 - \frac{a_3}{\kappa}W_2, D\frac{\partial^2 W_2}{\partial x^2} + \overline{c}\frac{\partial W_2}{\partial x} + \kappa a_2 W_1 - a_4 W_2\right). \tag{2.30}$$

Similarly as in Theorem 2.1, we find that

$$\Re\langle -\mathbf{A}^*W, W\rangle_{L_2(G)^2} \geq c\|W\|^2_{L_2(G)\times H^1(G)} \tag{2.31}$$

for all $W \in D(\mathbf{A}^*)$.

Corollary 2.2. *Let $c = \min\{a_2, a_4, D\}$. The operator $\mathbf{A}$ generates an exponentially bounded semigroup $T(t)$ on $L_2(G)^2$ with the exponential decay $-c$.*

Proof. The assertion follows immediately, for example, from [4, Corollary 2.2.3] and from estimates (2.24) and (2.31). □

In addition, we have the following corollary.

Corollary 2.3. *The operators $\mathbf{A}$ and $\mathbf{A}^*$ generate analytic semigroups $T(t)$ and $T(t)^*$ on $L_2(G)^2$.*

Proof. The assertion follows, for example, from [21, Theorem 5.2] and from estimates (2.24) and (2.31). □

Recall that the first output operator is given by $\mathbf{D}_1 W = W_2(1) = (pr_2(W))(1)$. By the Sobolev lemma, we find that

$$\begin{aligned} \left|\mathbf{D}_1 W\right|^2 = \left|W_2(1)\right|^2 \le C\|W_2\|^2_{H^1(G)} &\le \frac{C}{c}\left|\Re\langle -\mathbf{A}W, W\rangle_{L_2(G)^2}\right| \\ &\le \frac{C}{c}\left(\|\mathbf{A}W\|^2_{L_2(G)^2} + \|W\|^2_{L_2(G)^2}\right). \end{aligned} \tag{2.32}$$

Hence the operator $\mathbf{D}_1$ is $\mathbf{A}$-bounded.

2.3. *Riesz spectral property*

Consider the eigenvalue problem

$$DW_2'' - \overline{c}W_2' - a_4 W_2 = \mu W_2, \tag{2.33}$$

$$W_2'(0) - \frac{\overline{c}}{D}W_2(0) = 0, \qquad W_2'(1) = 0, \tag{2.34}$$

where $W_2 \in H^2(G)$. The Sturm-Liouville theory (e.g., [2, 15]) implies that the problem has countably many eigenvalues μ_j such that $\lim_{j\to\infty}\mu_j = -\infty$. We assume (see the note after Lemma 2.5) that the eigenvalues μ_j are *simple* (that is the algebraic multiplicity is one). The case where they are not simple can be treated in principle by the similar methods (see [4]) but it causes some complications. In addition, the (normalized) eigenfunctions w_j form a Riesz basis in $L_2(G)$. The eigenvalues of the adjoint problem

$$DW_2'' + \overline{c}W_2' - a_4 W_2 = \mu W_2, \tag{2.35}$$

$$W_2'(1) + \frac{\overline{c}}{D}W_2(1) = 0, \qquad W_2'(0) = 0 \tag{2.36}$$

are exactly $\overline{\mu}_j$. Denote the corresponding eigenfunctions by $\tilde{w}_j$.

The eigenvalue analysis of the operator $\mathbf{A}$ is based on the following technical lemma whose proof is omitted.

Lemma 2.4. (A) *The complex number λ is an eigenvalue of $\mathbf{A}$ if and only if*

$$\frac{a_2 a_3}{\lambda + a_1} + \lambda \in \{\mu_j \mid j \in \mathbb{N}\}. \tag{2.37}$$

The corresponding eigenvectors of $\mathbf{A}$ *are*

$$W_{l,j} = \left(\frac{\kappa a_2}{\lambda_{l,j} + a_1} w_j, w_j \right), \quad l = 1,2, \tag{2.38}$$

where $\lambda_{l,j}$ *are the (simple) roots of*

$$\frac{a_2 a_3}{\lambda + a_1} + \lambda = \mu_j. \tag{2.39}$$

(B) *The complex number* λ *is an eigenvalue of* $\mathbf{A}^*$ *if and only if* $\overline{\lambda}$ *is an eigenvalue of* $\mathbf{A}$. *The corresponding eigenvectors of* $\mathbf{A}^*$ *are*

$$\tilde{W}_{l,j} = \left(\frac{\kappa a_2}{\overline{\lambda}_{l,j} + a_1} \tilde{w}_j, \tilde{w}_j \right), \quad l = 1,2. \tag{2.40}$$

Note that

$$\lambda_{l,j} = \frac{a_1 - \mu_j \pm \sqrt{(a_1 - \mu_j)^2 - 4(a_2 a_3 - a_1 \mu_j)}}{2}, \quad l = 1,2. \tag{2.41}$$

The adjoint eigenvalue problem (2.36) (which we need below) can be solved as follows. The general solution of

$$D W_2'' + \overline{c} W_2' - (a_4 + \mu_j) W_2 = 0 \tag{2.42}$$

is

$$W_2(x) = C_1 e^{(-p + \beta(\mu_j))x} + C_2 e^{-(p + \beta(\mu_j))x}, \tag{2.43}$$

where $p = \overline{c}/2D$ and $\beta(\mu_j) = \sqrt{p^2 + (a_4 + \mu_j)/D}$. Matching the boundary conditions

$$W_2'(1) + \frac{\overline{c}}{D} W_2(1) = 0, \qquad W_2'(0) = 0 \tag{2.44}$$

leads to the requirement

$$(p - \beta(\mu_j))^2 e^{-\beta(\mu_j)} = (p + \beta(\mu_j))^2 e^{\beta(\mu_j)}. \tag{2.45}$$

Lemma 2.5. *The equation*

$$(p - \beta(\mu))^2 e^{-\beta(\mu)} = (p + \beta(\mu))^2 e^{\beta(\mu)}. \tag{2.46}$$

has only real roots μ. *In addition,* $\mu \leq -a_4 - D p^2$.

Proof. Let $\mu \in \mathbb{C}$ be the root of (2.46). Denote $\beta(\mu) = x + iy$. We find that

$$\left| \frac{p - x - iy}{p + x + iy} \right|^2 = \left| e^{2(x+iy)} \right| = e^{2x}. \tag{2.47}$$

Hence

$$\frac{(p-x)^2 + y^2}{(p+x)^2 + y^2} = e^{2x} \tag{2.48}$$

which implies that $x = 0$. Hence $\beta(\mu) = iy$ and then

$$-y^2 = p^2 + \frac{\mu + a_4}{D}. \tag{2.49}$$

Equation (2.49) gives that $\mu = -a_4 - Dp^2 - Dy^2$. Hence $\mu \in \mathbb{R}$ and $\mu \leq -a_4 - Dp^2$. □

Let $F(\mu) := (p - \beta(\mu))^2 e^{-\beta(\mu)} - (p + \beta(\mu))^2 e^{\beta(\mu)} = 0$ be the eigenvalue equation. One sees that $\Re F(\mu) = 0$ for any $\mu \leq -a_4 - Dp^2$. Hence μ is an eigenvalue if and only if $\Im F(\mu) = 0$. We have tested numerically the equation $\Im F(\mu) = 0$. Numerical results conjecture that the roots are simple for relevant parameter values. Hence the eigenvalues μ_j seem to be simple.

The corresponding eigenvectors for the adjoint problem are

$$\tilde{w}_j = C_{\mu_j}\Big[\big(-p + \beta(\mu_j)\big)e^{-(p+\beta(\mu_j))x} + \big(p + \beta(\mu_j)\big)e^{(-p+\beta(\mu_j))x}\Big], \tag{2.50}$$

where C_{μ_j} is an arbitrary constant (in the following we assume that $C_{\mu_j} = 1$). Similarly, we find the eigenvalues and eigenfunctions of problem (2.34).

From Theorem 2.1, we see that

$$\Re\langle \mathbf{A}W, W \rangle_{L_2(G)^2} \leq -c\|W\|_H^2 \tag{2.51}$$

and then

$$\Re\lambda_{l,j} \leq -c, \tag{2.52}$$

where $c = \min\{a_1, a_4, D\}$. The eigenvalues $\lambda_{l,j}$ can be calculated from the equation

$$\tilde{F}(\lambda) = F\left(\frac{a_2 a_3}{\lambda + a_1} + \lambda\right) = 0, \tag{2.53}$$

where $\Re\lambda \leq -c$. For relevant parameter values, $-a_4 - Dp^2 < -a_1 - 2\sqrt{a_2a_3}$. It is easy to see that under this condition, the eigenvalues $\lambda_{l,j}$ are in fact real. Plotting the function $\Im\tilde{F}(\lambda)$ as a function of $\lambda \leq -c$, one sees that also the eigenvalues $\lambda_{l,j}$ are simple. So it is reasonable to assume that the *eigenvalues $\lambda_{l,j}$ are simple*.

Corollary 2.6. *The sequence $\{W_{l,j} \mid l = 1,2,\ j \in \mathbb{N}\}$ forms a Riesz basis in $L_2(G)^2$.*

Proof. As we mentioned above, the Sturm-Liouville theory implies that the sequence $\{w_j \mid j \in \mathbb{N}\}$ forms a Riesz basis in $L_2(G)$, that is,

$$\overline{[\{w_j \mid j \in \mathbb{N}\}]} = L_2(G) \tag{2.54}$$

and there exist constants $c > 0$ and $C > 0$ such that, for all $N \in \mathbb{N}$ and $\alpha_j \in \mathbb{C}$,

$$c\sum_{j=1}^{N} |\alpha_j|^2 \leq \left\| \sum_{j=1}^{N} \alpha_j w_j \right\|_{L_2(G)}^2 \leq C\sum_{j=1}^{N} |\alpha_j|^2, \tag{2.55}$$

where [ ] denotes the linear hull of a set.

At first, we show that also

$$\overline{[\{W_{l,j} \mid l = 1,2,\ j \in \mathbb{N}\}]} = L_2(G)^2. \tag{2.56}$$

Let $f = (f_1, f_2) \in L_2(G)^2$ and let $\epsilon > 0$. Then by (2.54), there exists b_j, $c_j \in \mathbb{C}$ such that

$$\left\| f_1 - \sum_{j=1}^{p} b_j w_j \right\|_{L_2(G)} < \epsilon, \qquad \left\| f_2 - \sum_{j=1}^{q} c_j w_j \right\|_{L_2(G)} < \epsilon. \tag{2.57}$$

We may assume (after a slight modification if necessary) that $p = q$. Hence we see that f satisfies

$$\left\| f - \sum_{j=1}^{p} (b_j, c_j) w_j \right\|_{L_2(G)^2} < 2\epsilon. \tag{2.58}$$

Furthermore, the equation

$$A_{1,j}\left(\frac{\kappa a_2}{\lambda_{1,j}+a_1},1\right)+A_{2,j}\left(\frac{\kappa a_2}{\lambda_{2,j}+a_1},1\right)=(b_j,c_j) \tag{2.59}$$

has a solution $(A_{1,j},A_{2,j})$ since $\lambda_{1,j}\neq\lambda_{2,j}$. Hence

$$\left\| f-\sum_{j=1}^{p}\left(A_{1,j}W_{1,j}+A_{2,j}W_{2,j}\right)\right\|_{L_2(G)^2}<2\epsilon \tag{2.60}$$

as desired.

The estimate

$$c\sum_{j=1}^{N}\left|\alpha_{l,j}\right|^2\leq\left\|\sum_{j=1}^{N}\sum_{l=1}^{2}\alpha_{l,j}W_{l,j}\right\|_{L_2(G)}^2\leq C\sum_{j=1}^{N}\left|\alpha_{l,j}\right|^2 \tag{2.61}$$

follows from (2.55) and from the estimate (where $c_1>0$, $C_1>0$)

$$c_1\leq\left|\frac{\kappa a_2}{\lambda_{l,j}+a_1}\right|^2+1\leq C_1,\quad l=1,2,\ j\in\mathbb{N} \tag{2.62}$$

since $|\lambda_{l,j}+a_1|\geq c_2>0$ for all $l=1,2$ and $j\in\mathbb{N}$. This completes the proof. □

Let

$$c_{l,j}=\langle W_{l,j},\tilde{W}_{l,j}\rangle_{L_2(G)^2}^{-1}. \tag{2.63}$$

Since the eigenvalues $\lambda_{l,j}$ are simple, it follows that, after the multiplication of $W_{l,j}$ by $c_{l,j}$, the systems $\{W_{l,j}\}$ and $\{\tilde{W}_{l,j}\}$ are biorthogonal [4], that is,

$$\langle W_{l,j},\tilde{W}_{l,j}\rangle_{L_2(G)^2}=\delta_{(l,j),(l',j')}. \tag{2.64}$$

The semigroup $T(t)^*$ generated by $\mathbf{A}^*$ can be expressed as follows (see [4]):

$$T(t)^*=\sum_{j=1}^{\infty}\sum_{l=1}^{2}e^{\overline{\lambda}_{l,j}t}\langle\cdot,W_{l,j}\rangle_{L_2(G)^2}\tilde{W}_{l,j},\quad t>0. \tag{2.65}$$

Analogous result holds for the semigroup $T(t)$ generated by $\mathbf{A}$.

Remark 2.7. We find that $\mu = -a_4 - Dp^2$ if and only if $\beta(\mu) = 0$. It is easy to see that $\mu = -a_4 - Dp^2$ is not an eigenvalue.

3. Controllability and tracking

3.1. Exact and approximative controllability

Firstly, we consider the controllability of the system. Since S_a is not the adjustable variable (but the disturbance), we assume that S_a is constant and so $S = 0$. In this case, the system becomes (as mentioned above)

$$\frac{\partial W}{\partial t} = \mathbf{A}W + \mathbf{B}_{11}C + \mathbf{B}_{21}C'. \tag{3.1}$$

Since $\mathbf{A}$ generates an exponentially bounded semigroup $T(t)$, we get (again we assume that $W(\cdot,0) = 0$)

$$W = \int_0^t T(t-s)f_1C(s)ds + \int_0^t T(t-s)f_2C'(s)ds =: \mathbf{L}_{1,t}C + \mathbf{L}_{2,t}C, \tag{3.2}$$

where $f_1 = \binom{-\kappa a_2 Q_1}{-Q_2 + a_4 Q_1}$ and $f_2 = \binom{0}{Q_1}$. Expression (3.2) is called a mild solution of (3.1).

System (3.1) is *exactly controllable (approximately controllable)* on $[0,t_0]$ if

$$R\left(\mathbf{L}_{1,t_0} + \mathbf{L}_{2,t_0}\right) = L_2(G)^2, \qquad R\left(\overline{\mathbf{L}_{1,t_0} + \mathbf{L}_{2,t_0}}\right) = L_2(G)^2, \tag{3.3}$$

respectively.

The system is exactly (approximately) controllable if

$$\begin{gathered} \cup_{t_0>0}\left(R\left(\mathbf{L}_{1,t_0} + \mathbf{L}_{2,t_0}\right)\right) = L_2(G)^2, \\ \cup_{t_0>0}\left(R\left(\overline{\mathbf{L}_{1,t_0} + \mathbf{L}_{2,t_0}}\right)\right) = L_2(G)^2, \end{gathered} \tag{3.4}$$

respectively. Since our system is exponentially stable, we are able to characterize the concept of approximately controllability [4].

System (3.1) is approximately controllable if and only if

$$R\left(\overline{\mathbf{L}_{1,\infty} + \mathbf{L}_{2,\infty}}\right) = L_2(G)^2, \tag{3.5}$$

where $\mathbf{L}_{1,\infty}, \mathbf{L}_{2,\infty} : H_0^1([0,\infty[) \mapsto L_2(G)^2$ are the operators

$$\mathbf{L}_{1,\infty}C = \int_0^\infty T(t)f_1C(t)dt, \qquad \mathbf{L}_{2,\infty}C = \int_0^\infty T(t)f_2C'(t)dt. \tag{3.6}$$

Theorem 3.1. *System (3.1) is not exactly controllable on* $[0,t_0]$ *for any* $t_0 > 0$.

Proof. Denote the controllability mapping $\mathbf{L}_{1,t_0} + \mathbf{L}_{2,t_0}$ by K_{t_0}, that is, $K_{t_0} : H_0^1([0,t_0[) \mapsto L_2(G)^2$ is an operator

$$\begin{aligned} K_{t_0}C = \mathbf{L}_{1,t_0}C + \mathbf{L}_{2,t_0}C = &\int_0^{t_0} T(t_0 - s)f_1C(s)ds \\ &+ \int_0^{t_0} T(t_0 - s)f_2C'(s)ds. \end{aligned} \tag{3.7}$$

The operators $K_j : L_2(]0,t_0[) \mapsto L_2(G)^2$, defined by

$$K_jw = \int_0^{t_0} T(t_0 - s)f_jw(s)ds, \tag{3.8}$$

are compact (e.g., [4, Theorem 4.1.5]).

Since the imbedding $\iota : H_0^1([0,t_0[) \mapsto L_2(]0,t_0[)$ and the derivative operator $d/ds : H_0^1([0,t_0[) \mapsto L_2(]0,t_0[)$ are bounded and since $K_{t_0} = K_1 \circ \iota + K_2 \circ d/ds$, we find that also K_{t_0} is compact. Hence we can conclude that system (3.1) is not exactly controllable because the range of a compact operator is either finite dimensional or it is not closed (e.g., [4, Lemma A.3.22]). □

For the approximative controllability, we get the following theorem.

Theorem 3.2. *System (3.1) is approximately controllable if and only if the nonlinear system*

$$(p - \beta(\mu))^2 e^{-\beta(\mu)} = (p + \beta(\mu))^2 e^{\beta(\mu)}, \tag{3.9}$$

$$(\mu - a_4)Q_1 \frac{2ip\beta(\mu)}{p^2 - \beta(\mu)^2} + \gamma D(\mu) = 0 \tag{3.10}$$

has no solutions $\mu < -a_4 - p^2D$, *where*

$$\begin{aligned} D(\mu) = \frac{1}{2i(p^2 - \beta(\mu)^2)} \Big[&2q\Big(p\sinh(\beta(\mu)) - \beta(\mu)\cosh(\beta(\mu))\Big) \\ &+ \beta(\mu)\Big((q+p)e^q + (q-p)e^{-q}\Big)\Big]. \end{aligned} \tag{3.11}$$

Proof. Denote

$$K_\infty = \mathbf{L}_{1,\infty} + \mathbf{L}_{2,\infty} : H_0^1([0,\infty[) \longmapsto L_2(G)^2. \tag{3.12}$$

The proof is based on the fact that $\overline{R(K_\infty)} = L_2(G)^2$ if and only if $R(K_\infty)^\perp = N(K_\infty^*) = \{0\}$ (see [16]).

We must show that our algebraic condition is equivalent to the relation $N(K_\infty^*) = \{0\}$. The adjoint K_∞^* is a linear operator $L_2(G)^2 \mapsto H_0^1([0,\infty[)^*$ such that

$$\langle K_\infty C, v\rangle_{L_2(G)^2} = \langle C, K_\infty^* v\rangle_{L_2(]0,\infty[)} \tag{3.13}$$

for all $C \in H_0^1([0,\infty[)$ and $v \in D(K_\infty^*)$ for which $K_\infty^* v \in L_2(]0,\infty[)$. Since $\mathbf{A}^*$ generates an analytic semigroup $T(t)^*$, we know that the mapping $t \mapsto T(t)^* v$ is differentiable on $]0,\infty[$ and $tT(t)^*$ is bounded on $]0,\infty[$ (cf. [11, 21, 28]). Hence, for any $C \in H_0^1([0,\infty[)$, we can integrate (the functions in $H_0^1([0,\infty[)$ are absolutely continuous) by parts as follows:

$$\begin{aligned}\langle K_\infty C, v\rangle_{L_2(G)^2} &= \left\langle \int_0^\infty \left[T(t) f_1 C(t) + T(t) f_2 C'(t)\right], v\right\rangle_{L_2(G)^2}\\ &= \int_0^\infty C(t) \int_G \left[f_1^* T(t)^* v - f_2^* \frac{d}{dt}\left(T(t)^* v\right)\right] dx\, dt.\end{aligned} \tag{3.14}$$

Hence

$$\left(K_\infty^* v\right)(t) = \int_G \left[f_1^* T(t)^* v - f_2^* \frac{d}{dt}\left(T(t)^* v\right)\right] dx. \tag{3.15}$$

(A) First, suppose that system (3.9) has no solutions. Suppose that $\varphi \in N(K_\infty^*)$. We have to show that $\varphi = 0$. Denote $\xi = T(t)^* \varphi$. By (3.15), we find that

$$\frac{\partial \xi}{\partial t} - \mathbf{A}^* \xi = 0, \quad \xi \in D\left(\mathbf{A}^*\right), \tag{3.16}$$

$$\xi(\cdot, 0) = \varphi, \tag{3.17}$$

$$\int_G \left[f_{11} \xi_1(x,t) + f_{12}(x) \xi_2(x,t) - f_{22} \frac{\partial \xi_2}{\partial t}(x,t)\right] dx = 0, \quad \forall t \in [0,\infty[, \tag{3.18}$$

where $f_{11} = -\kappa a_2 Q_1$, $f_{12}(x) = -Q_2(x) + a_4 Q_1$, and $f_{22} = Q_1$. Note that the first two equations are equivalent to the relation $\xi = T(t)^* \varphi$. The requirement $\xi \in D(\mathbf{A}^*)$ is equivalent to $\xi := (\xi_1, \xi_2) \in L_2(G) \times H^2(G)$ and that ξ_2

satisfies the (adjoint) boundary conditions

$$\frac{\partial \xi_2}{\partial x}(1,t) + \frac{\overline{c}}{D}\xi_2(1,t) = 0, \qquad \frac{\partial \xi_2}{\partial x}(0,t) = 0. \tag{3.19}$$

Suppose, as above, that $\overline{\lambda}_{l,j}$ are the eigenvalues of $\mathbf{A}^*$ and that $\tilde{W}_{l,j} := (\tilde{W}^1_{l,j}, \tilde{W}^2_{l,j})$ are the corresponding eigenfunctions. By (2.65),

$$\begin{aligned} \xi = (\xi_1, \xi_2) = T(t)^*\varphi &= \sum_{j=1}^{\infty}\sum_{l=1}^{2} e^{\overline{\lambda}_{l,j}t}\langle \varphi, W_{l,j}\rangle_{L_2(G)^2}\tilde{W}_{l,j} \\ &= \sum_{j=1}^{\infty}\sum_{l=1}^{2} e^{\overline{\lambda}_{l,j}t}\langle \varphi, W_{l,j}\rangle_{L_2(G)^2}\left(\frac{\kappa a_2}{\overline{\lambda}_{l,j}+a_1}, 1\right)\tilde{\omega}_j. \end{aligned} \tag{3.20}$$

From (3.20), we get ξ_1, ξ_2, and $\partial\xi_2/\partial t$. Since $\Re\lambda_{l,j} \le -c < 0$, $|\tilde{\omega}_j(x)| \le 2(p + |\beta(\mu_j)|)e^{-px}$, and $|\langle \varphi, W_{l,j}\rangle_{L_2(G)^2}| \le C\|\varphi\|_{L_2(G)^2}$ (see [4, page 40]), we can see that the series for ξ_1, ξ_2, and $\partial\xi_2/\partial t$ are uniformly convergent in G for $t > 0$. Substituting ξ_1, ξ_2, and $\partial\xi_2/\partial t$ into (3.18) and changing the order of summation and integration, we see that the requirement (3.18) means that

$$\sum_{j=1}^{\infty}\sum_{l=1}^{2} e^{\overline{\lambda}_{l,j}t}\langle \varphi, W_{l,j}\rangle_{L_2(G)^2}\int_G \left[f_{11}\frac{\kappa a_2}{\overline{\lambda}_{l,j}+a_1} + f_{12}(x) - f_{22}\overline{\lambda}_{l,j}\right]\tilde{\omega}_j(x)dx = 0 \tag{3.21}$$

for $t > 0$. Similarly as in [4, pages 162–164], we find that by (3.21)

$$\langle \varphi, W_{l,j}\rangle_{L_2(G)^2}\int_G \left[\frac{f_{11}\kappa a_2}{\overline{\lambda}_{l,j}+a_1} + f_{12}(x) - f_{22}\overline{\lambda}_{l,j}\right]\tilde{\omega}_j(x)dx = 0, \quad \forall l,j. \tag{3.22}$$

As verified above, we know that $\mu_j < -a_4 - p^2D$. After tedious computations, we find that

$$\begin{aligned} \int_G \tilde{\omega}_j\, dx &= -\frac{2ip\beta(\mu_j)}{p^2 - \beta(\mu_j)^2}, \\ \int_G f_{12}\tilde{\omega}_j\, dx &= -a_4Q_1\frac{2ip\beta(\mu_j)}{p^2 - \beta(\mu_j)^2} - \gamma D(\mu_j). \end{aligned} \tag{3.23}$$

The above calculations imply that (3.22) holds if and only if

$$\begin{aligned}
&\langle \psi, W_{l,j}\rangle_{L_2(G)^2}\Bigg[\frac{f_{11}\kappa a_1}{\lambda_{l,j}+a_1}\cdot\left(-\frac{2ip\beta(\mu_j)}{p^2-\beta(\mu_j)^2}\right)\\
&\qquad\qquad+\left(-\frac{2ip\beta(\mu_j)}{p^2-\beta(\mu_j)^2}+\gamma D(\mu_j)\right)-f_{22}\lambda_{l,j}\left(-\frac{2ip\beta(\mu_j)}{p^2-\beta(\mu_j)^2}\right)\Bigg]\\
&\quad=\langle \psi, W_{l,j}\rangle_{L_2(G)^2}\left[(\mu_j-a_4)Q_1\frac{2ip\beta(\mu_j)}{p^2-\beta(\mu_j)^2}+\gamma D(\mu_j)\right]=0,
\end{aligned} \tag{3.24}$$

where $l=1,2$, $j\in\mathbb{N}$, and μ_j satisfy (2.45). By assumption, $\langle\psi,W_{l,j}\rangle_{L_2(G)^2}=0$ for all l,j and so, by Corollary 2.6, $\psi=0$.

(B) Conversely, suppose that $N(K_\infty^*)=\{0\}$. If system (3.9) has a solution μ_j, then by (2.64) $\tilde{W}_{l,j}\in N(K_\infty^*)$ which is a contradiction. This completes the proof. □

Let

$$H(\mu):=\left|F(\mu)\right|+\left|(\mu-a_4)Q_1\frac{2ip\beta(\mu)}{p^2-\beta(\mu)^2}+\gamma D(\mu)\right|. \tag{3.25}$$

Then μ is a solution of system (3.9) if and only if μ is a zero of H. The analytical consideration of the (transcendental) equation $H(\mu)$=0 is complicated. One possibility in this analysis is to verify that $\liminf_{\mu\to-\infty}H(\mu)\geq c'>0$ for some c', which limits the situation on a finite interval. We omit these considerations here. In Figure 3.1 we give a numerical test. These kind of numerical simulations conjecture that H has no zeros in the region $\mu<-a_4-Dp^2$ for a sample of relevant parameter values and so the system could be approximately controllable.

3.2. Output tracking

We now turn to consider the input-output system

$$\frac{\partial W}{\partial t}=\mathbf{A}W+\mathbf{B}_{11}C+\mathbf{B}_{21}C', \tag{3.26}$$

$$Y=\mathbf{D}_1W-Q_1C, \tag{3.27}$$

where again $W(\cdot,0)=C(0)=0$. Suppose that $Y^*\in L_2(]0,t_0[)$, the so-called *reference output*, is given. The problem: find the input $C\in H_0^1([0,t_0[)$ such that $Y=Y^*$, is called *output tracking problem*.

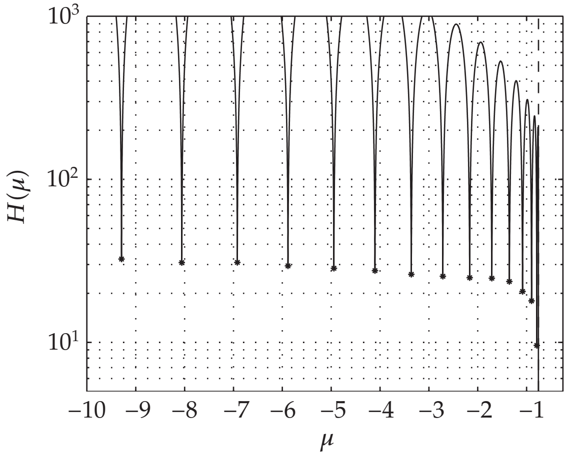

FIGURE 3.1 The graph of $H(\mu)$ for $\mu \in [-10, -a_4 - Dp^2[$. This conjectures that H has no zeros in the region $\mu < -a_4 - Dp^2$. The vertical axis is logarithmic, the stars $(*)$ denote local minimum values of H, and the dotted vertical line denotes the point $\mu = -a_4 - Dp^2$. One sees that the smallest local minimum value is approximately 10. The parameter values used in this plot are given in Table 3.1.

From the first equation (3.27), we can solve as above abstractly the state variable W,

$$W = \int_0^t T(t-s) f_1 C(s) ds + \int_0^t T(t-s) f_2 C'(s) ds. \tag{3.28}$$

Due to the Bochner's theorem, we can change the order of integration

$$\mathbf{D}_1 \left(\int_0^t T(t-s) f_1 C(s) ds \right) = \int_0^t \mathbf{D}_1 \big(T(t-s) f_1 \big) C(s) ds \tag{3.29}$$

and similarly for the integral $\int_0^t T(t-s) f_2 C'(s) ds$. Hence from (3.27), we get

$$\begin{aligned} Y = &\int_0^t pr_2 \big(T(t-s) f_1 \big)(1) C(s) ds \\ &+ \int_0^t pr_2 \big(T(t-s) f_2 \big)(1) C'(s) ds - Q_1 C. \end{aligned} \tag{3.30}$$

Denote

$$g_1(t,s) = pr_2 \big(T(t-s) f_1 \big)(1), \qquad g_2(t,s) = pr_2 \big(T(t-s) f_2 \big)(1). \tag{3.31}$$

TABLE 3.1

Parameter values used in the simulation			
$D = 0.005$	$k_1 = 0.4$	$k_2 = 0.4$	$k_d = 0.05$
$\mu_m = 0.35$	$\overline{c} = 0.1$	$\overline{S}_a = 5$	

Then the output Y is

$$Y = \int_0^t g_1(t-s)C(s)ds + \int_0^t g_2(t-s)C'(s)ds - Q_1 C. \tag{3.32}$$

The output and input *necessarily* satisfy (3.32).

THEOREM 3.3. *System (3.27) is not exactly trackable to all reference outputs $Y \in L_2(]0,t_0[)$.*

Proof. The semigroup $T(t)$ satisfies the estimate (see [28])

$$\|T(t)f\|_H \leq Ct^{-1/2}\|f\|_{L_2(G)^2}, \quad f \in L_2(G)^2. \tag{3.33}$$

Using the Sobolev imbedding theorem (2.19) and estimate (3.33), we find that

$$\begin{aligned} |g_j(t-s)| &= |(pr_2(T(t-s)f_j))(1)| \leq \|pr_2(T(t-s)f_j)\|_{H^1(G)} \\ &\leq \|T(t-s)f_j\|_H \leq \frac{C}{(t-s)^{1/2}}\|f_j\|_{L_2(G)^2}. \end{aligned} \tag{3.34}$$

Hence we find that the Volterra integral operators

$$K_j C = \int_0^t pr_2(T(t-s)f_j)(1)C(s)ds, \quad j = 1,2, \tag{3.35}$$

are compact operators $L_2(]0,t_0[) \mapsto L_2(]0,t_0[)$ (e.g., [10]). The imbedding $\iota : H_0^1([0,t_0[) \mapsto L_2(]0,t_0[)$ is compact (e.g., [21]) and the mapping $\partial_t : H_0^1([0,t_0[) \mapsto L_2(]0,t_0[)$ is bounded. Hence the operator

$$K := K_1 \circ \iota + K_2 \circ \partial_t - Q_1 \iota \tag{3.36}$$

is compact.

By (3.30), we know that the necessary condition for Y and C is that

$$Y = KC. \tag{3.37}$$

Since K is compact, the range $R(K)$ is either finite dimensional or the range $R(K)$ is not closed (e.g., [4, Lemma A.3.22]). Hence Y cannot be an arbitrary element in $L_2(]0,t_0[)$ which completes the proof. □

4. Parametrization and output tracking applying frequency-domain methods

4.1. Transfer function

The transfer function of system (2.18) is given by

$$G(\lambda) = \mathbf{D}_1(\lambda I - \mathbf{A})^{-1}(\mathbf{B}_1 + \lambda \mathbf{B}_2) + \mathbf{D}_2. \tag{4.1}$$

Denote

$$\begin{aligned} \mathbf{B}_{11} &= \begin{pmatrix} -\kappa a_2 Q_1 \\ -Q_2 + a_4 Q_1 \end{pmatrix}, \qquad \mathbf{B}_{12} = \begin{pmatrix} \kappa a_2 \\ -a_4 \end{pmatrix}, \\ \mathbf{B}_{21} &= \begin{pmatrix} 0 \\ Q_1 \end{pmatrix}, \qquad \mathbf{B}_{22} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}. \end{aligned} \tag{4.2}$$

We find that

$$G(\lambda) = \begin{pmatrix} G_1(\lambda) & G_2(\lambda) \end{pmatrix}, \tag{4.3}$$

where

$$\begin{aligned} G_1(\lambda) &= \mathbf{D}_1(\lambda I - \mathbf{A})^{-1}(\mathbf{B}_{11} + \lambda \mathbf{B}_{21}) - Q_1, \\ G_2(\lambda) &= \mathbf{D}_1(\lambda I - \mathbf{A})^{-1}(\mathbf{B}_{12} + \lambda \mathbf{B}_{22}) + 1. \end{aligned} \tag{4.4}$$

We have previously verified that (see [18, 29])

$$G_1(\lambda) = K \frac{b_1 r(\lambda)\cosh(r(\lambda)) + b_2 \sinh(r(\lambda)) + b_3 r(\lambda)^3 - b_4 r(\lambda)}{\left[(p^2 + r(\lambda)^2)\sinh(r(\lambda)) + 2pr(\lambda)\cosh(r(\lambda))\right](r(\lambda)^2 - q^2)}, \tag{4.5}$$

where

$$\begin{aligned} &b_1 = Pq, \qquad b_2 = Pqp, \qquad b_3 = \sinh q, \\ &b_4 = q^2 \sinh q + Pq \sinh q + Pq \cosh q, \\ &r(\lambda) = \sqrt{\frac{\lambda^2 + d_1\lambda + d_2}{D(\lambda + a_1)}}, \qquad d_1 = a_1\left(1 - \frac{k_1}{k_2}\right) + q^2 D, \qquad d_2 = D a_1 q^2. \end{aligned} \tag{4.6}$$

Similarly, we find that

$$G_2(\lambda) = -\frac{2pe^p r(\lambda)}{\left(p^2 + r(\lambda)^2\right)\sinh\left(r(\lambda)\right) + 2pr(\lambda)\cosh\left(r(\lambda)\right)} + 1. \tag{4.7}$$

We have also shown that $G_1(\lambda) \in \hat{A}_-(0)$ and that

$$\left|G_1(\lambda)\right| \leq C\frac{1}{|\lambda|} \quad \text{for } |\lambda| \geq r_0, \tag{4.8}$$

where r_0 is large enough [29]. Similarly, we can show that $G_2(\lambda)$ satisfies the same properties. Hence $G(\lambda) \in \hat{A}_-(0)$.

The total transfer function $G(\lambda)$ can be used in the refined study and controller design of the input-output system (e.g., disturb rejection, [3]).

4.2. A scheme for state space parametrization

We give a parametrization methods for infinite-dimensional systems where we use pseudodifferential operators. Parametrization can be considered as a kind of flatness property which has been treated for ordinary differential equation systems, for example, in [5, 8, 22]. The concept is also related to the so-called π-freeness of systems (see [9]). Roughly speaking, the parametrization means that, for a given system $\mathbf{Lv} = 0$, we can find an operator $\mathbf{S}$ such that $\mathbf{Lv} = 0$ if and only if $\mathbf{v} = \mathbf{Sf}$ where the components $f_1, \dots, f_m$ of $\mathbf{f}$ are *independent variables*. The methodologies below can be generalized also for infinite-dimensional MIMO systems. In the following, we however consider only the SISO system which occurs in our application.

Assume that the transfer function $G(\lambda)$ of the given system belongs to the Callier-Desoer class $\hat{B}(0)$ (see [4]). Then it has a coprime factorization over $\hat{A}_-(0)$

$$G = \frac{N}{M}, \qquad PN - QM = 1 \quad \text{(Bezout identity)}, \tag{4.9}$$

where $N, M, P, Q \in \hat{A}_-(0)$. Define

$$\hat{Z} = P\hat{Y} - Q\hat{C} \tag{4.10}$$

and put $Z = L^{-1}\hat{Z}$.

Let $u : [0,\infty[\mapsto \mathbb{R}$ be a function and let $e^+u : \mathbb{R} \mapsto \mathbb{R}$ be its extension by zero on $]-\infty, 0[$. The Paley-Wiener theory implies that $u \in L_2(]0,\infty[)$ if and only if the function $\alpha + i\xi \mapsto \widehat{e^+u}(\alpha + i\xi)$ is analytic in $\mathbb{C}_+$ and

$\sup_{\alpha>0}\int_{-\infty}^{\infty}|\widehat{e^{+}u}(\alpha+i\xi)|^2 d\xi<\infty$. In addition,

$$c\|u\|^2_{L_2(]0,\infty[)} \le \sup_{\alpha>0}\int_{-\infty}^{\infty}\left|\widehat{e^{+}u}(\alpha+i\xi)\right|^2 d\xi \le C\|u\|^2_{L_2(]0,\infty[)}. \tag{4.11}$$

Furthermore, $u\in H_0^k([0,\infty[)$, $k\in\mathbb{N}_0$, if and only if

$$\sum_{0\le l\le k}\sup_{\alpha>0}\int_{-\infty}^{\infty}|\alpha+i\xi|^{2l}\left|\widehat{e^{+}u}(\alpha+i\xi)\right|^2 d\xi<\infty,$$

$$c_1\|u\|^2_{H_0^k([0,\infty[)} \le \sum_{0\le l\le k}\sup_{\alpha>0}\int_{-\infty}^{\infty}|\alpha+i\xi|^{2l}\left|\widehat{e^{+}u}(\alpha+i\xi)\right|^2 d\xi \le c_2\|u\|^2_{H_0^k([0,\infty[)}. \tag{4.12}$$

A similar characterization holds for spaces $H_0^s([0,\infty[)$, where s is any real number.

Theorem 4.1. *Suppose that $G\in\hat{B}(0)$ and let its coprime factorization be (4.9). Furthermore, suppose that Y, $C\in H_0^k([0,\infty[)$, $k\ge 1$. Then $Z\in H_0^k([0,\infty[)$ and Z gives an input-output parametrization of the system in the sense that, for any (fixed) $\alpha>0$,*

$$\begin{aligned} C &= e^{\alpha t}M_\alpha(D)\left(e^{-\alpha t}Z\right),\\ Y &= e^{\alpha t}N_\alpha(D)\left(e^{-\alpha t}Z\right),\\ Z &= e^{\alpha t}P_\alpha(D)\left(e^{-\alpha t}Y\right)-e^{\alpha t}Q_\alpha(D)\left(e^{-\alpha t}C\right), \end{aligned} \tag{4.13}$$

where $M_\alpha(D)$, $N_\alpha(D)$, $P_\alpha(D)$, and $Q_\alpha(D)$ are (zero order) pseudodifferential operators.

Proof. Since P and Q are in $\hat{A}_-(0)$, they are bounded in $\mathbb{C}_+$. We find by (4.12) that

$$\begin{aligned} \|Z\|^2_{H_0^k([0,\infty[)} &\le \sum_{0\le l\le k}\sup_{\alpha>0}\int_{-\infty}^{\infty}|\alpha+i\xi|^{2l}\left|\left(P\hat{Y}-Q\hat{C}\right)(\alpha+i\xi)\right|^2 d\xi\\ &\le C_1c_1\|Y\|^2_{H^k([0,\infty[)}+C_2c_1\|C\|^2_{H^k([0,\infty[)}, \end{aligned} \tag{4.14}$$

where C_1 and C_2 are

$$C_1=\sup_{z\in\mathbb{C}_+}\left|P(z)\right|^2,\qquad C_2=\sup_{z\in\mathbb{C}_+}\left|Q(z)\right|^2. \tag{4.15}$$

Hence we find that $Z\in H_0^k([0,\infty[)$.

Since $\hat{Y} = (N/M)\hat{C}$ and $PN - QM = 1$, we find that

$$M\hat{Z} = MP\hat{Y} - MQ\hat{C} = (NP - MQ)\hat{C} = \hat{C} \tag{4.16}$$

and similarly $\hat{Y} = N\hat{Z}$.

Taking the inverse Laplace transform, we get, for $\alpha > 0$,

$$C = \frac{1}{2\pi}\int_{-\infty}^{\infty} M(\alpha + i\xi)\hat{Z}(\alpha + i\xi)e^{(\alpha+i\xi)t}d\xi. \tag{4.17}$$

We find that $\hat{Z}(\alpha + i\xi) = F(e^{+}(e^{-\alpha t}Z))(\xi)$, where F denotes the Fourier transform. Let $M_\alpha(D)$ be a pseudodifferential operator with symbol $M(\alpha + i\xi)$, that is,

$$M_\alpha(D)u(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} M(\alpha + i\xi)\big(F\big(e^{+}u\big)\big)(\xi)e^{i\xi t}d\xi \tag{4.18}$$

(see [13, 24]). Then we find that, for $t > 0$,

$$C = e^{\alpha t}M_\alpha(D)\big(e^{-\alpha t}Z\big). \tag{4.19}$$

Similarly,

$$\begin{aligned} Y &= e^{\alpha t}N_\alpha(D)\big(e^{-\alpha t}Z\big), \\ Z &= e^{\alpha t}P_\alpha(D)\big(e^{-\alpha t}Y\big) - e^{\alpha t}Q_\alpha(D)\big(e^{-\alpha t}C\big), \end{aligned} \tag{4.20}$$

where $N_\alpha(D)$, $P_\alpha(D)$, and $Q_\alpha(D)$ are pseudodifferential operators with symbols $N(\alpha + i\xi)$, $P(\alpha + i\xi)$, and $Q(\alpha + i\xi)$, respectively. This completes the proof. □

In many cases (such as in our application here, see [18]), the state can be solved from

$$\hat{W}(x,\lambda) = (\lambda I - \mathbf{A})^{-1}(f_1 + \lambda f_2)\hat{C}. \tag{4.21}$$

One knows that, under quite general assumptions (see [13]), the resolvent is of the form

$$(\lambda I - \mathbf{A})^{-1} = P(\cdot,\partial,\lambda) + G(\cdot,\partial,\lambda), \tag{4.22}$$

where $P(\cdot,\partial,\lambda)$ and $G(\cdot,\partial,\lambda)$ are parameter (λ) dependent pseudodifferential and singular Green operators, respectively. In this case, also the state has a parametrization

$$W = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left((\alpha+i\xi)-\mathbf{A}\right)^{-1}\left(f_1+(\alpha+i\xi)f_2\right)M(\alpha+i\xi)\hat{Z}(\alpha+i\xi)e^{(\alpha+i\xi)t}d\xi$$
$$= \frac{1}{2\pi}e^{\alpha t}\int_{-\infty}^{\infty}a_\alpha(x,\xi)F\left(e^{+}\left(e^{-\alpha t}Z\right)\right)(\xi)e^{i\xi t}d\xi, \tag{4.23}$$

where

$$a_\alpha(x,\xi) = \left[P(x,\partial,\alpha+i\xi)+G(x,\partial,\alpha+i\xi)\right]\left(f_1+(\alpha+i\xi)f_2\right)M(\alpha+i\xi). \tag{4.24}$$

Hence we find that

$$W = e^{\alpha t}K_\alpha\left(e^{-\alpha t}Z\right), \tag{4.25}$$

where K_α is the operator

$$K_\alpha u(x,t) = \frac{1}{2\pi}e^{\alpha t}\int_{-\infty}^{\infty}a_\alpha(x,\xi)F\left(e^{+}\left(e^{-\alpha t}u\right)\right)(\xi)e^{i\xi t}d\xi. \tag{4.26}$$

Note that in [9, Section 4.4] (in the context of Euler-Bernoulli equation) one in fact uses an expression like (4.21).

4.3. Calculation of reference input

The above parametrization can be applied in the design of output tracking as follows.

Theorem 4.2. *Suppose that $N \in \hat{A}_{-}(0)$ and that there exist $\alpha_0 \geq 0$ and $r \geq 0$ such that*

$$\frac{1}{\left|N(\alpha_0+z)\right|} \leq c\left(1+|z|\right)^r, \quad \forall z \in \mathbb{C}_+. \tag{4.27}$$

Denote $N_{\alpha_0}^{-1}(\xi) = 1/N(\alpha_0+i\xi)$ and let $k \geq 0$. Then for any Y for which $e^{-\alpha_0 t}Y \in H_0^{k+r}([0,\infty[)$, the equation

$$Y = e^{\alpha_0 t}N_{\alpha_0}(D)\left(e^{-\alpha_0 t}Z\right) \tag{4.28}$$

has a solution Z such that $e^{-\alpha_0 t}Z \in H_0^k([0,\infty[)$. The solution Z is given by

$$Z = e^{\alpha_0 t} N_{\alpha_0}^{-1}(D)\left(e^{-\alpha_0 t}\Upsilon\right). \tag{4.29}$$

Proof. We see immediately that (4.29) is the solution of (4.28). Furthermore, we find that

$$\begin{aligned}
&c_1\left\|e^{-\alpha_0 t}Z\right\|^2_{H_0^k([0,\infty[)}\\
&\quad\le \sum_{0\le l\le k}\sup_{\alpha>0}\int_{-\infty}^{\infty}|\alpha+i\xi|^{2l}\left|\hat{Z}(\alpha_0+\alpha+i\xi)\right|^2 d\xi\\
&\quad\le \sum_{0\le l\le k}\sup_{\alpha>0}\int_{-\infty}^{\infty}|\alpha+i\xi|^{2l}\left|\frac{\hat{\Upsilon}(\alpha_0+\alpha+i\xi)}{N(\alpha_0+\alpha+i\xi)}\right|^2 d\xi\\
&\quad\le c^2\sum_{0\le l\le k}\sup_{\alpha>0}\int_{-\infty}^{\infty}|\alpha+i\xi|^{2l}\left|(1+|\alpha+i\xi|)^{2r}\right|\widehat{e^{-\alpha_0 t}\Upsilon}(\alpha+i\xi)|^2 d\xi\\
&\quad\le C'\left\|e^{-\alpha_0 t}\Upsilon\right\|^2_{H_0^{k+r}([0,\infty[)}.
\end{aligned} \tag{4.30}$$

This completes the proof. □

We turn to the output tracking problem of our application. Assume that $S = 0$. Since $G_1 \in \hat{A}_-(0)$, it has the coprime factorization $G_1 = N/M$, $PN - QM = 1$. Actually, we can choose $N = G_1$, $M = 1$, $P = 0$, $Q = 1$, for example. In this case, $\hat{Z} = \hat{C}$ and $\hat{\Upsilon} = G_1\hat{C}$.

We have the following technical lemma whose proof is omitted.

Lemma 4.3. *There exist constants $\epsilon > 0$ and $R > 0$ such that*

$$\left|\frac{1}{G_1(\lambda)}\right| \le c(1+|\lambda|)^{3/2}, \quad \forall\lambda\in\mathbb{C}_{-\epsilon}; |\lambda|\ge R. \tag{4.31}$$

From Lemma 4.3, we see that there exists $\alpha_0 \ge 0$ such that

$$\left|\frac{1}{G_1(\alpha_0+z)}\right| \le c(1+|z|)^{3/2}, \quad \forall z\in\mathbb{C}_+. \tag{4.32}$$

Hence we are able to apply Theorem 4.2 with $r = 3/2$. We get that, at least for any Υ for which $e^{-\alpha_0 t}\Upsilon \in H_0^{3/2+1}([0,\infty[)$, there exists C such that $e^{-\alpha_0 t}C \in H_0^1([0,\infty[)$. In addition, C can be calculated from

$$C = e^{\alpha_0 t}\Phi(D)\left(e^{-\alpha_0 t}\Upsilon\right), \quad t > 0, \tag{4.33}$$

where $\Phi(D)$ is the pseudodifferential operator with symbol $\Phi(\xi) = 1/G_1(\alpha_0 + i\xi)$:

$$\Phi(D)u(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1}{G_1(\alpha_0 + i\xi)} \big(F(e^{+}u)\big)(\xi)e^{i\xi t}d\xi. \tag{4.34}$$

For a given reference output Y^*, we can compute the input by

$$C^* = e^{\alpha_0 t}\Phi(D)\big(e^{-\alpha_0 t}Y^*\big). \tag{4.35}$$

To improve the convergence of the integrals in computation, we can use the formula

$$C^* = \frac{d^m}{dt^m}\big[e^{\alpha_0 t}\Phi_m(D)\big(e^{-\alpha_0 t}Y^*\big)\big], \tag{4.36}$$

where $m \in \mathbb{N}_0$ and

$$\Phi_m(\xi) = \frac{1}{(\alpha_0 + i\xi)^m G_1(\alpha_0 + i\xi)}. \tag{4.37}$$

This formula easily follows from the fact that $L(\int_0^t \cdots \int_0^t C(s)ds)(\lambda) = (1/\lambda^m)\hat{C}(\lambda)$.

Remark 4.4. In the case, where $Y \in \cap_{k\in\mathbb{N}_0} H_0^k([0,\infty[)$, we formally have

$$\Phi(\xi) = \sum_{j=0}^{\infty} \frac{1}{j!}\Phi^{(j)}(0)(i\xi)^j. \tag{4.38}$$

Since $(i\xi)^j(Fu)(\xi) = F(u^{(j)})(\xi)$, we formally get

$$\Phi(D)u = \sum_{j=0}^{\infty} \frac{1}{j!}\Phi^{(j)}(0)u^{(j)}. \tag{4.39}$$

Hence, for $t > 0$,

$$C = e^{\alpha_0 t}\sum_{j=0}^{\infty} \frac{1}{j!}\Phi^{(j)}(0)\big(e^{-\alpha_0 t}Y\big)^{(j)}, \tag{4.40}$$

that is, we have obtained the C as an infinite linear combination of derivatives of the output Y. In this sense, Y is a flat output (cf. [9]). Truncating the series, one is able to get the input by a finite number of differentiations.

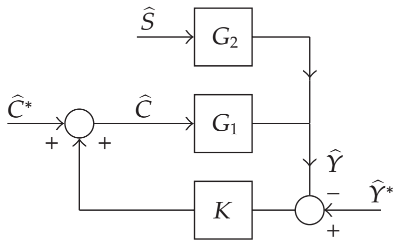

FIGURE 4.1 The closed loop system.

4.4. *Compensator design for tracking*

In the case where we have no disturbance ($S = 0$), we can calculate for a given reference output Y^* the required input C^* from (4.35). When the disturbance exists, that is, $S \neq 0$, we can construate the compensator, for example, as follows.

We consider the closed loop system illustrated in Figure 4.1.

By Section 4.1, $\hat{Y}^* = G_1\hat{C}^*$ and so $\hat{C}^* = \hat{Y}^*/G_1$ at least for $\Re\lambda \geq \alpha_0$, where α_0 is as in (4.32). The input $\hat{C}$ is

$$\hat{C} = K(\hat{Y}^* - \hat{Y}) + \hat{C}^*. \tag{4.41}$$

Taking into account that $Y = W_2(1,\cdot) - Q_1C + S$ and $\hat{C}^* = \hat{Y}^*/G_1$, we formally get after a small algebra that

$$\hat{C} = \frac{K + 1/G_1}{1 - Q_1K}\hat{Y}^* - \frac{K}{1 - Q_1K}(\hat{W}_2(1,\cdot) - \hat{d}), \tag{4.42}$$

where $\hat{d} = G_2\hat{S}$. Hence the input can be calculated from

$$C = e^{\alpha_0 t}\Phi_1(D)(e^{-\alpha_0 t}Y^*) - e^{\alpha_0 t}\Phi_2(D)(e^{-\alpha_0 t}(W_2(1,\cdot) - d)) =: \tilde{C}(W), \tag{4.43}$$

where $d = L^{-1}(G_1\hat{S})$ (disturbance in the output) and

$$\Phi_1(\xi) = \left(\frac{K + 1/G_1}{1 - Q_1K}\right)(\alpha_0 + i\xi), \qquad \Phi_2(\xi) = \left(\frac{K}{1 - Q_1K}\right)(\alpha_0 + i\xi). \tag{4.44}$$

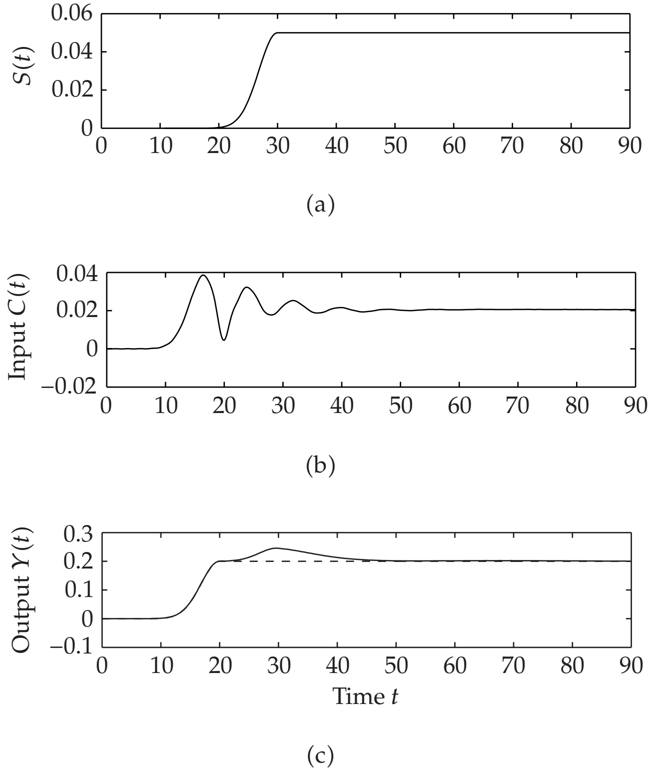

FIGURE 5.1 Closed loop simulation: (a) the input substrate concentration S, (b) the feedback control C, and (c) the reference output Y^* (dotted) and the resulting output Y (solid).

The compensator can be chosen to be $K(\lambda) = \epsilon/\lambda$ which gives an exponentially stable feedback dynamics. We have not tried to optimize the compensator design.

5. Numerical results

In simulations, we have used the following relevant parameter values (Table 3.1). The control law is computed by (4.43). The feedback equation

$$\frac{\partial W}{\partial t} = \mathbf{A}W + \mathbf{B}_{11}\tilde{C}(W) + \mathbf{B}_{21}\frac{\partial}{\partial t}\big(\tilde{C}(W)\big), \qquad W(0) = 0, \tag{5.1}$$

is solved by an iterative scheme.

At the moment $t = 10$, the reference output is changed rapidly from 0 (which corresponds to the chosen steady state) to $Y^* = 0.2$. At the moment $t = 20$, the substrate concentration $S_a = 5$ is changed to $S_a = 5.5$ (Figure 5.1a). This causes a disturbance $d = e^{\alpha_0 t}\Theta(D)(e^{-\alpha_0 t}S)$ in the output, where $\Theta(\xi) = G_2(\alpha_0 + i\xi)$. The closed loop control is seen in Figure 5.1b. The reference output Y^* and the computed output Y are plotted in Figure 5.1c. In $K(\lambda)$ we have chosen $\epsilon = 0.01$. The tracking is successful and stable. The input is oscillating smoothly but the oscillation is damping.

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