ANALYSIS OF A MULTIPLE-POROSITY MODEL FOR SINGLE-PHASE FLOW THROUGH NATURALLY FRACTURED POROUS MEDIA

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Received 28 May 2002 and in revised form 24 February 2003

A derivation of a multiple-porosity model for the flow of a single phase, slightly compressible fluid in a multiscale, naturally fractured reservoir is presented by means of recursive use of homagnetization theory. We obtain a model which generalizes the double-porosity model of Arbogast et al. (1990) to a flow system with an arbitrary finite number of scales.

1. Introduction

A model for single-phase flow in porous media that are hierarchically fissured in regular patterns was derived by a recursive asymptotic expansion technique in [16] and part of [24]. This work rigorously justifies that model. Through recursive homogenization, we extend the double-porosity model in [5], which has one fracture system and a matrix (rock) block system, to a triple-porosity model that has two levels of fracture systems and a matrix block system. See [9, 22] for an introduction to homogenization theory. Then, a multiple-porosity model with N levels of fracture systems and a matrix block system is derived, resulting in a general $(N+1)$-scale model.

A dual-porosity concept was first introduced in [8, 25] using a specific transmissibility function (see [7]) for the interaction of the matrix-fracture flow. For petroleum-reservoir engineering problems, a new treatment of the coupling of the flow through the fracture system with that in the matrix system was introduced over the past two decades in [2, 3, 4, 5, 6, 13, 14, 17, 18, 19]. The models discussed in this work are based on these ideas. Our focus on the nested levels of fracture systems is appropriate for further studies on high-level nuclear waste transport

Journal of Applied Mathematics 2003:7 (2003) 327–364
2000 Mathematics Subject Classification: 76S05
URL: http://dx.doi.org/10.1155/S1110757X03205143

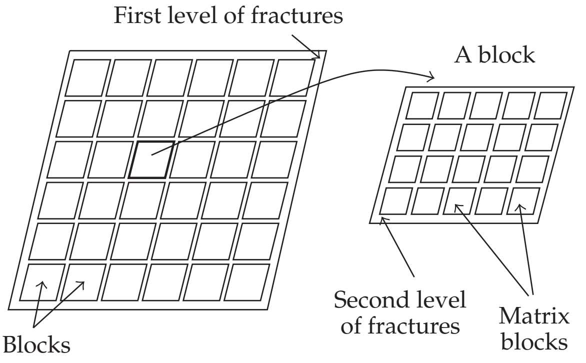

FIGURE 1.1. The periodic structure in a reservoir exhibiting two levels of fractures.

in porous media. It is the long-time scales, due to the length of the half-lives of some high-level nuclear elements, that allow for the possibility of nested levels of fracture systems in porous media (see [15, 20]).

Since we are interested in the mathematical details of the problem here, we refer to the introductions of [16, 24] for further details on the applicability of the models.

This work deals with modeling a single-phase, constant-compressibility fluid flowing in a geometrically complicated structure given (initially) by a naturally fractured reservoir that has a hierarchy of fracture systems, with the first being defined by an interconnected system of planar fractures dividing the reservoir into a collection of disjoint blocks. A second system of fractures divides each of the previous blocks into a collection of equally sized smaller blocks, and so forth, until a last level is reached in which the disjoint blocks behave as a collection of disjoint unfractured matrix blocks. The geometric structure is idealized by the assumption that each fracture system is periodic. See Figure 1.1 for a cross-sectional view of the idealized reservoir in the case of two levels of fracturing.

We begin by posing the flow equations on three different scales of the domain. This involves using three different porosity and permeability coefficients, one for each scale, since the fluid flows more readily through the fracture systems than it does through the matrix blocks. Via a parameter ε_1, which represents the linear size of a matrix block and half of its surrounding fractures, we first homogenize the flow equations on the smallest level of fractures and the matrix blocks. This gives an overall fracture flow in each of the fractured blocks. The porous matrix blocks provide a source term to the surrounding system of small-scale fractures which, after homogenization, are treated as a continuous porous medium. Thus, a continuous medium approach takes place between the smallest level of fractures and the matrix blocks. This is a scaled mesoscopic description since the equations depend on the parameter ε_0, which represents the linear size of the scaled fractured blocks. Following this averaging, we couple the equations for flow in the largest level of fractures with the fractured blocks, each of which is represented now by a double-porosity system. Averaging the flow in the first level of fractures then gives a macroscopic description of the flow in a three-sheeted covering of the domain; this extends the concepts of the models of [13]. Thus, the first level of fractures is now smoothed out to cover the entire domain and the blocks interact with the first level of fractures as sources, while the behavior of the flow on a first-level block is that of a double-porosity system. Overall, the system can be characterized as a triple-porosity model.

The $(N+1)$-scale analysis discussed in this paper can be used as a tool for analyzing problems with multiple scales of periodicity (i.e., homogeneous, hierarchically organized media). But in the presence of heterogeneities, [11] addresses two-scale convergence in the mean and includes applications to randomly fractured media [21]. However, such an approach does not apply immediately to heterogeneities with multiple scales of correlation, which is the case for many natural porous media. The extension of the $(N+1)$-scale approach introduced here is hence an alternative that may improve our understanding of the flow phenomena in some natural porous media.

We first present the triple-porosity model in order to illustrate, in a simpler and more readily understandable situation, the general techniques that are necessary for the $(N+1)$-scale model. However, intermediate source terms that are not present in the triple-porosity model appear in the $(N+1)$-scale model and they require additional arguments.

The organization of the paper is as follows. In Section 2, the assumptions, notation, and description of the triple-porosity reservoir are given. Also, two dilation and two location operators are defined. In Section 3, the microscale model, which involves both the intermediate and microscopic levels, is formulated using the parameters ε_1 and ε_0 described above. The coefficients are precisely defined on the appropriate parts of Ω. Next, in Section 4, the weak formulation of the microscale model is given, well-posedness is proven, a priori estimates are derived, and several technical lemmas regarding the dilation operators are presented. Then, the convergence results for the first homogenization ($\varepsilon_1 \to 0$) lead to a well-posed mesoscopic system of equations in Section 5. Then, in Section 6, a completely new well-posed problem is formulated in terms of the parameter ε_0, using the resulting model in Section 5 with a new boundary condition that conserves mass flux. In Section 7, a general

$(N+1)$-scale problem with N levels of fractures and the matrix (porous) level is presented. Previous lemmas and theorems are generalized in this section, with the double- and triple-porosity models serving as base cases for the homogenization procedure. The final well-posed system of equations is presented for the finite scale. Finally, in Section 8, concluding remarks on generalizations of the model are made.

2. Notation, assumptions, and preliminary lemmas

We begin this section by defining the nested periodic structure of the domain Ω in the presence of N levels of fractures. First, for $i=0,\ldots,N-1$, let Y_i be a parallelepiped and let $\delta_i \in (0,1)$ be such that $|Y_0| \ll \delta_0|\Omega|$ and, for $i=1,\ldots,N-1$, $|Y_i| \ll \delta_i|Y_{i-1}|$. Then, with $Y_{-1B}=\Omega$, define A^{if} to be a finite lattice containing the origin such that

$$\overline{Y}_{(i-1)B} = \bigcup_{c^i \in A^{if}} \left(\overline{Y}_i + c^i\right), \tag{2.1}$$

where

$$Y_i = Y_{iB} \cup \partial Y_{iB} \cup Y_{iF}. \tag{2.2}$$

Now, extend the lattice A_{if} into an infinite lattice A^i, containing the origin and define $\Omega_F^{\varepsilon_0}$, $\Omega_B^{\varepsilon_0}$ by

$$\Omega_F^{\varepsilon_0} = \Omega \cap \left(\bigcup_{c^0 \in A^0} \varepsilon_0\left(Y_{0F} + c^0\right)\right), \qquad \Omega_B^{\varepsilon_0} = \Omega \cap \left(\bigcup_{c^0 \in A^0} \varepsilon_0\left(Y_{0B} + c^0\right)\right), \tag{2.3}$$

and for $i=1,2,\ldots,N-1$, define

$$\Omega_{B,H}^{\varepsilon_0\cdots\varepsilon_i} = \Omega_{B,B}^{\varepsilon_0\cdots\varepsilon_{i-1}} \cap \left(\bigcup_{c^0\in A^0} \cdots \bigcup_{c^i \in A^i} \left(\varepsilon_0\cdots\varepsilon_i\left(Y_{iH} + c^i\right) + \varepsilon_0\cdots\varepsilon_{i-1}c^{i-1} + \cdots + \varepsilon_0 c^0\right)\right), \tag{2.4}$$

where $H = B$ or F (see Figure 2.1).

Since we are assuming that there are N levels of fractures in Ω, we let $\varepsilon_0,\ldots,\varepsilon_{N-1}$ be the parameters associated with the homogenization. In order to define dilation operators that incorporate each of these parameters, we proceed as follows. For $i=0,\ldots,N-1$, let

$$c^{i,\varepsilon_i} : Y_{(i-1)B} \longrightarrow \varepsilon_i A^i, \tag{2.5}$$

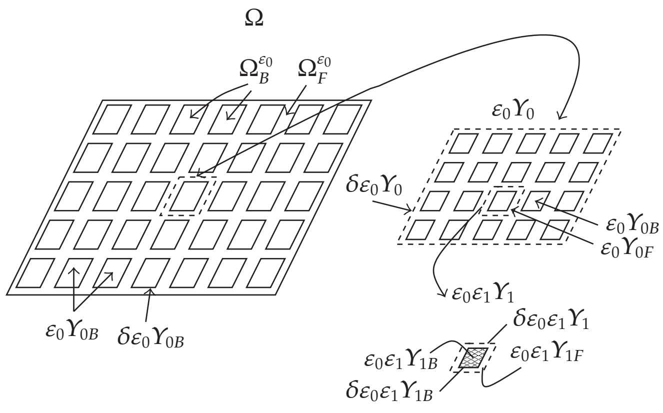

FIGURE 2.1. The parts of the reservoir.

where, for $x_i \in Y_{(i-1)B}$, $c^{i,\varepsilon_i} \in \varepsilon_i A^i$ is the lattice translation point of the $\varepsilon_i Y_i$-cell containing x_i, that is, c^{i,ε_i} is the lattice translation vector such that $x_i \in \varepsilon_i Y_i + c^{i,\varepsilon_i}(x_i)$.

Then define the dilation operator ${}^{\sim(i)}$, from the set of functions defined on $\Omega_{B,H}^{\varepsilon_0\cdots\varepsilon_{i-1}}$ $(H = B, F)$ to functions defined on $\Omega \times \cdots \times Y_{(i-2)B} \times Y_{(i-1)H}$, by

$$f^{\sim(i)}(x_0,\ldots,x_i) = f\big((\varepsilon_0\cdots\varepsilon_{i-1})x_i + (\varepsilon_0\cdots\varepsilon_{i-2})c^{(i-1),\varepsilon_{i-1}}(x_{i-1}) + \cdots + \varepsilon_0 c^{1,\varepsilon_1}(x_1) + c^{0,\varepsilon_0}(x_0)\big). \tag{2.6}$$

We also make heavy use of the definition of ${}^{\sim(i)}$ for functions defined on $\Omega_{B,B}^{\varepsilon_0\cdots\varepsilon_{i-2}}$ in the same way, except that in this case, ${}^{\sim(i)}$ maps to functions defined on $\Omega \times Y_{0B} \times \cdots \times Y_{(i-2)B} \times Y_{i-1}$.

For convenience, we recursively define the following location operators:

$$L_{-1}(x_0) = x_0, \quad L_0(x_0) = \frac{x_0 - c^{0,\varepsilon_0}(x_0)}{\varepsilon_0} \in \overline{Y}_0 \quad \text{for } x_0 \in \Omega, \tag{2.7}$$

and, in general, for $x_0 \in \Omega_{B,B}^{\varepsilon_0\cdots\varepsilon_{i-1}}$,

$$L_i(x_0) = \frac{L_{i-1}(x_0) - c^{i,\varepsilon_i}(L_{i-1}(x_0))}{\varepsilon_i} \in \overline{Y}_i \quad \text{for } i = 1,\ldots,N-1. \tag{2.8}$$

Let $\rho_i^{\varepsilon_0\cdots\varepsilon_i}$, for $i = 0,\ldots,N-1$, be the density of the fluid on the $(i+1)$st level of fractures, and let $\rho_N^{\varepsilon_0\cdots\varepsilon_{N-1}}$ be the density of the fluid in the porous matrix blocks. The following definitions reflect the nested periodic property of Ω. Let ϕ_0 and K_0 be the scalar porosity and scalar permeability, respectively, on the first level of fractures, extended throughout Ω. For

$i = 1, \dots, N-1$, we define the porosities ϕ_i and permeabilities K_i on the $(i+1)$st level of fractures as follows. First, we assume that ϕ_i and K_i are defined on $Y_{(i-1)B}$. So, $\phi_i = \phi_i(x_i)$ and $K_i = K_i(x_i)$, for $x_i \in Y_{(i-1)B}$. Then, we extend these definitions to all of $\Omega_{B,B}^{\varepsilon_0 \cdots \varepsilon_{i-1}}$ by defining $\phi_i^{\varepsilon_0 \cdots \varepsilon_{i-1}}(x_0) = \phi_i(L_{i-1}(x_0))$ and $K_i^{\varepsilon_0 \cdots \varepsilon_{i-1}}(x_0) = K_i(L_{i-1}(x_0))$. Similarly, we first define ϕ_N and $\mathbf{K}_N$ on $Y_{(N-1)B}$, and then extend their definitions to $\Omega_{B,B}^{\varepsilon_0 \cdots \varepsilon_{N-1}}$ by defining $\phi_N^{\varepsilon_0 \cdots \varepsilon_{N-1}}(x_0) = \phi_N(L_{N-1}(x_0))$ and $\mathbf{K}_N^{\varepsilon_0 \cdots \varepsilon_{N-1}}(x_0) = \mathbf{K}_N(L_{N-1}(x_0))$. All coefficients are uniformly positive and bounded, and $\mathbf{K}_N$ is a bounded, symmetric, positive-definite tensor.

In order to carry out our recursive homogenization process, we require that the fracture and matrix geometry satisfy

$$\partial\Omega \subseteq \partial\Omega_F^{\varepsilon_0}, \qquad \partial\Omega_{B,F}^{\varepsilon_0 \cdots \varepsilon_{i-1}} \subseteq \partial\Omega_{B,F}^{\varepsilon_0 \cdots \varepsilon_i}, \quad i = 1, \dots, N-1. \tag{2.9}$$

Let $J = (0,T)$ be the time interval of interest. Also, throughout this paper, we denote by n_D the outward unit normal to the boundary of D, where D is the relevant domain.

We begin the study of our model at the microscopic level, which consists of equations describing Darcy flow on all parts of Ω, that is, we will pose the flow equations separately on the disjoint regions that compose the domain. For the homogenization process, the equations on the different parts of Ω will have to be scaled appropriately to conserve flow, just as was necessary in the derivation of the double-porosity model (see [13]). Actually, it is convenient to scale the equations on $\Omega_{B,B}^{\varepsilon_0\varepsilon_1}$ with respect to the equations on $\Omega_{B,F}^{\varepsilon_0\varepsilon_1}$ and then again with respect to those on $\Omega_F^{\varepsilon_0}$. This will allow us to derive, recursively, a triple-porosity model through rigorous homogenization.

For convenience, assume that gravity is negligible. This assumption is only used to simplify the presentation. A note regarding the inclusion of gravity is made just after the derivation of the equations in terms of the density of the fluid. Assume that the fluid has viscosity μ and constant (small) compressibility c, so that the equation of state is given by

$$d\rho = c\rho\, dp, \tag{2.10}$$

where ρ is the density of the fluid and p is its pressure. In a single-porosity model, if $\mathbf{K}$ is the permeability (which can be a tensor), then the volumetric flow rate v of the fluid is given by Darcy's law:

$$v = -\frac{\mathbf{K}}{\mu}\nabla p. \tag{2.11}$$

If ϕ denotes the porosity of the medium, the conservation of mass requires that

$$\phi \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho v) = S, \tag{2.12}$$

where S is the external source. Rewriting this in terms of ρ, we obtain

$$\phi \frac{\partial \rho}{\partial t} - \nabla \cdot \left(\frac{\mathbf{K}}{\mu c} \nabla \rho \right) = S. \tag{2.13}$$

We remark that if the gravity term $-\nabla \cdot ((\mathbf{K}/\mu c)(cg\rho^2))$ is added to the left-hand side of the above equation, then everything that follows holds if we linearize the equation as in [5] by defining a reference density ρ_{ref} and approximating the effects of gravity by $\rho^2 \approx \rho_{\text{ref}}(2\rho - \rho_{\text{ref}})$.

The verification of our homogenization procedure will make a crucial use of the following technical lemmas. In the interest of brevity, we omit their proofs.

Lemma 2.1. *For* $\psi, \varphi \in L^2(\Omega_{B,r}^{\varepsilon_0 \cdots \varepsilon_{i-1}})$, *where* $r = B, F$, *or blank, and* $\Omega_{B,}^{\varepsilon_0 \cdots \varepsilon_{i-1}} \equiv \Omega_{B,B}^{\varepsilon_0 \cdots \varepsilon_{i-2}}$,

$$\begin{aligned}
(\psi^{\sim(i)}, \varphi^{\sim(i)})_{\Omega \times Y_{0B} \times \cdots \times Y_{(i-2)B} \times Y_{(i-1)r}} &= |Y_0||Y_1| \cdots |Y_{i-1}| (\psi, \varphi)_{\Omega_{B,r}^{\varepsilon_0 \cdots \varepsilon_{i-1}}}, \\
\nabla_{x_i} \psi^{\sim(i)} &= \varepsilon_0 \cdots \varepsilon_{i-1} (\nabla \psi)^{\sim(i)}, \\
\|\psi^{\sim(i)}\|_{L^2(\Omega \times Y_{0B} \times \cdots \times Y_{(i-1)r})} &= (|Y_0||Y_1| \cdots |Y_{i-1}|)^{1/2} \|\psi\|_{L^2(\Omega_{B,r}^{\varepsilon_0 \cdots \varepsilon_{i-1}})}, \\
\|\nabla_{x_i} \psi^{\sim(i)}\|_{L^2(\Omega \times Y_{0B} \times \cdots \times Y_{(i-2)B} \times Y_{(i-1)r})} &= \varepsilon_0 \cdots \varepsilon_{i-1} (|Y_0||Y_1| \cdots |Y_{i-1}|)^{1/2} \\
&\quad \times \|\nabla \psi\|_{L^2(\Omega_{B,r}^{\varepsilon_0 \cdots \varepsilon_{i-1}})},
\end{aligned} \tag{2.14}$$

for $i = 1, \ldots, N$.

Lemma 2.2. *If* $\psi \in L^2(\Omega)$, *then the following holds* strongly *in* $L^2(\Omega \times Y_{0B} \times \cdots \times Y_{(i-2)B} \times Y_{(i-1)r})$:

$$\lim_{\varepsilon_{i-1} \to 0} \psi^{\sim(i)} = \psi^{\sim(i-1)}, \tag{2.15}$$

for $i = 1, \ldots, N$, *where* $^{\sim(0)}$ *is the identity operator.*

In what follows, $\mathbf{e}_j$ denotes the jth standard basis vector in the appropriate Euclidean space.

Lemma 2.3. *Let* $f \in L^2(\Omega)$ *and* $g \in L^2(\Omega \times Y_{0B} \times Y_{1B} \times \cdots \times Y_{(i-2)B} \times Y_{i-1})$. *Then, for* $i = 2,3,\ldots,$

$$\int_{\Omega\times Y_{0B}\times\cdots\times Y_{(i-2)B}\times Y_{i-1}} \widetilde{f}^{(i)}(x_0,x_1,\ldots,x_i)g(x_0,x_1,\ldots,x_i)dx_i\cdots dx_0 = \int_{\Omega_{B,B}^{\varepsilon_0\cdots\varepsilon_{i-2}}\times Y_0\times Y_1\times\cdots\times Y_{i-1}} f(x_0)g\left(\sum_{k=0}^{i-1}\left(\varepsilon_k x_{k+1} + c^{k,\varepsilon_k}(L_{k-1}(x_0))\right)\mathbf{e}_k + L_{i-1}(x_0)\mathbf{e}_i\right)dx_i\cdots dx_0. \tag{2.16}$$

Lemma 2.4. *Let* $f \in L^2(\Omega)$ *and* $g \in L^2(\Omega \times Y_{0B} \times \cdots \times Y_{(i-1)B})$. *Then, for* $i = 1,2,\ldots,$

$$\int_{\Omega\times Y_{0B}\times\cdots\times Y_{(i-1)B}} \widetilde{f}^{(i)}(x_0,x_1,\ldots,x_i)g(x_0,x_1,\ldots,x_i)dx_i\cdots dx_0 = \int_{\Omega_{B,B}^{\varepsilon_0\cdots\varepsilon_{i-1}}\times Y_0\times\cdots\times Y_{i-1}} f(x_0)g\left(\sum_{k=0}^{i-1}\left(\varepsilon_k x_{k+1} + c^{k,\varepsilon_k}(L_{k-1}(x_0))\right)\mathbf{e}_k + L_{i-1}(x_0)\mathbf{e}_i\right)dx_i\cdots dx_0. \tag{2.17}$$

Lemma 2.5. *Let* $f,g \in L^2(Y_{(i-2)B})$. *Then, with* $Y_{-1B} \equiv \Omega$, *for* $i = 1,2,\ldots,$ *the following equation holds:*

$$\int_{Y_{(i-2)B}\times Y_{i-1}} f\left(\varepsilon_{i-1}y_i + c^{(i-1),\varepsilon_{i-1}}(x_{i-1})\right)g(x_{i-1})dy_i\,dx_{i-1} = \int_{Y_{(i-2)B}\times Y_{i-1}} f(x_{i-1})g\left(\varepsilon_{i-1}y_i + c^{(i-1),\varepsilon_{i-1}}(x_{i-1})\right)dy_i\,dx_{i-1}. \tag{2.18}$$

3. The initial microscopic equations for a triple-porosity model

Denote by $\rho^{\varepsilon_0}(x_0,t)$ the density on $\Omega_F^{\varepsilon_0}$. Equations involving ρ^{ε_0} will be posed once the initial homogenization has been completed (i.e., after letting $\varepsilon_1 \to 0$). Let $\sigma^{\varepsilon_0\varepsilon_1}(x_0,t)$ and $\theta^{\varepsilon_0\varepsilon_1}(x_0,t)$ denote the densities on $\Omega_{B,F}^{\varepsilon_0\varepsilon_1}$ and $\Omega_{B,B}^{\varepsilon_0\varepsilon_1}$, respectively.

The assumptions made above lead to the sets (3.1) and (3.2) of equations for the micromodel. The scaling rules are explained immediately after the equations.

On the second level of fractures,

$$\begin{aligned}
\phi_\sigma^{\varepsilon_0}\frac{\partial\sigma^{\varepsilon_0\varepsilon_1}}{\partial t}-\varepsilon_0^2\nabla\cdot\left(\frac{K_\sigma^{\varepsilon_0}}{\mu c}\nabla\sigma^{\varepsilon_0\varepsilon_1}\right)=0 \quad &\text{in } \Omega_{B,F}^{\varepsilon_0\varepsilon_1}\times J,\\
\frac{K_\sigma^{\varepsilon_0}}{\mu c}\nabla\sigma^{\varepsilon_0\varepsilon_1}\cdot n_{\Omega_{B,B}^{\varepsilon_0\varepsilon_1}}=\varepsilon_1^2\frac{\mathbf{K}_\theta^{\varepsilon_0\varepsilon_1}}{\mu c}\nabla\theta^{\varepsilon_0\varepsilon_1}\cdot n_{\Omega_{B,B}^{\varepsilon_0\varepsilon_1}} \quad &\text{on } \partial\Omega_{B,B}^{\varepsilon_0\varepsilon_1}\times J,\\
\frac{K_\sigma^{\varepsilon_0}}{\mu c}\nabla\sigma^{\varepsilon_0\varepsilon_1}\cdot n_{\Omega_B^{\varepsilon_0}}=0 \quad \text{on } \partial\Omega_B^{\varepsilon_0}\times J, \qquad \sigma^{\varepsilon_0\varepsilon_1}=\rho_{\text{init}} \quad &\text{in } \Omega_{B,F}^{\varepsilon_0\varepsilon_1}\times\{0\}.
\end{aligned} \tag{3.1}$$

On the second level of matrix blocks,

$$\begin{aligned}
\phi_\theta^{\varepsilon_0\varepsilon_1}\frac{\partial\theta^{\varepsilon_0\varepsilon_1}}{\partial t}-\varepsilon_0^2\varepsilon_1^2\nabla\cdot\left(\frac{\mathbf{K}_\theta^{\varepsilon_0\varepsilon_1}}{\mu c}\nabla\theta^{\varepsilon_0\varepsilon_1}\right)=0 \quad &\text{in } \Omega_{B,B}^{\varepsilon_0\varepsilon_1}\times J,\\
\theta^{\varepsilon_0\varepsilon_1}=\sigma^{\varepsilon_0\varepsilon_1} \quad \text{on } \partial\Omega_{B,B}^{\varepsilon_0\varepsilon_1}\times J, \qquad \theta^{\varepsilon_0\varepsilon_1}=\rho_{\text{init}} \quad &\text{in } \Omega_{B,B}^{\varepsilon_0\varepsilon_1}\times\{0\}.
\end{aligned} \tag{3.2}$$

The aim of this work is to use homogenization theory to rigorously determine the equations that describe the flow. Since we let $\varepsilon_1 \to 0$ first, we do not, at this stage, consider the fluid flow across $\partial\Omega_B^{\varepsilon_0}$. Instead, we assume no-flow boundary conditions because we are interested in determining the interior behavior of the flow on $\Omega_B^{\varepsilon_0}$. Then, once the equations are discovered, we will impose boundary conditions on $\Omega_B^{\varepsilon_0}$ and develop a completely new system of partial differential equations that describe the flow. Then, we let $\varepsilon_0 \to 0$ to obtain the final model. The case in which a flow across $\partial\Omega_B^{\varepsilon_0}$ is considered at the microscopic level will be taken up elsewhere.

4. Preliminary analysis of the microscopic model

Multiply (3.1) by a test function $\varphi \in H^1(\Omega_B^{\varepsilon_0})$, and multiply (3.2) by a test function $\psi \in H_0^1(\Omega_{B,B}^{\varepsilon_0\varepsilon_1})$. Then, use the boundary conditions and the divergence theorem to find that the weak form of the microscale model for fixed ε_1 is given by

$$\begin{aligned}
\left(\phi_\sigma^{\varepsilon_0}\sigma_t^{\varepsilon_0\varepsilon_1},\varphi\right)_{\Omega_{B,F}^{\varepsilon_0\varepsilon_1}}+\varepsilon_0^2\left(\frac{K_\sigma^{\varepsilon_0}}{\mu c}\nabla\sigma^{\varepsilon_0\varepsilon_1},\nabla\varphi\right)_{\Omega_{B,F}^{\varepsilon_0\varepsilon_1}}+\varepsilon_0^2\varepsilon_1^2\left(\frac{\mathbf{K}_\theta^{\varepsilon_0\varepsilon_1}}{\mu c}\nabla\theta^{\varepsilon_0\varepsilon_1},\nabla\varphi\right)_{\Omega_{B,B}^{\varepsilon_0\varepsilon_1}}&\\
+\left(\phi_\theta^{\varepsilon_0\varepsilon_1}\theta_t^{\varepsilon_0\varepsilon_1},\varphi\right)_{\Omega_{B,B}^{\varepsilon_0\varepsilon_1}}=0 \quad \forall\varphi\in H^1(\Omega_B^{\varepsilon_0}),&
\end{aligned} \tag{4.1}$$

$$\left(\phi_\theta^{\varepsilon_0\varepsilon_1}\theta_t^{\varepsilon_0\varepsilon_1},\psi\right)_{\Omega_{B,B}^{\varepsilon_0\varepsilon_1}}+\varepsilon_0^2\varepsilon_1^2\left(\frac{\mathbf{K}_\theta^{\varepsilon_0\varepsilon_1}}{\mu c}\nabla\theta^{\varepsilon_0\varepsilon_1},\nabla\psi\right)_{\Omega_{B,B}^{\varepsilon_0\varepsilon_1}}=0 \quad \forall\psi\in H_0^1(\Omega_{B,B}^{\varepsilon_0\varepsilon_1}). \tag{4.2}$$

Now, let

$$\beta^{\varepsilon_0\varepsilon_1} = \begin{cases} \sigma^{\varepsilon_0\varepsilon_1} & \text{for } x_0 \in \Omega_{B,F}^{\varepsilon_0\varepsilon_1}, \\ \theta^{\varepsilon_0\varepsilon_1} & \text{for } x_0 \in \Omega_{B,B}^{\varepsilon_0\varepsilon_1}. \end{cases} \tag{4.3}$$

Then, (4.1) is the weak form of

$$\begin{gathered} \alpha^{\varepsilon_0\varepsilon_1}\beta_t^{\varepsilon_0\varepsilon_1} - \nabla\cdot\left(\boldsymbol{\kappa}^{\varepsilon_0\varepsilon_1}\nabla\beta^{\varepsilon_0\varepsilon_1}\right) = 0 \quad \text{on } \Omega_B^{\varepsilon_0}\times J, \\ \boldsymbol{\kappa}^{\varepsilon_0\varepsilon_1}\nabla\beta^{\varepsilon_0\varepsilon_1}\cdot n_{\Omega_B^{\varepsilon_0}} = 0 \quad \text{on } \partial\Omega_B^{\varepsilon_0}\times J, \qquad \beta^{\varepsilon_0\varepsilon_1} = \rho_{\text{init}} \quad \text{on } \Omega_B^{\varepsilon_0}\times\{0\}, \end{gathered} \tag{4.4}$$

where

$$\begin{gathered} \alpha^{\varepsilon_0\varepsilon_1} = \chi_{\Omega_{B,F}^{\varepsilon_0\varepsilon_1}}\phi_\sigma^{\varepsilon_0} + \chi_{\Omega_{B,B}^{\varepsilon_0\varepsilon_1}}\phi_\theta^{\varepsilon_0\varepsilon_1}, \\ \boldsymbol{\kappa}^{\varepsilon_0\varepsilon_1} = \varepsilon_0^2\chi_{\Omega_{B,F}^{\varepsilon_0\varepsilon_1}}\frac{K_\sigma^{\varepsilon_0}}{\mu c}\mathbf{I} + \varepsilon_0^2\varepsilon_1^2\chi_{\Omega_{B,B}^{\varepsilon_0\varepsilon_1}}\frac{\mathbf{K}_\theta^{\varepsilon_0\varepsilon_1}}{\mu c}. \end{gathered} \tag{4.5}$$

The initial-boundary value problem (4.4) is a standard parabolic problem with Neumann boundary conditions and it is well known that it has a unique weak solution in $H^1(J;L^2(\Omega_B^{\varepsilon_0}))\cap L^\infty(J;H^1(\Omega_B^{\varepsilon_0}))$. By restricting $\beta^{\varepsilon_0\varepsilon_1}$, we have the following theorem.

Theorem 4.1. *Assume that $\rho_{\text{init}}\in H^1(\Omega)$. Then, for each ε_1, there exists a unique solution to the microscopic model posed on $\Omega_B^{\varepsilon_0}\times J$, and $\sigma^{\varepsilon_0\varepsilon_1}\in H^1(J;L^2(\Omega_{B,F}^{\varepsilon_0\varepsilon_1}))\cap L^\infty(J;H^1(\Omega_{B,F}^{\varepsilon_0\varepsilon_1}))$ and $\theta^{\varepsilon_0\varepsilon_1}\in H^1(J;L^2(\Omega_{B,B}^{\varepsilon_0\varepsilon_1}))\cap L^\infty(J;H^1(\Omega_{B,B}^{\varepsilon_0\varepsilon_1}))$.*

Lemma 4.2. *There exists a constant $C>0$, independent of ε_0 and ε_1, such that*

$$\begin{gathered} \left\|\sigma_t^{\varepsilon_0\varepsilon_1}\right\|_{L^2(J\times\Omega_{B,F}^{\varepsilon_0\varepsilon_1})} + \left\|\sigma^{\varepsilon_0\varepsilon_1}\right\|_{L^\infty(J;H^1(\Omega_{B,F}^{\varepsilon_0\varepsilon_1}))} \le C\left(1+\varepsilon_0\varepsilon_1\right), \\ \left\|\nabla\theta^{\varepsilon_0\varepsilon_1}\right\|_{L^\infty(J;L^2(\Omega_{B,B}^{\varepsilon_0\varepsilon_1}))} \le C\left(1+\left(\varepsilon_0\varepsilon_1\right)^{-1}\right), \\ \left\|\theta_t^{\varepsilon_0\varepsilon_1}\right\|_{L^2(J\times\Omega_{B,B}^{\varepsilon_0\varepsilon_1})} + \left\|\theta^{\varepsilon_0\varepsilon_1}\right\|_{L^\infty(J;L^2(\Omega_{B,B}^{\varepsilon_0\varepsilon_1}))} \le C\left(1+\varepsilon_0\varepsilon_1\right). \end{gathered} \tag{4.6}$$

Proof. These are the standard parabolic energy estimates for the weak form (4.1) on $\Omega_B^{\varepsilon_0}$. To derive these estimates, start by taking $\varphi=\theta^{\varepsilon_0\varepsilon_1}$ and then $\varphi=\theta_t^{\varepsilon_0\varepsilon_1}$ on a smooth dense subspace. □

5. Homogenization as $\varepsilon_1\to 0$ for fixed $\varepsilon_0>0$

We now begin to find the unique weak solution of the limit problem as $\varepsilon_1\to 0$. Throughout, we use C to denote a generic positive constant that is independent of ε_1 and which can be different at different occurrences.

For fixed $\varepsilon_0 > 0$, it follows from the a priori estimates in Lemma 4.2 that

$$\begin{gathered}\chi_{\Omega_{B,F}^{\varepsilon_0\varepsilon_1}}\sigma^{\varepsilon_0\varepsilon_1} \text{ is bounded in } H^1(J;L^2(\Omega)),\\ \chi_{\Omega_{B,F}^{\varepsilon_0\varepsilon_1}}\frac{K_\sigma^{\varepsilon_0}}{\mu c}\nabla\sigma^{\varepsilon_0\varepsilon_1} \text{ is bounded in } L^2(\Omega\times J),\\ \chi_{\Omega_{B,B}^{\varepsilon_0\varepsilon_1}}\theta^{\varepsilon_0\varepsilon_1} \text{ is bounded in } H^1(J;L^2(\Omega)),\\ \varepsilon_1\chi_{\Omega_{B,B}^{\varepsilon_0\varepsilon_1}}\nabla\theta^{\varepsilon_0\varepsilon_1} \text{ is bounded in } L^2(\Omega\times J).\end{gathered} \tag{5.1}$$

It follows from (5.1) and Lemma 2.1 that

$$(\theta^{\varepsilon_0\varepsilon_1})^{\sim(2)} \text{ is bounded in } L^2(\Omega\times Y_{0B};H^1(Y_{1B}\times J)), \tag{5.2}$$

$$(\sigma^{\varepsilon_0\varepsilon_1})^{\sim(2)} \text{ is bounded in } L^2(\Omega\times Y_{0B};H^1(Y_{1F}\times J)), \tag{5.3}$$

$$\left\|\nabla_{x_2}(\sigma^{\varepsilon_0\varepsilon_1})^{\sim(2)}\right\|_{L^2(\Omega\times Y_{0B}\times Y_{1F}\times J)} \le (C_{\varepsilon_0})\varepsilon_1. \tag{5.4}$$

Hence, upon passing to a subsequence in ε_1, as $\varepsilon_1 \to 0$, the following limits take place *weakly* in the indicated spaces:

$$\chi_{\Omega_{B,F}^{\varepsilon_0\varepsilon_1}}\sigma^{\varepsilon_0\varepsilon_1} \rightharpoonup \widehat{\sigma}^{\varepsilon_0} \quad \text{in } H^1(J;L^2(\Omega)), \tag{5.5}$$

$$\chi_{\Omega_{B,F}^{\varepsilon_0\varepsilon_1}}\frac{K_\sigma^{\varepsilon_0}}{\mu c}\nabla\sigma^{\varepsilon_0\varepsilon_1} \rightharpoonup \xi^{\varepsilon_0} \quad \text{in } L^2(\Omega\times J), \tag{5.6}$$

$$(\sigma^{\varepsilon_0\varepsilon_1})^{\sim(2)} \rightharpoonup \sigma^{\varepsilon_0} \quad \text{in } L^2(\Omega\times Y_{0B};H^1(Y_{1F}\times J)), \tag{5.7}$$

$$(\theta^{\varepsilon_0\varepsilon_1})^{\sim(2)} \rightharpoonup \theta^{\varepsilon_0} \quad \text{in } L^2(\Omega\times Y_{0B};H^1(Y_{1B}\times J)). \tag{5.8}$$

From (5.4) and the connectedness of Y_{1F}, σ^{ε_0} is independent of x_2.

LEMMA 5.1. *The following relation holds:*

$$(\widehat{\sigma}^{\varepsilon_0})^{\sim(1)} = \frac{|Y_{1F}|}{|Y_1|}\sigma^{\varepsilon_0}. \tag{5.9}$$

Proof. We first note that for $R = B, F$, or blank,

$$\left(\chi_{\Omega_{B,R}^{\varepsilon_0\varepsilon_1}}\right)^{\sim(2)}(x_0,x_1,x_2) = \chi_{\Omega\times Y_{0B}\times Y_{1R}}(x_0,x_1,x_2). \tag{5.10}$$

Now, let $\varphi \in C^{\infty}(\Omega \times Y_{0B} \times J)$. Then by (5.7),

$$\begin{aligned} I^{\varepsilon_0\varepsilon_1} &= \int_{J\times\Omega\times Y_{0B}\times Y_{1F}} \left(\sigma^{\varepsilon_0\varepsilon_1}\right)^{\sim(2)} \varphi \, dx_2 \, dx_1 \, dx_0 \, dt \\ &\longrightarrow \int_{J\times\Omega\times Y_{0B}\times Y_{1F}} \sigma^{\varepsilon_0} \varphi \, dx_2 \, dx_1 \, dx_0 \, dt \\ &= \left|Y_{1F}\right| \int_{J\times\Omega\times Y_{0B}} \sigma^{\varepsilon_0} \varphi \, dx_1 \, dx_0 \, dt \end{aligned} \tag{5.11}$$

since σ^{ε_0}, φ do not depend on x_2.

On the other hand,

$$\begin{aligned} I^{\varepsilon_0\varepsilon_1} &= \left(\left(\sigma^{\varepsilon_0\varepsilon_1}\right)^{\sim(2)}, \chi_{\Omega\times Y_{0B}\times Y_{1F}} \varphi\right)_{J\times\Omega\times Y_{0B}\times Y_1\times J} \\ &= \left(\left(\chi_{\Omega_{B,F}^{\varepsilon_0\varepsilon_1}} \sigma^{\varepsilon_0\varepsilon_1}\right)^{\sim(2)}, \varphi\right)_{\Omega\times Y_{0B}\times Y_1\times J} \\ &= \int_{J\times\Omega_B^{\varepsilon_0}\times Y_0\times Y_1} \chi_{\Omega_{B,F}^{\varepsilon_0\varepsilon_1}} \sigma^{\varepsilon_0\varepsilon_1} \varphi\big(\varepsilon_0 x_1 + c^{0,\varepsilon_0}(x_0), \varepsilon_1 x_2 \\ &\qquad + c^{1,\varepsilon_1}(L_0(x_0)), t\big) dx_2 \, dx_1 \, dx_0 \, dt, \end{aligned} \tag{5.12}$$

where (5.10) and Lemma 2.3 were used.

Now, let $\varepsilon_1 \to 0$; by Lemma 2.2,

$$\begin{aligned} &\varphi\left(\varepsilon_0 x_1 + c^{0,\varepsilon_0}(x_0), \varepsilon_1 x_2 + c^{1,\varepsilon_1}(L_0(x_0)), t\right) \\ &\qquad \longrightarrow \varphi\left(\varepsilon_0 x_1 + c^{0,\varepsilon_0}(x_0), L_0(x_0), t\right) \end{aligned} \tag{5.13}$$

strongly in $L^2(\Omega_B^{\varepsilon_0} \times Y_0 \times Y_1 \times J)$, and this, combined with (5.5), (5.12), and Lemma 2.4, yields

$$\begin{aligned} I^{\varepsilon_0\varepsilon_1} &\longrightarrow \int_{J\times\Omega_B^{\varepsilon_0}\times Y_0\times Y_1} \widehat{\sigma}^{\varepsilon_0}(x_0,t) \varphi\left(\varepsilon_0 x_1 + c^{0,\varepsilon_0}(x_0), L_0(x_0), t\right) dx_2 \, dx_1 \, dx_0 \, dt \\ &= \left|Y_1\right| \int_{J\times\Omega_B^{\varepsilon_0}\times Y_0} \widehat{\sigma}^{\varepsilon_0}(x_0,t) \varphi\left(\varepsilon_0 x_1 + c^{0,\varepsilon_0}(x_0), L_0(x_0), t\right) dx_1 \, dx_0 \, dt \\ &= \left|Y_1\right| \int_{J\times\Omega\times Y_{0B}} \left(\widehat{\sigma}^{\varepsilon_0}\right)^{\sim(1)}(x_0,x_1,t) \varphi(x_0,x_1,t) \, dx_1 \, dx_0 \, dt. \end{aligned} \tag{5.14}$$

Since φ is arbitrary, (5.11) and (5.14) imply the lemma. □

We now derive an equation satisfied by θ^{ε_0}. Let $\psi \in L^2(\Omega \times Y_{0B} \times J; H_0^1(Y_{1B}))$. Set

$$\widehat{\psi}(x_0,x_1,z,t) = \begin{cases} \psi\left(x_0,x_1,\dfrac{z-\varepsilon_0 c^{1,\varepsilon_1}(x_1)-c^{0,\varepsilon_0}(x_0)}{\varepsilon_0\varepsilon_1},t\right) \\ \qquad \text{if } z \in \varepsilon_0\varepsilon_1 Y_{1B}+\varepsilon_0 c^{1,\varepsilon_1}(x_1)+c^{0,\varepsilon_0}(x_0), \\ 0 \quad \text{elsewhere.} \end{cases} \tag{5.15}$$

Then $\widehat{\psi} \in L^2(\Omega \times Y_{0B} \times J; H_0^1(\varepsilon_0\varepsilon_1 Y_{1B} + \varepsilon_0 c^{1,\varepsilon_1}(x_1) + c^{0,\varepsilon_0}(x_0)))$. For fixed $(x_0,x_1) \in \Omega \times Y_{0B}$, use $\widehat{\psi}$ as a test function in (4.2) to obtain

$$\begin{aligned} &\int_{\varepsilon_0\varepsilon_1 Y_{1B}+\varepsilon_0 c^{1,\varepsilon_1}(x_1)+c^{0,\varepsilon_0}(x_0)} \phi_\theta^{\varepsilon_0\varepsilon_1}(z)\theta_t^{\varepsilon_0\varepsilon_1}(z)\psi \\ &\qquad\qquad \times \left(x_0,x_1,\frac{z-\varepsilon_0 c^{1,\varepsilon_1}(x_1)-c^{0,\varepsilon_0}(x_0)}{\varepsilon_0\varepsilon_1},t\right)dz \\ &+\varepsilon_0^2\varepsilon_1^2\int_{\varepsilon_0\varepsilon_1 Y_{1B}+\varepsilon_0 c^{1,\varepsilon_1}(x_1)+c^{0,\varepsilon_0}(x_0)} \frac{\mathbf{K}_\theta^{\varepsilon_0\varepsilon_1}}{\mu c}(z)\nabla_z\theta^{\varepsilon_0\varepsilon_1}(z)\cdot\nabla_z\psi \\ &\qquad\qquad \times \left(x_0,x_1,\frac{z-\varepsilon_0 c^{1,\varepsilon_1}(x_1)-c^{0,\varepsilon_0}(x_0)}{\varepsilon_0\varepsilon_1},t\right)dz = 0. \end{aligned} \tag{5.16}$$

Use the dilation $z \mapsto \varepsilon_0\varepsilon_1 x_2 + \varepsilon_0 c^{1,\varepsilon_1}(x_1) + c^{0,\varepsilon_0}(x_0)$ and integrate over $\Omega \times Y_{0B} \times J$ to get

$$\begin{aligned} &\int_{J\times\Omega\times Y_{0B}\times Y_{1B}} \phi_\theta(x_2)\left(\theta_t^{\varepsilon_0\varepsilon_1}\right)^{\sim(2)}\psi(x_0,x_1,x_2,t)\,dx_2\,dx_1\,dx_0\,dt \\ &\quad+\int_{J\times\Omega\times Y_{0B}\times Y_{1B}} \frac{\mathbf{K}_\theta}{\mu c}(x_2)\nabla_{x_2}\left(\theta^{\varepsilon_0\varepsilon_1}\right)^{\sim(2)}\cdot\nabla_{x_2}\psi(x_0,x_1,x_2,t)\,dx_2\,dx_1\,dx_0\,dt = 0, \end{aligned} \tag{5.17}$$

where we used

$$\left(\phi_\theta^{\varepsilon_0\varepsilon_1}\right)^{\sim(2)} = \chi_{\Omega\times Y_{0B}\times Y_{1B}}\phi_\theta, \qquad \left(\frac{\mathbf{K}_\theta^{\varepsilon_0\varepsilon_1}}{\mu c}\right)^{\sim(2)} = \chi_{\Omega\times Y_{0B}\times Y_{1B}}\frac{\mathbf{K}_\theta}{\mu c}. \tag{5.18}$$

Next, let $\varepsilon_1 \to 0$ and use the weak limits to get

$$\left(\phi_\theta\theta_t^{\varepsilon_0},\psi\right)_{\Omega\times Y_{0B}\times Y_{1B}\times J} + \left(\frac{\mathbf{K}_\theta}{\mu c}\nabla_{x_2}\theta^{\varepsilon_0},\nabla_{x_2}\psi\right)_{\Omega\times Y_{0B}\times Y_{1B}\times J} = 0, \tag{5.19}$$

that is, θ^{ε_0} is a weak solution of

$$\phi_\theta \theta_t^{\varepsilon_0} - \nabla_{x_2} \cdot \left(\frac{\mathbf{K}_\theta}{\mu c} \nabla_{x_2} \theta^{\varepsilon_0} \right) = 0 \tag{5.20}$$

in $L^2(\Omega \times \Upsilon_{0B}; H^1(\Upsilon_{1B} \times J))$.

We now begin to derive an equation for $\widehat{\sigma}^{\varepsilon_0}$. Let $\varphi \in L^2(J; H^1(\Omega_B^{\varepsilon_0}))$ and integrate (4.1) over J to get

$$\begin{aligned} &(\phi_\sigma^{\varepsilon_0} \sigma_t^{\varepsilon_0\varepsilon_1}, \varphi)_{J\times\Omega_{B,F}^{\varepsilon_0\varepsilon_1}} + \varepsilon_0^2 \left(\frac{K_\sigma^{\varepsilon_0}}{\mu c} \nabla \sigma^{\varepsilon_0\varepsilon_1}, \varphi \right)_{J\times\Omega_{B,F}^{\varepsilon_0\varepsilon_1}} \\ &\quad + \varepsilon_0^2 \varepsilon_1^2 \left(\frac{\mathbf{K}_\theta^{\varepsilon_0\varepsilon_1}}{\mu c} \nabla \theta^{\varepsilon_0\varepsilon_1}, \nabla \varphi \right)_{J\times\Omega_{B,B}^{\varepsilon_0\varepsilon_1}} + (\phi_\theta^{\varepsilon_0\varepsilon_1} \theta_t^{\varepsilon_0\varepsilon_1}, \varphi)_{J\times\Omega_{B,B}^{\varepsilon_0\varepsilon_1}} \\ &= T_1 + T_2 + T_3 + T_4 = 0. \end{aligned} \tag{5.21}$$

We now let $\varepsilon_1 \longrightarrow 0$;

$$\begin{aligned} &T_1 \longrightarrow \int_{J\times\Omega_B^{\varepsilon_0}} \phi_\sigma^{\varepsilon_0} \widehat{\sigma}_t^{\varepsilon_0} \varphi \, dx_0 \, dt \quad \text{by (5.5)}, \\ &T_2 \longrightarrow \varepsilon_0^2 \int_{J\times\Omega_B^{\varepsilon_0}} \xi^{\varepsilon_0} \cdot \nabla \varphi \, dx_0 \, dt \quad \text{by (5.6)}, \\ &T_3 \longrightarrow 0 \quad \text{since the term is bounded by a multiple of } \varepsilon_1^2. \end{aligned} \tag{5.22}$$

We now investigate the convergence of T_4. By Lemmas 2.1, 2.2, (5.8), and Lemma 2.4, we have

$$\begin{aligned} T_4 &= (|\Upsilon_0||\Upsilon_1|)^{-1} \left((\phi_\theta^{\varepsilon_0\varepsilon_1})^{\sim(2)} (\theta_t^{\varepsilon_0\varepsilon_1})^{\sim(2)}, \varphi^{\sim(2)} \right)_{J\times\Omega\times\Upsilon_{0B}\times\Upsilon_{1B}} \\ &= (|\Upsilon_0||\Upsilon_1|)^{-1} \left(\phi_\theta (\theta_t^{\varepsilon_0\varepsilon_1})^{\sim(2)}, \varphi^{\sim(2)} \right)_{J\times\Omega\times\Upsilon_{0B}\times\Upsilon_{1B}} \\ &\longrightarrow (|\Upsilon_0||\Upsilon_1|)^{-1} (\phi_\theta \theta_t^{\varepsilon_0}, \varphi^{\sim(1)})_{J\times\Omega\times\Upsilon_{0B}\times\Upsilon_{1B}} \\ &= (|\Upsilon_0||\Upsilon_1|)^{-1} \int_{J\times\Omega_B^{\varepsilon_0}\times\Upsilon_0\times\Upsilon_{1B}} \phi_\theta \theta_t^{\varepsilon_0} (\varepsilon_0 x_1 + c^{0,\varepsilon_0}(x_0), L_0(x_0), x_2, t) \\ &\qquad \times \varphi(x_0, t) \, dx_2 \, dx_1 \, dx_0 \, dt. \end{aligned} \tag{5.23}$$

It follows from (5.22) and (5.23) that

$$\begin{aligned}\int_{J\times\Omega_B^{\varepsilon_0}} \phi_\sigma^{\varepsilon_0}\widehat{\sigma}_t^{\varepsilon_0}\varphi\,dx_0\,dt + \varepsilon_0^2\int_{J\times\Omega_B^{\varepsilon_0}} \zeta^{\varepsilon_0}\cdot\nabla\varphi\,dx_0\,dt \\ + \left(|Y_0||Y_1|\right)^{-1}\int_{J\times\Omega_B^{\varepsilon_0}\times Y_0\times Y_{1B}} \phi_\theta\theta_t^{\varepsilon_0}\left(\varepsilon_0 x_1 + c^{0,\varepsilon_0}(x_0), L_0(x_0), x_2, t\right) \\ \times\varphi(x_0,t)\phi_\theta(x_2)\,dx_2\,dx_1\,dx_0\,dt \\ = 0 \quad \forall\varphi\in L^2(J;H^1(\Omega)).\end{aligned} \tag{5.24}$$

We now relate ζ^{ε_0} to $\widehat{\sigma}^{\varepsilon_0}$. For $j = 1,2,3$, let $\omega_j = \omega_j(x_2)$ be the Y_1-periodic solution, modulo constants, to the Neumann problem

$$\Delta_{x_2}\omega_j = 0 \quad \text{in } Y_{1F}, \qquad \nabla_{x_2}\omega_j\cdot\nu = -\mathbf{e}_j\cdot\nu = -\nu_j \quad \text{on } \partial Y_{1B}, \tag{5.25}$$

where ν is the outer unit normal to ∂Y_{1B}. Define $\omega_j^{\varepsilon_0\varepsilon_1}\in H^1(\Omega_B^{\varepsilon_0})$ by

$$\omega_j^{\varepsilon_0\varepsilon_1}(x_0) = \varepsilon_0\varepsilon_1\mathcal{E}\omega_j\left(L_1(x_0)\right), \tag{5.26}$$

where $\mathcal{E}: H^1(Y_{1F})\to H^1(Y_1)$ is a bounded extension operator [12].

Hence $(\omega_j^{\varepsilon_0\varepsilon_1})^{\sim(2)}(x_0,x_1,x_2) = \varepsilon_0\varepsilon_1(\mathcal{E}\omega_j)(x_2)$. A similar argument shows that

$$\left(\nabla_{x_0}\omega_j^{\varepsilon_0\varepsilon_1}\right)^{\sim(2)}(x_0,x_1,x_2) = \left(\nabla_{x_2}\mathcal{E}\omega_j\right)(x_2). \tag{5.27}$$

Now let

$$\omega_{ij} = \frac{1}{|Y_1|}\int_{Y_{1F}} \partial_i\omega_j(x_2)\,dx_2, \tag{5.28}$$

where $\partial_i = \partial/\partial x_{2,i}$.

An argument as in the proof of [5, Lemma 4.3] can be used to verify the following lemma.

Lemma 5.2. *As* $\varepsilon_1\to 0$,

$$\begin{gathered}\omega_j^{\varepsilon_0\varepsilon_1} \longrightarrow 0 \text{ strongly in } L^2(\Omega_B^{\varepsilon_0}), \\ \varepsilon_0\varepsilon_1\nabla\omega_j^{\varepsilon_0\varepsilon_1} \longrightarrow 0 \text{ strongly in } L^2(\Omega_B^{\varepsilon_0}), \\ \chi_{\Omega_{B,F}^{\varepsilon_0\varepsilon_1}}\partial_i\omega_j^{\varepsilon_0\varepsilon_1} \longrightarrow \omega_{ij} \text{ weakly in } L^2(\Omega_B^{\varepsilon_0}).\end{gathered} \tag{5.29}$$

If $\varphi \in H^1(\Omega_{B,F}^{\varepsilon_0\varepsilon_1})$, then

$$\left(\nabla\omega_j^{\varepsilon_0\varepsilon_1} + \mathbf{e}_j, \nabla\varphi\right)_{\Omega_{B,F}^{\varepsilon_0\varepsilon_1}} = 0 \tag{5.30}$$

since ω_j solves the above Neumann problem. Now, for $\varphi \in C^\infty((\Omega_B^{\varepsilon_0} \times J)^-)$, take $\varphi = \sigma^{\varepsilon_0\varepsilon_1}(K_\sigma^{\varepsilon_0}/\mu c)\varphi$ in (5.30) and integrate in time to get

$$\begin{aligned}
0 &= \left(\nabla\omega_j^{\varepsilon_0\varepsilon_1}, \frac{K_\sigma^{\varepsilon_0}}{\mu c}\varphi\nabla\left(\sigma^{\varepsilon_0\varepsilon_1}\right)\right)_{J\times\Omega_{B,F}^{\varepsilon_0\varepsilon_1}} + \left(\nabla\omega_j^{\varepsilon_0\varepsilon_1}, \sigma^{\varepsilon_0\varepsilon_1}\nabla\left(\frac{K_\sigma^{\varepsilon_0}}{\mu c}\varphi\right)\right)_{J\times\Omega_{B,F}^{\varepsilon_0\varepsilon_1}} \\
&\quad + \left(\mathbf{e}_j, \frac{K_\sigma^{\varepsilon_0}}{\mu c}\varphi\nabla\left(\sigma^{\varepsilon_0\varepsilon_1}\right)\right)_{J\times\Omega_{B,F}^{\varepsilon_0\varepsilon_1}} + \left(\mathbf{e}_j, \sigma^{\varepsilon_0\varepsilon_1}\nabla\left(\frac{K_\sigma^{\varepsilon_0}}{\mu c}\varphi\right)\right)_{J\times\Omega_{B,F}^{\varepsilon_0\varepsilon_1}} \\
&= T_5 + T_6 + T_7 + T_8.
\end{aligned} \tag{5.31}$$

We proceed to let $\varepsilon_1 \to 0$ in each term of (5.31). We begin with term T_5. Use $\omega_j^{\varepsilon_0\varepsilon_1}\varphi$ as a test function in (4.1) to obtain

$$\begin{aligned}
&\varepsilon_0^2\left(\frac{K_\sigma^{\varepsilon_0}}{\mu c}\nabla\sigma^{\varepsilon_0\varepsilon_1}, \varphi\nabla\left(\omega_j^{\varepsilon_0\varepsilon_1}\right)\right)_{\Omega_{B,F}^{\varepsilon_0\varepsilon_1}} \\
&\quad = -\left(\phi_\sigma^{\varepsilon_0}\sigma_t^{\varepsilon_0\varepsilon_1}, \omega_j^{\varepsilon_0\varepsilon_1}\varphi\right)_{\Omega_{B,F}^{\varepsilon_0\varepsilon_1}} - \varepsilon_0^2\varepsilon_1^2\left(\frac{\mathbf{K}_\theta^{\varepsilon_0\varepsilon_1}}{\mu c}\nabla\theta^{\varepsilon_0\varepsilon_1}, \omega_j^{\varepsilon_0\varepsilon_1}\nabla(\varphi)\right)_{\Omega_{B,B}^{\varepsilon_0\varepsilon_1}} \\
&\qquad - \varepsilon_0^2\varepsilon_1^2\left(\frac{\mathbf{K}_\theta^{\varepsilon_0\varepsilon_1}}{\mu c}\nabla\theta^{\varepsilon_0\varepsilon_1}, \varphi\nabla\left(\omega_j^{\varepsilon_0\varepsilon_1}\right)\right)_{\Omega_{B,B}^{\varepsilon_0\varepsilon_1}} - \left(\phi_\theta^{\varepsilon_0\varepsilon_1}\theta_t^{\varepsilon_0\varepsilon_1}, \omega_j^{\varepsilon_0\varepsilon_1}\varphi\right)_{\Omega_{B,B}^{\varepsilon_0\varepsilon_1}} \\
&\qquad - \varepsilon_0^2\left(\frac{K_\sigma^{\varepsilon_0}}{\mu c}\nabla\sigma^{\varepsilon_0\varepsilon_1}, \omega_j^{\varepsilon_0\varepsilon_1}\nabla(\varphi)\right)_{\Omega_{B,F}^{\varepsilon_0\varepsilon_1}}.
\end{aligned} \tag{5.32}$$

It follows from (5.1), Lemma 5.2, and the boundedness of $\nabla\omega_j^{\varepsilon_0\varepsilon_1}$ that

$$T_5 \longrightarrow 0 \quad \text{as } \varepsilon_1 \longrightarrow 0. \tag{5.33}$$

For term T_6, we have

$$\begin{aligned}
T_6 &= \left(|Y_0||Y_1|\right)^{-1}\left(\nabla_{x_2}\omega_j, \left(\sigma^{\varepsilon_0\varepsilon_1}\right)^{\sim(2)}\left[\nabla_{x_0}\left(\frac{K_\sigma^{\varepsilon_0}}{\mu c}\varphi\right)\right]^{\sim(2)}\right)_{J\times\Omega\times Y_{0B}\times Y_{1F}} \\
&\xrightarrow{\varepsilon_1\to 0} \left(|Y_0||Y_1|\right)^{-1}\left(\nabla_{x_2}\omega_j, \sigma^{\varepsilon_0}\left[\nabla_{x_0}\left(\frac{K_\sigma^{\varepsilon_0}}{\mu c}\varphi\right)\right]^{\sim(1)}\right)_{J\times\Omega\times Y_{0B}\times Y_{1F}}
\end{aligned}$$

$$= (|Y_0||Y_1|)^{-1} \int_{J\times\Omega\times Y_{0B}\times Y_{1F}} \sigma^{\varepsilon_0}(\varepsilon_0 x_1 + c^{0,\varepsilon_0}(x_0), L_0(x_0), t) \times \nabla_{x_2}\omega_j \cdot \nabla_{x_0}\left(\frac{K_\sigma^{\varepsilon_0}}{\mu c}\varphi\right) dx_2\, dx_1\, dx_0\, dt$$

$$= (|Y_0|)^{-1} \int_{J\times\Omega\times Y_{0B}} \chi_{\Omega_B^{\varepsilon_0}} \sigma^{\varepsilon_0}(\varepsilon_0 x_1 + c^{0,\varepsilon_0}(x_0), L_0(x_0), t) \times \left(\sum_i \omega_{ij}\partial_i\left(\frac{K_\sigma^{\varepsilon_0}}{\mu c}\varphi\right)(x_0)\right) dx_1\, dx_0\, dt \tag{5.34}$$

by Lemma 2.1, (5.7), Lemmas 2.2, 2.4, and the definition of ω_{ij}.

But we observe from Lemma 5.1 that

$$\sigma^{\varepsilon_0}(\varepsilon_0 x_1 + c^{0,\varepsilon_0}(x_0), L_0(x_0), t) = \frac{|Y_1|}{|Y_{1F}|}\widehat{\sigma}^{\varepsilon_0}(x_0, t). \tag{5.35}$$

Hence

$$T_6 \xrightarrow{\varepsilon_1\to 0} \frac{|Y_1|}{|Y_0||Y_{1F}|} \int_{J\times\Omega\times Y_0} \chi_{\Omega_B^{\varepsilon_0}}(x_0)\widehat{\sigma}^{\varepsilon_0}(x_0, t) \times \left(\sum_i \omega_{ij}\partial_i\left(\frac{K_\sigma^{\varepsilon_0}}{\mu c}\varphi(x_0)\right)\right) dx_0\, dt. \tag{5.36}$$

For the term T_8, we have by (5.5)

$$T_8 \xrightarrow{\varepsilon_1\to 0} \left(\mathbf{e}_j, \widehat{\sigma}^{\varepsilon_0}\nabla\left(\frac{K_\sigma^{\varepsilon_0}}{\mu c}\varphi\right)\right)_{J\times\Omega_B^{\varepsilon_0}} = \int_{J\times\Omega_B^{\varepsilon_0}} \widehat{\sigma}^{\varepsilon_0}(x_0, t)\left(\sum_i \delta_{ij}\partial_i\left(\frac{K_\sigma^{\varepsilon_0}}{\mu c}\varphi(x_0)\right)\right) dx_0\, dt. \tag{5.37}$$

Hence, by (5.6), (5.31), (5.33), (5.36), and (5.37),

$$T_7 \xrightarrow{\varepsilon_1\to 0} (\xi_j^{\varepsilon_0}, \varphi)_{J\times\Omega_B^{\varepsilon_0}} = \int_{J\times\Omega_B^{\varepsilon_0}} \sum_i \left(\frac{|Y_1|}{|Y_{1F}|}\omega_{ij} + \delta_{ij}\right)\partial_i(\widehat{\sigma}^{\varepsilon_0}(x_0, t))\frac{K_\sigma^{\varepsilon_0}}{\mu c}\varphi(x_0)\, dx_0\, dt. \tag{5.38}$$

So, define

$$(\mathbf{K}_\sigma^{h\varepsilon_0})_{ij} \equiv K_\sigma^{\varepsilon_0}\left(\omega_{ij} + \frac{|Y_{1F}|}{|Y_1|}\delta_{ij}\right). \tag{5.39}$$

Then, as in [5], $\mathbf{K}_\sigma^{h\varepsilon_0}$ is a bounded, symmetric, positive-definite tensor. Then we can write the equality in (5.38) as

$$(\xi^{\varepsilon_0},\varphi)_{J\times\Omega_B^{\varepsilon_0}} = \left(\frac{|Y_1|}{|Y_{1F}|}\frac{\mathbf{K}_\sigma^{h\varepsilon_0}}{\mu c}\nabla\widehat{\sigma}^{\varepsilon_0},\varphi\right)_{J\times\Omega_B^{\varepsilon_0}}, \tag{5.40}$$

that is,

$$\xi^{\varepsilon_0} = \frac{|Y_1|}{|Y_{1F}|}\frac{\mathbf{K}_\sigma^{h\varepsilon_0}}{\mu c}\nabla\widehat{\sigma}^{\varepsilon_0} \quad \text{in } \Omega_B^{\varepsilon_0}\times J. \tag{5.41}$$

Next define

$$r^{\varepsilon_0} \equiv \frac{|Y_1|}{|Y_{1F}|}\widehat{\sigma}^{\varepsilon_0}, \qquad \phi_\sigma^{h\varepsilon_0} \equiv \frac{|Y_{1F}|}{|Y_1|}\phi_\sigma^{\varepsilon_0}. \tag{5.42}$$

Then we can rewrite (5.24) as

$$\begin{aligned}\int_{J\times\Omega_B^{\varepsilon_0}} &\phi_\sigma^{h\varepsilon_0} r_t^{\varepsilon_0}\varphi\,dx_0\,dt + \varepsilon_0^2\int_{J\times\Omega_B^{\varepsilon_0}}\frac{\mathbf{K}_\sigma^{h\varepsilon_0}}{\mu c}\nabla r^{\varepsilon_0}\cdot\nabla\varphi\,dx_0\,dt \\ &+ (|Y_0||Y_1|)^{-1}\int_{J\times\Omega_B^{\varepsilon_0}\times Y_0\times Y_{1B}}\phi_\theta(x_2)\theta_t^{\varepsilon_0}(\varepsilon_0 x_1 + c^{0,\varepsilon_0}(x_0),L_0(x_0),x_2,t) \\ &\qquad\qquad \times\varphi(x_0,t)\,dx_2\,dx_1\,dx_0\,dt \\ &= 0 \quad \forall\varphi\in L^2(J;H^1(\Omega_B^{\varepsilon_0})),\end{aligned} \tag{5.43}$$

which is a weak form of the following partial differential equation:

$$\phi_\sigma^{h\varepsilon_0} r_t^{\varepsilon_0} + \varepsilon_0^2\nabla\cdot\left(\frac{\mathbf{K}_\sigma^{h\varepsilon_0}}{\mu c}\nabla r^{\varepsilon_0}\right) + f_{B,B}^{\varepsilon_0} = 0 \quad \text{in } \Omega_B^{\varepsilon_0}\times J, \tag{5.44}$$

where

$$f_{B,B}^{\varepsilon_0} = (|Y_0||Y_1|)^{-1}\int_{Y_0\times Y_{1B}}\phi_\theta(x_2)\theta_t^{\varepsilon_0}(\varepsilon_0 x_1 + c^{0,\varepsilon_0}(x_0),L_0(x_0),x_2,t)\,dx_2\,dx_1. \tag{5.45}$$

We now determine the initial and boundary conditions for θ^{ε_0} and $\widehat{\sigma}^{\varepsilon_0}$. We begin with the following lemma, which can be established by means of Lemmas 2.1, 2.2, and 2.4.

LEMMA 5.3. *The following is true:*

$$\chi_{\Omega_{B,F}^{\varepsilon_0\varepsilon_1}} \xrightarrow{\varepsilon_1\to 0} \frac{|Y_{1F}|}{|Y_1|} \quad \textit{weak* in } L^\infty(\Omega_B^{\varepsilon_0}). \tag{5.46}$$

By Lemma 5.3,

$$\chi_{\Omega_{B,F}^{\varepsilon_0\varepsilon_1}} \rho_{\text{init}} \longrightarrow \frac{|Y_{1F}|}{|Y_1|}\rho_{\text{init}} \tag{5.47}$$

weakly in $L^2(\Omega_B^{\varepsilon_0})$ as $\varepsilon_1 \to 0$. Also, by (5.5) and weak continuity of the appropriate trace map,

$$\chi_{\Omega_{B,F}^{\varepsilon_0\varepsilon_1}} \sigma^{\varepsilon_0\varepsilon_1}(x_0,0) \longrightarrow \hat{\sigma}^{\varepsilon_0}(x_0,0) \tag{5.48}$$

weakly in $L^2(\Omega_B^{\varepsilon_0})$ as $\varepsilon_1 \to 0$. Therefore, since $\sigma^{\varepsilon_0\varepsilon_1}(x_0,0) = \rho_{\text{init}}(x_0)$, it must be true that

$$\frac{|Y_{1F}|}{|Y_1|}\rho_{\text{init}} = \hat{\sigma}^{\varepsilon_0} \quad \text{in } \Omega_B^{\varepsilon_0} \times \{0\}. \tag{5.49}$$

A more convenient way of writing this is

$$r^{\varepsilon_0} = \rho_{\text{init}} \quad \text{in } \Omega_B^{\varepsilon_0} \times \{0\}. \tag{5.50}$$

To obtain the initial condition for θ^{ε_0}, let $\mathcal{T}_0 : H^1(Y_{1B} \times J) \to H^{1/2}(Y_{1B} \times \{0\})$ denote the trace map. Then, as $\varepsilon_1 \to 0$,

$$\mathcal{T}_0\Big(\big(\theta^{\varepsilon_0\varepsilon_1}\big)^{\sim(2)}\Big) \longrightarrow \mathcal{T}_0\big(\theta^{\varepsilon_0}\big) \tag{5.51}$$

weakly in $L^2(\Omega \times Y_{0B} \times Y_{1B})$. But

$$\mathcal{T}_0\Big(\big(\theta^{\varepsilon_0\varepsilon_1}\big)^{\sim(2)}\Big) = \big(\rho_{\text{init}}\big)^{\sim(2)} \quad \forall \varepsilon_0, \varepsilon_1. \tag{5.52}$$

Also, by Lemma 2.2, $(\rho_{\text{init}})^{\sim(2)} \to (\rho_{\text{init}})^{\sim(1)}$ strongly as $\varepsilon_1 \to 0$. Hence

$$\theta^{\varepsilon_0} = \rho_{\text{init}}^{\sim(1)} \quad \text{in } \Omega \times Y_{0B} \times Y_{1B} \times \{0\}. \tag{5.53}$$

To obtain a boundary condition on ∂Y_{1B}, for θ^{ε_0}, let $\mathcal{T} : H^1(Y_{1B}) \to H^{1/2}(\partial Y_{1B} \times J)$ denote the trace map. Then

$$\begin{aligned} &\mathcal{T}\Big(\big(\sigma^{\varepsilon_0\varepsilon_1}\big)^{\sim(2)}\Big) \longrightarrow \mathcal{T}\big(\sigma^{\varepsilon_0}\big) = \frac{|Y_1|}{|Y_{1F}|}\mathcal{T}\Big(\big(\hat{\sigma}^{\varepsilon_0}\big)^{\sim(1)}\Big), \\ &\mathcal{T}\Big(\big(\theta^{\varepsilon_0\varepsilon_1}\big)^{\sim(2)}\Big) \longrightarrow \mathcal{T}\big(\theta^{\varepsilon_0}\big) \end{aligned} \tag{5.54}$$

weakly in $L^2(\Omega \times Y_{0B} \times \partial Y_{1B} \times J)$. Since, from the boundary condition in (3.2),

$$\tau\Big(\big(\sigma^{\varepsilon_0\varepsilon_1}\big)^{\sim(2)}\Big) = \tau\Big(\big(\theta^{\varepsilon_0\varepsilon_1}\big)^{\sim(2)}\Big), \tag{5.55}$$

it follows that

$$\tau\big(\theta^{\varepsilon_0}\big) = \frac{|Y_1|}{|Y_{1F}|}\tau\Big(\big(\widehat{\sigma}^{\varepsilon_0}\big)^{\sim(1)}\Big), \tag{5.56}$$

that is,

$$\theta^{\varepsilon_0} = \frac{|Y_1|}{|Y_{1F}|}\big(\widehat{\sigma}^{\varepsilon_0}\big)^{\sim(1)} \equiv \big(r^{\varepsilon_0}\big)^{\sim(1)} \quad \text{on } \Omega \times Y_{0B} \times \partial Y_{1B} \times J. \tag{5.57}$$

Thus, θ^{ε_0} is a solution of problems (5.20), (5.57), and (5.53) and r^{ε_0} is a solution of problems (5.44) and (5.50) with the boundary condition

$$\frac{\mathbf{K}_\sigma^{h\varepsilon_0}}{\mu c}\nabla r^{\varepsilon_0} \cdot n_{\Omega_B^{\varepsilon_0}} = 0 \quad \text{on } \partial\Omega_B^{\varepsilon_0} \times J. \tag{5.58}$$

The fact that these problems determine θ^{ε_0} and $\widehat{\sigma}^{\varepsilon_0}$ (and therefore r^{ε_0}) uniquely is a special case of Theorem 7.2, which is proved in Section 7.

Our results so far are summarized in the following theorem.

Theorem 5.4. *As $\varepsilon_1 \to 0$, the following weak limits hold in the indicated spaces:*

$$\begin{aligned}
\big(\theta^{\varepsilon_0\varepsilon_1}\big)^{\sim(2)} &\rightharpoonup \theta^{\varepsilon_0} \quad \textit{in } L^2\big(\Omega; L^2\big(Y_{0B}; H^1\big(Y_{1B} \times J\big)\big)\big), \\
\chi_{\Omega_{B,F}^{\varepsilon_0\varepsilon_1}}\sigma^{\varepsilon_0\varepsilon_1} &\rightharpoonup \widehat{\sigma}^{\varepsilon_0} \quad \textit{in } H^1\big(J; L^2\big(\Omega_B^{\varepsilon_0}\big)\big), \\
\chi_{\Omega_{B,F}^{\varepsilon_0\varepsilon_1}}\frac{K_\sigma^{\varepsilon_0}}{\mu c}\nabla\sigma^{\varepsilon_0} &\rightharpoonup \frac{|Y_1|}{|Y_{1F}|}\frac{\mathbf{K}_\sigma^{h\varepsilon_0}}{\mu c}\nabla\widehat{\sigma}^{\varepsilon_0} \quad \textit{in } L^2\big(\Omega_B^{\varepsilon_0} \times J\big),
\end{aligned} \tag{5.59}$$

and if

$$r^{\varepsilon_0} = \frac{|Y_1|}{|Y_{1F}|}\widehat{\sigma}^{\varepsilon_0},$$

$$\begin{gathered}
f_{B,B}^{\varepsilon_0} = \big(|Y_0||Y_1|\big)^{-1}\int_{Y_0 \times Y_{1B}} \phi_\theta(x_2)\theta_t^{\varepsilon_0}\big(\varepsilon_0 x_1 + c^{0,\varepsilon_0}(x_0), L_0(x_0), x_2, t\big)\,dx_2\,dx_1, \\
\phi_\sigma^{h\varepsilon_0} \equiv \frac{|Y_{1F}|}{|Y_1|}\phi_\sigma^{\varepsilon_0}, \qquad \big(\mathbf{K}_\sigma^{h\varepsilon_0}\big)_{ij} \equiv \mathbf{K}_\sigma^{\varepsilon_0}\left(\omega_{ij} + \frac{|Y_{1F}|}{|Y_1|}\delta_{ij}\right),
\end{gathered} \tag{5.60}$$

then $(r^{\varepsilon_0},\theta^{\varepsilon_0})$ *is the unique weak solution to the following coupled initial-boundary value problems:*

$$\phi_\sigma^{h\varepsilon_0} r_t^{\varepsilon_0} + \varepsilon_0^2 \nabla \cdot \left(\frac{\mathbf{K}_\sigma^{h\varepsilon_0}}{\mu c} \nabla r^{\varepsilon_0} \right) + f_{B,B}^{\varepsilon_0} = 0 \quad \text{in } \Omega_B^{\varepsilon_0} \times J, \tag{5.61}$$

$$\frac{\mathbf{K}_\sigma^{h\varepsilon_0}}{\mu c} \nabla r^{\varepsilon_0} \cdot n_{\Omega_B^{\varepsilon_0}} = 0 \quad \text{on } \partial\Omega_B^{\varepsilon_0} \times J, \tag{5.62}$$

$$r^{\varepsilon_0} = \rho_{\text{init}} \quad \text{in } \Omega_B^{\varepsilon_0} \times \{0\}, \tag{5.63}$$

$$\phi_\theta \theta_t^{\varepsilon_0} - \nabla_{x_2} \cdot \left(\frac{\mathbf{K}_\theta}{\mu c} \nabla_{x_2} \theta^{\varepsilon_0} \right) = 0 \quad \text{in } \Omega \times Y_{0B} \times Y_{1B} \times J, \tag{5.64}$$

$$\theta^{\varepsilon_0} = \left(r^{\varepsilon_0} \right)^{\sim(1)} \quad \text{on } \Omega \times Y_{0B} \times \partial Y_{1B} \times J, \tag{5.65}$$

$$\theta^{\varepsilon_0} = \left(\rho_{\text{init}} \right)^{\sim(1)} \quad \text{in } \Omega \times Y_{0B} \times Y_{1B} \times \{0\}. \tag{5.66}$$

6. Homogenization as $\varepsilon_0 \longrightarrow 0$

With an external source term S defined on $\Omega \times J$, we now create a completely new problem on all of Ω. In order to do this, we use the partial differential equations for the system (5.61), (5.62), (5.63), (5.64), (5.65), and (5.66), but we change the boundary condition (5.62) in order to conserve mass flux and pressure on $\partial\Omega_B^{\varepsilon_0}$. Thus, we have new unknowns now, namely, $\rho^{\varepsilon_0}, \sigma^{h(\varepsilon_0)}$, and $\theta^{h(\varepsilon_0)}$. We pose the following problem based on the previous homogenization:

$$\phi_\rho \rho_t^{\varepsilon_0} - \nabla \cdot \left(\frac{K_\rho}{\mu c} \nabla \rho^{\varepsilon_0} \right) = S \quad \text{in } \Omega_F^{\varepsilon_0} \times J, \tag{6.1}$$

$$\frac{K_\rho}{\mu c} \nabla \rho^{\varepsilon_0} \cdot n_\Omega = 0 \quad \text{on } \partial\Omega \times J, \tag{6.2}$$

$$\frac{K_\rho}{\mu c} \nabla \rho^{\varepsilon_0} \cdot n_{\Omega_B^{\varepsilon_0}} = \varepsilon_0^2 \frac{\mathbf{K}_\sigma^{h\varepsilon_0}}{\mu c} \nabla \sigma^{h(\varepsilon_0)} \cdot n_{\Omega_B^{\varepsilon_0}} \quad \text{on } \partial\Omega_B^{\varepsilon_0} \times J, \tag{6.3}$$

$$\rho^{\varepsilon_0} = \rho_{\text{init}} \quad \text{in } \Omega_F^{\varepsilon_0} \times \{0\}, \tag{6.4}$$

$$\phi_\sigma^{h\varepsilon_0} \sigma_t^{h(\varepsilon_0)} - \varepsilon_0^2 \nabla \cdot \left(\frac{\mathbf{K}_\sigma^{h\varepsilon_0}}{\mu c} \nabla \sigma^{h(\varepsilon_0)} \right) + f_{B,B}^{h\varepsilon_0} = 0 \quad \text{in } \Omega_B^{\varepsilon_0} \times J, \tag{6.5}$$

$$\sigma^{h(\varepsilon_0)} = \rho^{\varepsilon_0} \quad \text{on } \partial\Omega_B^{\varepsilon_0} \times J, \tag{6.6}$$

$$\sigma^{h(\varepsilon_0)} = \rho_{\text{init}} \quad \text{in } \Omega_B^{\varepsilon_0} \times \{0\}, \tag{6.7}$$

$$\phi_\theta \theta_t^{h(\varepsilon_0)} - \nabla_{x_2} \cdot \left(\frac{\mathbf{K}_\theta}{\mu c} \nabla_{x_2} \theta^{h(\varepsilon_0)} \right) = 0 \quad \text{in } \Omega \times Y_{0B} \times Y_{1B} \times J, \tag{6.8}$$

$$\theta^{h(\varepsilon_0)} = \left(\sigma^{h(\varepsilon_0)}\right)^{\sim(1)} \quad \text{on } \Omega \times Y_{0B} \times \partial Y_{1B} \times J, \tag{6.9}$$

$$\theta^{h(\varepsilon_0)} = \left(\rho_{\text{init}}\right)^{\sim(1)} \quad \text{in } \Omega \times Y_{0B} \times Y_{1B} \times \{0\}, \tag{6.10}$$

where

$$f_{B,B}^{h\varepsilon_0} = \left(|Y_0||Y_1|\right)^{-1} \int_{Y_0 \times Y_{1B}} \phi_\theta(x_2) \times \theta_t^{h\varepsilon_0}\left(\varepsilon_0 x_1 + c^{0,\varepsilon_0}(x_0), L_0(x_0), x_2, t\right) dx_2\, dx_1. \tag{6.11}$$

Theorem 6.1. *Problem (6.1), (6.2), (6.3), (6.4), (6.5), (6.6), (6.7), (6.8), (6.9), and (6.10) is well posed in*

$$L^\infty(J;H^1(\Omega_F^{\varepsilon_0})) \cap L^2(\Omega_F^{\varepsilon_0};H^1(J)) \times L^\infty(J;H^1(\Omega_B^{\varepsilon_0})) \cap L^2(\Omega_B^{\varepsilon_0};H^1(J)) \times L^\infty(J;L^2(\Omega \times Y_{0B};H^1(Y_{1B}))) \cap L^2(\Omega \times Y_{0B} \times Y_{1B};H^1(J)). \tag{6.12}$$

This is a specific case of Theorem 7.1 in the more general setting of Section 7. We prove it there, and we also deduce from it the following important estimates. There exists $C > 0$ that is independent of ε_0 such that

$$\|\rho^{\varepsilon_0}\|_{L^\infty(J;H^1(\Omega_F^{\varepsilon_0}))} + \|\rho_t^{\varepsilon_0}\|_{L^2(\Omega_F^{\varepsilon_0} \times J)} \le C\left(\|S\|_{L^2(\Omega \times J)} + \|\rho_{\text{init}}\|_{H^1(\Omega)}\right), \tag{6.13}$$

$$\|\sigma^{h(\varepsilon_0)}\|_{L^2(\Omega_B^{\varepsilon_0} \times J)} + \|\sigma_t^{h(\varepsilon_0)}\|_{L^2(\Omega_B^{\varepsilon_0} \times J)} \le C\|\rho_{\text{init}}\|_{H^1(\Omega)}, \tag{6.14}$$

$$\varepsilon_0\|\nabla\sigma^{h(\varepsilon_0)}\|_{L^\infty(J;L^2(\Omega_B^{\varepsilon_0} \times J))} \le C\|\rho_{\text{init}}\|_{H^1(\Omega)}, \tag{6.15}$$

$$\|\theta^{h(\varepsilon_0)}\|_{L^\infty(J;L^2(\Omega \times Y_{0B});H^1(Y_{1B}))} + \|\theta_t^{h(\varepsilon_0)}\|_{L^2(\Omega \times Y_{0B} \times Y_{1B} \times J)} \le C\|\rho_{\text{init}}\|_{H^1(\Omega)}. \tag{6.16}$$

We now proceed to determine the limit of $(\rho^{\varepsilon_0}, \sigma^{h(\varepsilon_0)}, \theta^{h(\varepsilon_0)})$ as $\varepsilon_0 \to 0$. By virtue of (6.13), (6.14), (6.15), and (6.16), we can pass to a subsequence and deduce that as $\varepsilon_0 \to 0$, we have the following weak limits in the indicated spaces:

$$\chi_{\Omega_F^{\varepsilon_0}} \phi_\rho \rho^{\varepsilon_0} \rightharpoonup \frac{|Y_{0F}|}{|Y_0|} \phi_\rho \rho \quad \text{in } H^1(J;L^2(\Omega)), \tag{6.17}$$

$$\chi_{\Omega_F^{\varepsilon_0}} \frac{K_\rho}{\mu c} \nabla\rho^{\varepsilon_0} \rightharpoonup \xi \quad \text{in } L^2(\Omega \times J), \tag{6.18}$$

$$\left(\sigma^{h(\varepsilon_0)}\right)^{\sim(1)} \rightharpoonup \sigma \quad \text{in } L^2(\Omega;H^1(Y_{0B} \times J)), \tag{6.19}$$

$$\theta^{h(\varepsilon_0)} \rightharpoonup \theta \quad \text{in } L^2(\Omega \times Y_{0B};H^1(Y_{1B} \times J)). \tag{6.20}$$

The partial differential equations satisfied by (ρ,σ,θ) will be derived next. To obtain the equation for θ, we pass to the limit in the equation defining the weak form of (6.7) by virtue of (6.20) to obtain

$$\begin{aligned}(\phi_\theta \theta_t,\psi)_{\Omega\times Y_{0B}\times Y_{1B}\times J} + \left(\frac{\mathbf{K}_\theta}{\mu c}\nabla_{x_2}\theta,\psi\right)_{\Omega\times Y_{0B}\times Y_{1B}\times J}& \\ = 0 \quad \forall \psi \in L^2(\Omega\times Y_{0B}\times J; H_0^1(Y_{1B})).&\end{aligned} \tag{6.21}$$

To find the equation for σ, we deduce from the weak form of (6.5) and the argument in [5, page 831] that

$$\begin{aligned}&\left(\phi_\sigma^h \big(\sigma_t^{h(\varepsilon_0)}\big)^{\sim(1)},\psi\right)_{\Omega\times Y_{0B}\times J} + \left(\frac{\mathbf{K}_\sigma^h}{\mu c}\nabla_{x_1}\big(\sigma_t^{h(\varepsilon_0)}\big)^{\sim(1)},\nabla_{x_1}\psi\right)_{\Omega\times Y_{0B}\times J} \\ &\quad + \left(\big(f_{B,B}^{h\varepsilon_0}\big)^{\sim(1)},\psi\right)_{\Omega\times Y_{0B}\times J} \\ &= T_9 + T_{10} + T_{11} = 0 \quad \forall \psi \in L^2(\Omega; L^2(J; H_0^1(Y_{0B}))).\end{aligned} \tag{6.22}$$

Let $\varepsilon_0 \longrightarrow 0$; from (6.19),

$$T_9 \longrightarrow (\phi_\sigma^h \sigma_t,\psi)_{\Omega\times Y_{0B}\times J}, \qquad T_{10} \longrightarrow \left(\frac{\mathbf{K}_\sigma^h}{\mu c}\nabla_{x_1}\sigma,\nabla_{x_1}\psi\right)_{\Omega\times Y_{0B}\times J}. \tag{6.23}$$

For term T_{11}, we write

$$\begin{aligned}&\big(f_{B,B}^{h\varepsilon_0}\big)^{\sim(1)}(x_0,y,t) \\ &\quad = f_{B,B}^{h\varepsilon_0}(\varepsilon_0 y + c^{0,\varepsilon_0}(x_0),t) \\ &\quad = \frac{1}{|Y_0||Y_1|}\int_{Y_0\times Y_{1B}} \phi_\theta(x_2)\theta_t^{h(\varepsilon_0)}(\varepsilon_0 x_1 + c^{0,\varepsilon_0}(x_0),y,x_2,t)\,dx_2\,dx_1,\end{aligned} \tag{6.24}$$

where we have used the fact that $c^{0,\varepsilon_0}(\varepsilon_0 x_1 + c^{0,\varepsilon_0}(x_0)) = c^{0,\varepsilon_0}(x_0)$. Hence,

$$\begin{aligned}T_{11} &= \frac{1}{|Y_0||Y_1|}\int_{J\times\Omega\times Y_{0B}}\left(\int_{Y_0\times Y_{1B}} \phi_\theta(x_2)\theta_t^{h(\varepsilon_0)}(\varepsilon_0 x_1 + c^{0,\varepsilon_0}(x_0),y,x_2,t)\right. \\ &\qquad\qquad \left.\times \psi(x_0,y,t)\,dx_2\,dx_1\right)dy\,dx_0\,dt \\ &= \frac{1}{|Y_0||Y_1|}\int_{J\times Y_{0B}}\left[\int_{Y_{1B}}\phi_\theta(x_2)\left(\int_{\Omega\times Y_0}\theta_t^{h(\varepsilon_0)}(\varepsilon_0 x_1 + c^{0,\varepsilon_0}(x_0),y,x_2,t)\right.\right. \\ &\qquad\qquad \left.\left.\times \psi(x_0,y,t)\,dx_1\,dx_0\right)dx_2\right]dy\,dt.\end{aligned} \tag{6.25}$$

Hence, by Lemmas 2.1, 2.2 and (6.20),

$$
\begin{aligned}
T_{11} &= \frac{1}{|Y_0||Y_1|}\int_{J\times Y_0\times\Omega\times Y_{0B}\times Y_{1B}} \phi_\theta(x_2)\theta_t^{h(\varepsilon_0)}(x_0,y,x_2,t)\\
&\qquad\qquad \times\varphi(\varepsilon_0 x_1 + c^{0,\varepsilon_0}(x_0),y,t)\,dx_2\,dy\,dx_0\,dx_1\,dt\\
&\xrightarrow{\varepsilon_0\to 0} \frac{1}{|Y_0||Y_1|}\int_{J\times Y_0\times\Omega\times Y_{0B}\times Y_{1B}} \phi_\theta(x_2)\theta_t(x_0,y,x_2,t)\\
&\qquad\qquad \times\varphi(x_0,y,t)\,dx_2\,dy\,dx_0\,dx_1\,dt\\
&= \frac{|Y_0|}{|Y_0||Y_1|}\int_{J\times\Omega\times Y_{0B}\times Y_{1B}} \phi_\theta(x_2)\theta_t(x_0,x_1,x_2,t)\\
&\qquad\qquad \times\varphi(x_0,x_1,t)\,dx_2\,dy\,dx_0\,dx_1\,dt\\
&= \frac{1}{|Y_1|}(\phi_\theta\theta_t,\varphi)_{J\times\Omega\times Y_{0B}\times Y_{1B}}.
\end{aligned}
\tag{6.26}
$$

Hence, by (6.22), (6.23), and (6.26),

$$
\begin{aligned}
&(\phi_\sigma^h\sigma_t,\psi)_{\Omega\times Y_{0B}\times J} + \left(\frac{\mathbf{K}_\sigma^h}{\mu c}\nabla_{x_1}\sigma,\nabla_{x_1}\psi\right)_{\Omega\times Y_{0B}\times J}\\
&\quad + \frac{1}{|Y_1|}(\phi_\theta\theta_t,\psi)_{J\times\Omega\times Y_{0B}\times Y_{1B}} = 0 \quad \forall\psi\in L^2(\Omega;L^2(J;H_0^1(Y_{0B}))).
\end{aligned}
\tag{6.27}
$$

In order to derive an equation for ρ, we use the weak form of (6.1), (6.2), and (6.3) to write

$$
\begin{aligned}
&T_{12}+T_{13}+T_{14}+T_{15}+T_{16}\\
&\quad = (\phi_\rho\rho_t^{\varepsilon_0},\varphi)_{\Omega_F^{\varepsilon_0}\times J} + (\phi_\sigma^{h\varepsilon_0}\sigma_t^{h(\varepsilon_0)},\varphi)_{\Omega_B^{\varepsilon_0}\times J} + \left(\frac{K_\rho}{\mu c}\nabla\rho^{\varepsilon_0},\nabla\varphi\right)_{\Omega_F^{\varepsilon_0}\times J}\\
&\qquad + \varepsilon_0^2\left(\frac{\mathbf{K}_\sigma^{h\varepsilon_0}}{\mu c}\nabla\sigma^{h(\varepsilon_0)},\nabla\varphi\right)_{\Omega_B^{\varepsilon_0}\times J} + (f_{B,B}^{h\varepsilon_0},\varphi)_{\Omega_B^{\varepsilon_0}\times J}\\
&\quad = (S,\varphi)_{\Omega_F^{\varepsilon_0}\times J} = T_{17} \quad \forall\varphi\in L^2(J;H^1(\Omega)).
\end{aligned}
\tag{6.28}
$$

As $\varepsilon_0\to 0$, the following convergence results take place by (6.17) for T_{12}; Lemma 2.1, (6.19), and Lemma 2.2 for T_{13}; (6.18) for T_{14}; (6.15) for T_{15};

Lemma 2.4 and (6.20) for T_{16}; and Lemmas 2.1, 2.2 for T_{17}:

$$\begin{aligned}
T_{12} &\longrightarrow \left(\frac{|Y_{0F}|}{|Y_0|}\phi_\rho\rho_t,\varphi\right)_{\Omega\times J},\\
T_{13} &= |Y_0|^{-1}\left(\phi_\sigma^h\left(\sigma_t^{h(\varepsilon_0)}\right)^{\sim(1)},\varphi^{\sim(1)}\right)_{\Omega\times Y_{0B}\times J} \longrightarrow |Y_0|^{-1}\left(\phi_\sigma^h\sigma_t,\varphi\right)_{\Omega\times Y_{0B}\times J},\\
T_{14} &\longrightarrow (\xi,\nabla\varphi)_{\Omega\times J}, \qquad T_{15}\longrightarrow 0,\\
T_{16} &= \frac{1}{|Y_0||Y_1|}\int_{J\times\Omega_B^{\varepsilon_0}\times Y_0\times Y_{1B}}\phi_\theta(x_2)\theta_t^{h(\varepsilon_0)}\left(\varepsilon_0x_1+c^{0,\varepsilon_0}(x_0),L_0(x_0),x_2,t\right)\\
&\quad\times\varphi(x_0,t)\,dx_2\,dx_1\,dx_0\,dt\\
&= \frac{1}{|Y_0||Y_1|}\int_{J\times Y_{1B}}\left(\int_{\Omega\times Y_{0B}}\phi_\theta(x_2)\theta_t^{h(\varepsilon_0)}(x_0,x_1,x_2,t)\right.\\
&\quad\left.\times\varphi^{\sim(1)}(x_0,x_1,t)\,dx_1\,dx_0\right)dx_2\,dt\\
&= \frac{1}{|Y_0||Y_1|}\left(\phi_\theta\theta_t^{h(\varepsilon_0)},\varphi^{\sim(1)}\right)_{J\times\Omega\times Y_{0B}\times Y_{1B}} \longrightarrow \frac{1}{|Y_0||Y_1|}\left(\phi_\theta\theta_t,\varphi\right)_{J\times\Omega\times Y_{0B}\times Y_{1B}},\\
T_{17} &= \frac{1}{|Y_0|}\left(S^{\sim(1)},\varphi^{\sim(1)}\right)_{\Omega\times Y_{0F}\times J}\longrightarrow\frac{1}{|Y_0|}(S,\varphi)_{\Omega\times Y_{0F}\times J}\\
&= \frac{|Y_{0F}|}{|Y_0|}(S,\varphi)_{\Omega\times J}.
\end{aligned}\tag{6.29}$$

We now apply the arguments of [5, pages 831–833] to identify ξ. We need only to show that the interchange term $f_{B,B}^{h\varepsilon_0}$ between the micro- and mesoscales is bounded for $\varepsilon_0>0$. To this end, we estimate it using the Cauchy-Schwartz inequality, Lemma 2.4, and (6.20) as follows:

$$\begin{aligned}
&\left\|f_{B,B}^{h\varepsilon_0}\right\|^2_{\Omega_B^{\varepsilon_0}\times J}\\
&\qquad\le C\int_{J\times\Omega_B^{\varepsilon_0}\times Y_0\times Y_{1B}}\left|\theta_t^{h(\varepsilon_0)}\left(\varepsilon_0x_1+c^{0,\varepsilon_0}(x_0),L_0(x_0),x_2,t\right)\right|^2dx_2\,dx_1\,dx_0\,dt\\
&\qquad= C\int_{J\times\Omega\times Y_{0B}\times Y_{1B}}1^{\sim(1)}\left|\theta_t^{h(\varepsilon_0)}\right|dx_2\,dx_1\,dx_0\,dt\\
&\qquad= C\left\|\theta_t^{h(\varepsilon_0)}\right\|^2_{L^2(\Omega\times Y_{0B}\times Y_{1B}\times J)}\le C.
\end{aligned}\tag{6.30}$$

It follows that

$$(\xi,\nabla\varphi)_{\Omega\times J}=\left(\frac{\mathbf{K}_\rho^h}{\mu c}\nabla\rho,\nabla\varphi\right)_{\Omega\times J},\tag{6.31}$$

where $\mathbf{K}_\rho^h$ is the homogenized permeability tensor corresponding to K_ρ, as defined in [5, page 827] with the Q appearing there replaced by Y_0.

Hence, by (6.28), (6.29), and (6.31),

$$\frac{|Y_{0F}|}{|Y_0|}(\phi_\rho \rho_t, \varphi)_{\Omega\times J} + \frac{1}{|Y_0|}(\phi_\sigma^h \sigma_t, \varphi)_{\Omega\times Y_{0B}\times J} + \left(\frac{\mathbf{K}_\rho^h}{\mu c}\nabla\rho, \nabla\varphi\right)_{\Omega\times J} + \frac{1}{|Y_0||Y_1|}(\phi_\theta \theta_t, \varphi)_{\Omega\times Y_{0B}\times Y_{1B}\times J} = \frac{|Y_{0F}|}{|Y_0|}(S,\varphi)_{\Omega\times J} \quad \forall \varphi \in L^2(J; H^1(\Omega)). \tag{6.32}$$

We now determine the initial and boundary conditions satisfied by (ρ,σ,θ). By weak continuity of the appropriate trace operators and Lemma 2.2, we can pass to the limit in (6.9), (6.10) as in Section 5 to get

$$\theta = \sigma \quad \text{on } \Omega\times Y_{0B}\times \partial Y_{1B}\times J, \qquad \theta = \rho_{\text{init}} \quad \text{in } \Omega\times Y_{0B}\times Y_{1B}\times\{0\}, \tag{6.33}$$

and from (6.4), (6.6), (6.8), and the argument on [5, page 833], it follows that

$$\begin{gathered} \rho = \sigma \quad \text{on } \Omega\times \partial Y_{0B}\times J, \qquad \rho = \rho_{\text{init}} \quad \text{in } \Omega\times\{0\}, \\ \sigma = \rho_{\text{init}} \quad \text{in } \Omega\times Y_{0B}\times\{0\}. \end{gathered} \tag{6.34}$$

We have shown that (ρ,σ,θ) satisfies [16, problem (3.29)–(3.31)]. Hence, by [16, Theorem 5.2], (ρ,σ,θ) is uniquely determined by (6.21), (6.27), (6.32), (6.33), and (6.34). Hence, the limits in (6.17), (6.18), (6.19), and (6.20) hold as $\varepsilon_0 \to 0$ through its full range of values.

The results of this section are summarized in the following theorem.

Theorem 6.2. *As $\varepsilon_0 \to 0$, the following limits hold weakly in the indicated spaces:*

$$\begin{gathered} \chi_{\Omega_F^{\varepsilon_0}} \phi_\rho \rho^{\varepsilon_0} \rightharpoonup \frac{|Y_{0F}|}{|Y_0|}\phi_\rho \rho \quad \text{in } H^1(J; L^2(\Omega)), \\ \chi_{\Omega_F^{\varepsilon_0}} \frac{K_\rho}{\mu c}\nabla\rho^{\varepsilon_0} \rightharpoonup \frac{\mathbf{K}_\rho^h}{\mu c}\nabla\rho \quad \text{in } L^2(\Omega\times J), \\ \left(\sigma^{h(\varepsilon_0)}\right)^{\sim(1)} \rightharpoonup \sigma \quad \text{in } L^2(\Omega; H^1(Y_{0B}\times J)), \\ \theta^{h(\varepsilon_0)} \rightharpoonup \theta \quad \text{in } L^2(\Omega\times Y_{0B}; H^1(Y_{1B}\times J)), \end{gathered} \tag{6.35}$$

and if

$$\begin{gathered}\phi_\rho^h = \frac{|Y_{0F}|}{|Y_0|}\phi_\rho, \qquad \phi_\sigma^h = \frac{|Y_{1F}|}{|Y_1|}\phi_\sigma, \\ (\mathbf{K}_\sigma^h)_{ij} \equiv K_\sigma\left(\omega_{ij} + \frac{|Y_{1F}|}{|Y_1|}\delta_{ij}\right), \qquad S^h = \frac{|Y_{0F}|}{|Y_0|}S,\end{gathered} \tag{6.36}$$

then (ρ,σ,θ) *is the unique solution to the following system of coupled initial-boundary value problems:* ρ *solves*

$$\begin{gathered}\phi_\rho^h\rho_t - \nabla_{x_0}\cdot\left(\frac{\mathbf{K}_\rho^h}{\mu c}\nabla_{x_0}\rho\right) + \frac{1}{|Y_0|}\int_{Y_{0B}}\phi_\sigma^h\sigma_t\,dx_1 \\ + \frac{1}{|Y_0||Y_1|}\int_{Y_{0B}\times Y_{1B}}\phi_\theta\theta_t\,dx_2\,dx_1 = S^h \quad \text{on } \Omega\times J, \\ \frac{\mathbf{K}_\rho^h}{\mu c}\nabla_{x_0}\rho\cdot n_\Omega = 0 \quad \text{on } \partial\Omega\times J, \qquad \rho = \rho_{\text{init}} \quad \text{on } \Omega\times\{0\},\end{gathered} \tag{6.37}$$

where to each $x_0 \in \Omega$ *a block* Y_{0B} *is associated such that* σ *solves*

$$\begin{gathered}\phi_\sigma^h\sigma_t - \nabla_{x_1}\cdot\left(\frac{\mathbf{K}_\sigma^h}{\mu c}\nabla_{x_1}\sigma\right) + \frac{1}{|Y_1|}\int_{Y_{1B}}\phi_\theta\theta_t\,dx_2 = 0 \quad \text{in } \Omega\times Y_{0B}\times J, \\ \sigma = \rho \quad \text{on } \Omega\times\partial Y_{0B}\times J, \qquad \sigma = \rho_{\text{init}} \quad \text{in } \Omega\times Y_{0B}\times\{0\},\end{gathered} \tag{6.38}$$

where to each $x_0 \in \Omega$ *a block* Y_{0B} *is associated, and to each* $x_1 \in Y_{0B}$ *a block* Y_{1B} *is associated such that* θ *solves*

$$\begin{gathered}\phi_\theta\theta_t - \nabla_{x_2}\cdot\left(\frac{\mathbf{K}_\theta}{\mu c}\nabla_{x_2}\theta\right) = 0 \quad \text{in } \Omega\times Y_{0B}\times Y_{1B}\times J, \\ \theta = \sigma \quad \text{on } \Omega\times Y_{0B}\times\partial Y_{1B}\times J, \qquad \theta = \rho_{\text{init}} \quad \text{in } \Omega\times Y_{0B}\times Y_{1B}\times\{0\}.\end{gathered} \tag{6.39}$$

7. $(N+1)$-scale porosity model

In this section, we generalize the triple-porosity model to an $(N+1)$-scale porosity model. For the recursive homogenization procedure, we hold $\varepsilon_0,\ldots,\varepsilon_{m-1}$ constant and let $\varepsilon_m \to 0$, recursively starting with $m = N-1$ and ending with $m = 0$. Generalizing the techniques used to derive the triple-porosity model, we recursively derive a flow model for $N > 2$ levels of fractures. We begin by assuming that $\varepsilon_0,\ldots,\varepsilon_{i-1}$ are held constant and ε_i is sent to zero, for a fixed $i \geq 1$, giving the following homogenized system of coupled partial differential equations (where, for $i = 1$, we set $\Omega^{\varepsilon_0}_{B,H} \equiv \Omega^{\varepsilon_0}_{H}$, $H = B, F$):

$$\begin{aligned}
\phi_i^{h(\varepsilon_0\cdots\varepsilon_{i-1})}\frac{\partial \rho_i^{h(\varepsilon_0\cdots\varepsilon_{i-1})}}{\partial t} - \varepsilon_0^2\cdots\varepsilon_{i-1}^2 \nabla_{x_0}\cdot\left(\frac{\mathbf{K}_i^{h(\varepsilon_0\cdots\varepsilon_{i-1})}}{\mu c}\nabla_{x_0}\rho_i^{h(\varepsilon_0\cdots\varepsilon_{i-1})}\right)&\\
+ L_i^{h(\varepsilon_0\cdots\varepsilon_{i-1})} = 0 \quad \text{in } \Omega^{\varepsilon_0\cdots\varepsilon_{i-1}}_{B,B}\times J,&\\
\frac{\mathbf{K}_i^{h(\varepsilon_0\cdots\varepsilon_{i-1})}}{\mu c}\nabla_{x_0}\rho_i^{h(\varepsilon_0\cdots\varepsilon_{i-1})}\cdot n_{\Omega^{\varepsilon_0\cdots\varepsilon_{i-1}}_{B,B}} = 0 \quad \text{in } \partial\Omega^{\varepsilon_0\cdots\varepsilon_{i-1}}_{B,B}\times J,&\\
\rho_i^{h(\varepsilon_0\cdots\varepsilon_{i-1})} = \rho_{\text{init}} \quad \text{in } \Omega^{\varepsilon_0\cdots\varepsilon_{i-1}}_{B,B}\times\{0\},&
\end{aligned} \tag{7.1}$$

where

$$\begin{aligned}
&L_i^{h(\varepsilon_0\cdots\varepsilon_{i-1})}\\
&= \sum_{\alpha=i}^{N-1}\frac{1}{|Y_0|\cdots|Y_\alpha|}\\
&\quad\times\int_{Y_0\times\cdots\times Y_{i-1}\times Y_{iB}\times\cdots\times Y_{\alpha B}}\phi_{\alpha+1}^h(y_{\alpha+1})\frac{\partial\rho_{\alpha+1}^{h(\varepsilon_0\cdots\varepsilon_{i-1})}}{\partial t}\\
&\qquad\times\left(\sum_{k=0}^{i-1}\left(\varepsilon_k y_{k+1} + c^{k,\varepsilon_k}(L_{k-1}(x_0))\right)\mathbf{e}_k\right.\\
&\qquad\quad\left. + L_{i-1}(x_0)\mathbf{e}_i + \sum_{k=i+1}^{\alpha+1} y_k\mathbf{e}_k + t\mathbf{e}_{\alpha+2}\right)dy_{\alpha+1}\cdots dy_1.
\end{aligned} \tag{7.2}$$

We have the following system of coupled initial-boundary value problems that are coupled with the above initial-boundary value problem. We state each of them in terms of j, where $j = i+1,\ldots,N$, as follows. For every $x_0 \in \Omega$, there exists a block Y_{0B}, and for every $x_1 \in Y_{0B}$, there exists a block Y_{2B}, and so forth, and for every $x_j \in Y_{(j-1)B}$, the following initial-boundary value problem is satisfied:

$$\begin{aligned}
\phi_j^h\frac{\partial\rho_j^{h(\varepsilon_0\cdots\varepsilon_{i-1})}}{\partial t} - \nabla_{x_j}\cdot\left(\frac{\mathbf{K}_j^h}{\mu c}\nabla_{x_j}\rho_j^{h(\varepsilon_0\cdots\varepsilon_{i-1})}\right)&\\
+ S_j^{h(\varepsilon_0\cdots\varepsilon_{i-1})} = 0 \quad \text{in } \Omega\times Y_{0B}\times\cdots\times Y_{(j-1)B}\times J,&
\end{aligned}$$

$$\rho_j^{h(\varepsilon_0\cdots\varepsilon_{i-1})} = \begin{cases} \left(\rho_{j-1}^{h(\varepsilon_0\cdots\varepsilon_{i-1})}\right)^{\sim(i)} & \text{if } j = i+1, \\ \rho_{j-1}^{h(\varepsilon_0\cdots\varepsilon_{i-1})} & \text{otherwise,} \end{cases}$$

$$\text{in } \Omega \times Y_{0B} \times \cdots \times Y_{(j-2)B} \times \partial Y_{(j-1)B} \times J,$$

$$\rho_j^{h(\varepsilon_0\cdots\varepsilon_{i-1})} = \left(\rho_{\text{init}}\right)^{\sim(i)} \quad \text{in } \Omega \times Y_{0B} \times \cdots \times Y_{(j-1)B} \times \{0\}, \tag{7.3}$$

where, for $j = i+1,\ldots,N-1$,

$$\begin{aligned} &S_j^{h(\varepsilon_0\cdots\varepsilon_{i-1})} \\ &= \sum_{\alpha=j}^{N-1}\left(\frac{|Y_i||Y_{i+1}|\cdots|Y_{j-1}|}{|Y_0||Y_1|\cdots|Y_\alpha|}\right) \\ &\quad\times \int_{Y_0\times\cdots\times Y_{i-1}\times Y_{jB}\times\cdots\times Y_{\alpha B}} \phi_{\alpha+1}^h(y_{\alpha+1})\frac{\partial\rho_{\alpha+1}^{h(\varepsilon_0\cdots\varepsilon_{i-1})}}{\partial t} \\ &\qquad\times\left(\sum_{k=0}^{i-1}\left(\varepsilon_k y_{k+1} + c^{k,\varepsilon_k}(x_k)\right)\mathbf{e}_k + \sum_{k=i}^{j} x_k\mathbf{e}_k \right. \\ &\qquad\left. + \sum_{k=j+1}^{\alpha+1} y_k\mathbf{e}_k + t\mathbf{e}_{\alpha+2}\right) dy_{\alpha+1}\cdots dy_{j+1}\,dy_i\cdots dy_1, \end{aligned} \tag{7.4}$$

and $S_N^{h(\varepsilon_0\cdots\varepsilon_{i-1})} \equiv 0$. Also, $\phi_N^h \equiv \phi_N$ and $\mathbf{K}_N^h \equiv \mathbf{K}_N$.

If we proceed to recursively homogenize the problem at the next level, that is, if we want to let $\varepsilon_{i-1} \to 0$, then we first change the no-flow boundary condition on $\partial\Omega_{B,B}^{\varepsilon_0\cdots\varepsilon_{i-1}} \times J$ in (7.1) to the following boundary condition:

$$\rho_i^{h(\varepsilon_0\cdots\varepsilon_{i-1})} = \rho_{i-1}^{(\varepsilon_0\cdots\varepsilon_{i-1})} \quad \text{in } \partial\Omega_{B,B}^{\varepsilon_0\cdots\varepsilon_{i-1}} \times J, \tag{7.5}$$

Then, we impose the following initial-boundary value problem in $\Omega_{B,F}^{\varepsilon_0\cdots\varepsilon_{i-1}} \times J$:

$$\begin{aligned} &\phi_{i-1}^{\varepsilon_0\cdots\varepsilon_{i-2}}\frac{\partial\rho_{i-1}^{(\varepsilon_0\cdots\varepsilon_{i-1})}}{\partial t} - \varepsilon_0^2\cdots\varepsilon_{i-2}^2\nabla_{x_0}\cdot\left(\frac{K_{i-1}^{\varepsilon_0\cdots\varepsilon_{i-2}}}{\mu c}\nabla_{x_0}\rho_{i-1}^{(\varepsilon_0\cdots\varepsilon_{i-1})}\right) \\ &\quad = \delta_{i,1}S(x_0,t) \quad \text{in } \Omega_{B,F}^{\varepsilon_0\cdots\varepsilon_{i-1}} \times J, \end{aligned}$$

$$\begin{aligned}
&\frac{K_{i-1}^{\varepsilon_0\cdots\varepsilon_{i-2}}}{\mu c}\nabla_{x_0}\rho_{i-1}^{(\varepsilon_0\cdots\varepsilon_{i-1})}\cdot n_{\Omega_{B,B}^{\varepsilon_0\cdots\varepsilon_{i-1}}}\\
&\qquad = \varepsilon_{i-1}^2\frac{\mathbf{K}_i^{h(\varepsilon_0\cdots\varepsilon_{i-1})}}{\mu c}\nabla_{x_0}\rho_i^{h(\varepsilon_0\cdots\varepsilon_{i-1})}\cdot n_{\Omega_{B,B}^{\varepsilon_0\cdots\varepsilon_{i-1}}}\quad \text{on } \partial\Omega_{B,B}^{\varepsilon_0\cdots\varepsilon_{i-1}}\times J,\\
&\frac{K_{i-1}^{\varepsilon_0\cdots\varepsilon_{i-2}}}{\mu c}\nabla_{x_0}\rho_{i-1}^{(\varepsilon_0\cdots\varepsilon_{i-1})}\cdot n_{\Omega_{B,F}^{\varepsilon_0\cdots\varepsilon_{i-1}}}=0\quad \text{on } \partial\Omega_{B,B}^{\varepsilon_0\cdots\varepsilon_{i-2}}\times J,\\
&\rho_{i-1}^{(\varepsilon_0\cdots\varepsilon_{i-1})}=\rho_{\text{init}}\quad \text{in } \Omega_{B,F}^{\varepsilon_0\cdots\varepsilon_{i-1}}\times\{0\}.
\end{aligned}\tag{7.6}$$

We now prove that the model is well posed.

THEOREM 7.1. *The problem at stage* $i-1$*, namely, (7.1) (with the new boundary condition (7.5)), (7.3), and (7.6) is well posed in*

$$\begin{aligned}
&L^\infty(J;H^1(\Omega_{B,F}^{\varepsilon_0\cdots\varepsilon_{i-1}}))\cap L^2(\Omega_{B,F}^{\varepsilon_0\cdots\varepsilon_{i-1}};H^1(J))\times L^\infty(J;H^1(\Omega_{B,B}^{\varepsilon_0\cdots\varepsilon_{i-1}}))\\
&\quad\cap L^2(\Omega_{B,B}^{\varepsilon_0\cdots\varepsilon_{i-1}};H^1(J))\\
&\qquad\times\big[\times_{j=i+1}^N L^\infty(J;L^2(\Omega\times Y_{0B}\times\cdots\times Y_{(j-2)B};H^1(Y_{(j-1)B}\times J)))\\
&\qquad\quad\cap L^2(\Omega\times Y_{0B}\times\cdots\times Y_{(j-1)B};H^1(J))\big],
\end{aligned}\tag{7.7}$$

and there exists $C>0$*, independent of* ε_{i-1}*, such that*

$$\begin{aligned}
&\|\rho_{i-1}^{(\varepsilon_0\cdots\varepsilon_{i-1})}\|_{L^\infty(J;H^1(\Omega_{B,F}^{\varepsilon_0\cdots\varepsilon_{i-1}}))}+\|\rho_{i-1,t}^{(\varepsilon_0\cdots\varepsilon_{i-1})}\|_{L^2(\Omega_{B,F}^{\varepsilon_0\cdots\varepsilon_{i-1}}\times J)}\\
&\qquad\le C\Big(\delta_{i,1}\|S\|_{L^2(\Omega\times J)}+\|\rho_{\text{init}}\|_{H^1(\Omega)}\Big),\\
&\|\rho_i^{h(\varepsilon_0\cdots\varepsilon_{i-1})}\|_{L^2(\Omega_{B,B}^{\varepsilon_0\cdots\varepsilon_{i-1}}\times J)}+\|\rho_{i,t}^{h(\varepsilon_0\cdots\varepsilon_{i-1})}\|_{L^2(\Omega_{B,B}^{\varepsilon_0\cdots\varepsilon_{i-1}}\times J)}\le C\|\rho_{\text{init}}\|_{H^1(\Omega)},\\
&\varepsilon_{i-1}\|\nabla\rho_i^{h(\varepsilon_0\cdots\varepsilon_{i-1})}\|_{L^\infty(J;L^2(\Omega^{\varepsilon_0\cdots\varepsilon_{i-1}}\times J))}\le C\|\rho_{\text{init}}\|_{H^1(\Omega)},
\end{aligned}\tag{7.8}$$

and for $j=i+1,\dots,N$*,*

$$\begin{aligned}
&\|\rho_j^{h(\varepsilon_0\cdots\varepsilon_{i-1})}\|_{L^\infty(J;L^2(\Omega\times Y_{0B}\times\cdots\times Y_{(j-2)B};H^1(Y_{(j-1)B}\times J)))}\\
&\qquad+\|\rho_{j,t}^{h(\varepsilon_0\cdots\varepsilon_{i-1})}\|_{L^2(\Omega\times Y_{0B}\times\cdots\times Y_{(j-1)B})}\le C\|\rho_{\text{init}}\|_{H^1(\Omega)}.
\end{aligned}\tag{7.9}$$

Proof. This can be established by a straightforward application of [23, Propositions I.4.1, III.2.1, III.2.5], a standard fixed-point argument, and the usual energy and Dirichlet problem estimates for parabolic equations. □

A similar kind of arguments also works to prove the following theorem.

THEOREM 7.2. *Problem (7.1), (7.3) is well posed in*

$$\begin{aligned} &L^\infty(J;H^1(\Omega_{B,B}^{\varepsilon_0\cdots\varepsilon_{i-1}}))\cap L^2(\Omega_{B,B}^{\varepsilon_0\cdots\varepsilon_{i-1}};H^1(J)) \\ &\quad\times\Bigg[\prod_{j=i+1}^{N} L^\infty(J;L^2(\Omega\times Y_{0B}\times\cdots\times Y_{(j-2)B};H^1(Y_{(j-1)B}\times J))) \\ &\qquad\cap L^2(\Omega\times Y_{0B}\times\cdots\times Y_{(j-1)B};H^1(J))\Bigg]. \end{aligned}\tag{7.10}$$

In order to identify the intermediate source terms which do not appear in the triple-porosity model, the following lemma is essential.

LEMMA 7.3. *Let $F^{\varepsilon_0\cdots\varepsilon_{i-1}},\varphi\in L^2(\Omega\times Y_{0B}\times\cdots\times Y_{(j-1)B}\times J)$. Assume that $F^{\varepsilon_0\cdots\varepsilon_{i-1}}\rightharpoonup F^{\varepsilon_0\cdots\varepsilon_{i-2}}$ weakly in $L^2(\Omega\times Y_{0B}\times\cdots\times Y_{(j-1)B}\times J)$. Then, for $j=i,\ldots,N-1$,*

$$\begin{aligned} &\int_{J\times\Omega\times Y_{0B}\times\cdots\times Y_{(j-1)B}}\Bigg[\int_{Y_0\times\cdots\times Y_{i-1}}F^{\varepsilon_0\cdots\varepsilon_{i-1}}\Bigg(\sum_{k=0}^{i-1}(\varepsilon_k y_{k+1}+c^{k,\varepsilon_k}(x_k))\mathbf{e}_k \\ &\qquad\qquad+\sum_{k=i}^{j}x_k\mathbf{e}_k+t\mathbf{e}_{j+1}\Bigg)dy_i\cdots dy_1\Bigg] \\ &\qquad\times\varphi(x_0,\ldots,x_j,t)\,dx_j\cdots dx_0\,dt\xrightarrow{\varepsilon_{i-1}\to 0}|Y_{i-1}| \\ &\quad\times\int_{J\times\Omega\times Y_{0B}\times\cdots\times Y_{(j-1)B}}\Bigg[\int_{Y_0\times\cdots\times Y_{i-2}}F^{\varepsilon_0\cdots\varepsilon_{i-2}} \\ &\qquad\times\Bigg(\sum_{k=0}^{i-2}(\varepsilon_k y_{k+1}+c^{k,\varepsilon_k}(x_k))\mathbf{e}_k \\ &\qquad\qquad+\sum_{k=i-1}^{j}x_k\mathbf{e}_k+t\mathbf{e}_{j+1}\Bigg)dy_{i-1}\cdots dy_1\Bigg] \\ &\qquad\times\varphi(x_0,\ldots,x_j,t)\,dx_j\cdots dx_0\,dt. \end{aligned}\tag{7.11}$$

Proof. Using Fubini's theorem, we interchange the integration spaces in the following convenient way:

$$\int_{J\times\Omega\times Y_{0B}\times\cdots\times Y_{(j-1)B}}\int_{Y_0\times\cdots\times Y_{i-1}} F^{\varepsilon_0\cdots\varepsilon_{i-1}}\Bigg(\sum_{k=0}^{i-1}\big(\varepsilon_k y_{k+1}+c^{k,\varepsilon_k}(x_k)\big)\mathbf{e}_k + \sum_{k=i}^{j} x_k\mathbf{e}_k + t\mathbf{e}_{j+1}\Bigg)dy_i\cdots dy_1$$
$$\times\,\varphi(x_0,\dots,x_j,t)\,dx_j\cdots dx_0\,dt$$
$$= \int_{J\times\Omega\times Y_{0B}\times\cdots\times Y_{(i-3)B}\times Y_{(i-1)B}\times\cdots\times Y_{(j-1)B}}$$
$$\times\int_{Y_0\times\cdots\times Y_{i-2}}\Bigg[\iint_{Y_{(i-2)B}\times Y_{(i-1)}} F^{\varepsilon_0\cdots\varepsilon_{i-1}}\big(\dots,\varepsilon_{i-1}y_i + c^{(i-1),\varepsilon_{i-1}}(x_{i-1}),\dots\big)$$
$$\times\,\varphi(\dots,x_{i-1},\dots)\,dy_i\,dx_{i-1}\Bigg]$$
$$\times\, dy_{i-1}\cdots dy_1\,dx_j\cdots dx_i\,dx_{i-2}\cdots dx_0\,dt$$
$$= \int_{J\times\Omega\times Y_{0B}\times\cdots\times Y_{(i-3)B}\times Y_{(i-1)B}\times\cdots\times Y_{(j-1)B}}$$
$$\times\int_{Y_0\times Y_{0B}\times\cdots\times Y_{i-2}}\Bigg[\iint_{Y_{(i-2)B}\times Y_{(i-1)}} F^{\varepsilon_0\cdots\varepsilon_{i-1}}(\dots,x_{i-1},\dots)$$
$$\times\,\varphi\big(\dots,\varepsilon_{i-1}y_i + c^{(i-1),\varepsilon_{i-1}}(x_{i-1}),$$
$$\dots\big)\,dy_i\,dx_{i-1}\Bigg]$$
$$\times\, dy_{i-1}\cdots dy_1\,dx_j\cdots dx_i\,dx_{i-2}\cdots dx_0\,dt$$
$$\xrightarrow{\varepsilon_{i-1}\to 0} \int_{J\times\Omega\times Y_{0B}\times\cdots\times Y_{(i-3)B}\times Y_{(i-1)B}\times\cdots\times Y_{(j-1)B}}$$
$$\times\int_{Y_0\times\cdots\times Y_{i-2}}\Bigg[\iint_{Y_{(i-2)B}\times Y_{(i-1)}} F^{\varepsilon_0\cdots\varepsilon_{i-2}}(\dots,x_{i-1},\dots)$$
$$\times\,\varphi(\dots,x_{i-1},\dots)\,dy_i\,dx_{i-1}\Bigg]$$
$$\times\, dy_{i-1}\cdots dy_1\,dx_j\cdots dx_i\,dx_{i-2}\cdots dx_0\,dt$$
$$= |Y_{i-1}|\int_{J\times\Omega\times Y_{0B}\times\cdots\times Y_{(i-3)B}\times Y_{(i-1)B}\times\cdots\times Y_{(j-1)B}}$$
$$\times\int_{Y_0\times\cdots\times Y_{i-2}}\Bigg[\iint_{Y_{(i-2)B}} F^{\varepsilon_0\cdots\varepsilon_{i-2}}(\dots,x_{i-1},\dots)$$
$$\times\,\varphi(\dots,x_{i-1},\dots)\,dx_{i-1}\Bigg]$$
$$\times\, dy_{i-1}\cdots dy_1\,dx_j\cdots dx_i\,dx_{i-2}\cdots dx_0\,dt$$

$$
\begin{aligned}
&= |Y_{i-1}| \\
&\quad \times \int_{J\times\Omega\times\cdots\times\cdots\times Y_{(j-1)B}} \Bigg[\int_{Y_0\times\cdots\times Y_{i-2}} F^{\varepsilon_0\cdots\varepsilon_{i-2}} \Bigg(\sum_{k=0}^{i-2} \big(\varepsilon_k y_{k+1} + c^{k,\varepsilon_k}(x_k)\big)\mathbf{e}_k \\
&\qquad\qquad\qquad + \sum_{k=i-1}^{j} x_k \mathbf{e}_k + t\mathbf{e}_{j+1} \Bigg) dy_{i-1}\cdots dy_1 \Bigg] \\
&\qquad\qquad \times \varphi(x_0,\ldots,x_j,t)\, dx_j\cdots dx_0\, dt,
\end{aligned}
\tag{7.12}
$$

where Lemma 2.5 gives the second equality, and the weak convergence of $F^{\varepsilon_0\cdots\varepsilon_{i-1}}$ to $F^{\varepsilon_0\cdots\varepsilon_{i-2}}$ and the strong convergence of $\phi(x_0,\ldots,\varepsilon_{i-1}y_i + c^{(i-1),\varepsilon_{i-1}}(x_{i-1}),\ldots,x_j,t)$ to $\phi(x_0,\ldots,x_{i-1},\ldots,x_j,t)$ as $\varepsilon_{i-1}\to 0$ give the convergence result. The volume $|Y_{i-1}|$ appears as a result of the integrand's independence on the variable y_{i-2} after $\varepsilon_{i-1}\to 0$. Fubini's theorem gives the final equality. □

Corollary 7.4. *With the definition of* $S_j^{h(\varepsilon_0\cdots\varepsilon_{i-1})}$ *in* (7.4),

$$
S_j^{h(\varepsilon_0\cdots\varepsilon_{i-1})} \rightharpoonup S_j^{h(\varepsilon_0\cdots\varepsilon_{i-2})} \text{ weakly as } \varepsilon_{i-1}\longrightarrow 0. \tag{7.13}
$$

Notice now that

$$
\begin{aligned}
&\big(L_i^{h(\varepsilon_0\cdots\varepsilon_{i-1})}\big)^{\sim(i)} \\
&\quad = \sum_{\alpha=i}^{N-1} \frac{1}{|Y_0|\cdots|Y_\alpha|} \\
&\qquad \times \int_{Y_0\times\cdots\times Y_{i-1}\times Y_{iB}\times\cdots\times Y_{\alpha B}} \phi_{\alpha+1}^h(y_{\alpha+1}) \frac{\partial \rho_{\alpha+1}^{h(\varepsilon_0\cdots\varepsilon_{i-1})}}{\partial t} \\
&\qquad\qquad \times \Bigg(\sum_{k=0}^{i-1} \big(\varepsilon_k y_{k+1} + c^{k,\varepsilon_k}(x_k)\big)\mathbf{e}_k + x_i\mathbf{e}_i \\
&\qquad\qquad\qquad + \sum_{k=i+1}^{\alpha+1} y_k\mathbf{e}_k + t\mathbf{e}_{\alpha+2} \Bigg) dy_{\alpha+1}\cdots dy_1 \\
&\quad \equiv S_i^{h(\varepsilon_0\cdots\varepsilon_{i-1})}.
\end{aligned}
\tag{7.14}
$$

We observe that this definition of $S_i^{h(\varepsilon_0\cdots\varepsilon_{i-1})}$ agrees with the one given by (7.4), with the empty product in the numerator of the coefficient there replaced by 1. Thus, by Lemma 7.3, with $j = i$, we have the following weak convergence result in $L^2(D)$:

$$(L_i^{h(\varepsilon_0\cdots\varepsilon_{i-1})})^{\sim(i)} \xrightarrow{\varepsilon_{i-1}\to 0} \sum_{\alpha=i}^{N-1} \frac{|Y_{i-1}|}{|Y_0|\cdots|Y_\alpha|} \times \int_{Y_0\times\cdots\times Y_{i-2}\times Y_{iB}\times\cdots\times Y_{\alpha B}} \phi_{\alpha+1}^h(y_{\alpha+1}) \frac{\partial \rho_{\alpha+1}^{h(\varepsilon_0\cdots\varepsilon_{i-1})}}{\partial t} \times \left(\sum_{k=0}^{i-2} (\varepsilon_k y_{k+1} + c^{k,\varepsilon_k}(x_k))\mathbf{e}_k + \sum_{k=i-1}^{i} x_k \mathbf{e}_k + \sum_{k=i+1}^{\alpha+1} y_k \mathbf{e}_k + t\mathbf{e}_{\alpha+2} \right) dy_{\alpha+1}\cdots dy_1 \equiv S_i^{h(\varepsilon_0\cdots\varepsilon_{i-2})}. \tag{7.15}$$

The permeability tensors that arise in the intermediate stages of homogenization are given by

$$(\mathbf{K}_{i-1}^{h(\varepsilon_0\cdots\varepsilon_{i-2})})_{\alpha\beta} = K_{i-1}^{\varepsilon_0\cdots\varepsilon_{i-2}} \left(\omega_{\alpha\beta}^{i-1} + \frac{|Y_{(i-1)F}|}{|Y_{i-1}|} \delta_{\alpha\beta} \right). \tag{7.16}$$

In the above definition, the geometry of the fractures enters the equations via $\boldsymbol{\omega}^{i-1} \equiv (\omega^{i-1})_{\alpha\beta}$, which is defined by the Y_{i-1}-periodic solution ω_β^{i-1}, modulo constants, of the following Neumann problem:

$$\Delta_{x_i} w_\beta^{i-1} = 0 \quad \text{in } Y_{(i-1)F}, \qquad \nabla_{x_i} \omega_\beta^{i-1} \cdot \nu = -\mathbf{e}_\beta \cdot \nu = -\nu_\beta \quad \text{on } \partial Y_{(i-1)B},$$
$$\omega_{\alpha\beta}^{i-1} = \frac{1}{|Y_{i-1}|} \int_{Y_{(i-1)F}} \partial_{i,\alpha} \omega_\beta^{i-1}(x_i)\, dx_i, \tag{7.17}$$

where $\partial_{i,\alpha} = \partial/\partial x_{i,\alpha}$.

Finally, define

$$\phi_{i-1}^{h(\varepsilon_0\cdots\varepsilon_{i-2})} \equiv \frac{|Y_{(i-1)F}|}{|Y_{i-1}|} \phi_{i-1}^{\varepsilon_0\cdots\varepsilon_{i-2}}, \qquad \rho_{i-1}^{h(\varepsilon_0\cdots\varepsilon_{i-2})} \equiv \frac{|Y_{i-1}|}{|Y_{(i-1)F}|} \widehat{\rho}_{i-1}^{\varepsilon_0\cdots\varepsilon_{i-2}}. \tag{7.18}$$

By adapting the arguments employed in the homogenization of the triple-porosity model, using in addition Corollary 7.4 and (7.15), we arrive at the following theorem.

Theorem 7.5. *For $i = 1,\ldots,N$, let $(\rho_{i-1}^{(\varepsilon_0\cdots\varepsilon_{i-1})}, \rho_i^{h(\varepsilon_0\cdots\varepsilon_{i-1})}, \ldots, \rho_N^{h(\varepsilon_0\cdots\varepsilon_{i-1})})$ be the unique weak solution to problem (7.1) (with the new boundary condition (7.5)),* (7.3), and (7.6). Then, as $\varepsilon_{i-1} \to 0$, the following limits hold weakly in the indicated spaces:

$$\begin{gathered}
\chi_{\Omega_{B,F}^{\varepsilon_0\cdots\varepsilon_{i-1}}} \rho_{i-1}^{(\varepsilon_0\cdots\varepsilon_{i-1})} \rightharpoonup \widehat{\rho}_{i-1}^{\varepsilon_0\cdots\varepsilon_{i-2}} \quad \text{in } H^1(J; L^2(\Omega_{B,B}^{\varepsilon_0\cdots\varepsilon_{i-2}})), \\
\chi_{\Omega_{B,F}^{\varepsilon_0\cdots\varepsilon_{i-1}}} \frac{K_{i-1}^{\varepsilon_0\cdots\varepsilon_{i-2}}}{\mu c} \nabla_{x_0} \rho_{i-1}^{(\varepsilon_0\cdots\varepsilon_{i-1})} \rightharpoonup \frac{|Y_{i-1}|}{|Y_{(i-1)F}|} \mathbf{K}_{i-1}^{h(\varepsilon_0\cdots\varepsilon_{i-2})} \nabla_{x_0} \widehat{\rho}_{i-1}^{\varepsilon_0\cdots\varepsilon_{i-2}} \\
\text{in } L^2(\Omega_{B,B}^{\varepsilon_0\cdots\varepsilon_{i-2}} \times J), \\
\left(\rho_i^{h(\varepsilon_0\cdots\varepsilon_{i-1})}\right)^{\sim(i)} \rightharpoonup \rho_i^{h(\varepsilon_0\cdots\varepsilon_{i-2})} \quad \text{in } L^2(\Omega \times Y_{0B} \times \cdots \times Y_{(i-2)B}; H^1(Y_{(i-1)B} \times J)),
\end{gathered} \tag{7.19}$$

and for $j = i+1, \ldots, N$,

$$\rho_j^{h(\varepsilon_0\cdots\varepsilon_{i-1})} \rightharpoonup \rho_j^{h(\varepsilon_0\cdots\varepsilon_{i-2})} \quad \text{in } L^2(\Omega \times Y_{0B} \times \cdots \times Y_{(j-2)B}; H^1(Y_{(j-1)B} \times J)). \tag{7.20}$$

Moreover, if

$$\rho_{i-1}^{h(\varepsilon_0\cdots\varepsilon_{i-2})} = \frac{|Y_{i-1}|}{|Y_{(i-1)F}|} \widehat{\rho}_{i-1}^{\varepsilon_0\cdots\varepsilon_{i-2}}, \tag{7.21}$$

then $(\rho_{i-1}^{h(\varepsilon_0\cdots\varepsilon_{i-2})}, \ldots, \rho_N^{h(\varepsilon_0\cdots\varepsilon_{i-2})})$ is the unique weak solution of problem (7.1), (7.3) with i replaced by $i-1$. In particular, as $\varepsilon_0 \to 0$, the following weak limits hold:

$$\begin{gathered}
\chi_{\Omega_F^{\varepsilon_0}} \rho_0^{\varepsilon_0} \rightharpoonup \widehat{\rho}_0 \quad \text{in } H^1(J; L^2(\Omega)), \\
\chi_{\Omega_B^{\varepsilon_0}} \frac{K_0^{\varepsilon_0}}{\mu c} \nabla \rho_0^{\varepsilon_0} \rightharpoonup \frac{|Y_0|}{|Y_{0F}|} \mathbf{K}_0^h \nabla \widehat{\rho}_0 \quad \text{in } L^2(\Omega \times J), \\
\left(\rho_1^{h(\varepsilon_0)}\right)^{\sim(1)} \rightharpoonup \rho_1 \quad \text{in } L^2(\Omega; H^1(Y_{0B} \times J)),
\end{gathered} \tag{7.22}$$

and for $j = 2, \ldots, N$,

$$\rho_j^{h(\varepsilon_0)} \rightharpoonup \rho_j \quad \text{in } L^2(\Omega \times Y_{0B} \times \cdots \times Y_{(j-2)B}; H^1(Y_{(j-1)B} \times J)). \tag{7.23}$$

Moreover, if

$$\rho_0 = \frac{|Y_0|}{|Y_{0F}|} \widehat{\rho}_0, \qquad S^h = \frac{|Y_{0F}|}{|Y_0|} S, \tag{7.24}$$

then $(\rho_0, \dots, \rho_N)$ *is the unique weak solution of the following system of final homogenized equations:*

$$\begin{aligned} \phi_0^h \frac{\partial \rho_0}{\partial t} - \nabla_{x_0} \cdot \left(\frac{\mathbf{K}_0^h}{\mu c} \nabla_{x_0} \rho_0 \right) \\ + \sum_{k=1}^{N} \frac{1}{|Y_0| \cdots |Y_{k-1}|} \int_{Y_{0B} \times \cdots \times Y_{(k-1)B}} \phi_k^h \frac{\partial \rho_k}{\partial t} dx_k \cdots dx_1 = S^h \quad \text{in } \Omega \times J, \\ \frac{\mathbf{K}_0^h}{\mu c} \nabla_{x_0} \rho_0 \cdot n_\Omega = 0 \quad \text{on } \partial\Omega \times J, \\ \rho_0 = \rho_{\text{init}} \quad \text{in } \Omega \times \{0\}, \end{aligned} \tag{7.25}$$

and for every $x_{j-1} \in Y_{(j-2)B}$*, there exists a block* $Y_{(j-1)B}$ *such that* ρ_j *solves the following initial-boundary value problem for* $j = 1, \dots, N$*:*

$$\begin{aligned} \phi_j^h \frac{\partial \rho_j}{\partial t} - \nabla_{x_j} \cdot \left(\frac{\mathbf{K}_j^h}{\mu c} \nabla_{x_j} \rho_j \right) \\ + \sum_{k=j}^{N-1} \frac{1}{|Y_j| \cdots |Y_k|} \int_{Y_{jB} \times \cdots \times Y_{kB}} \phi_{k+1}^h \frac{\partial \rho_{k+1}}{\partial t} dx_{k+1} \cdots dx_{j+1} \\ = 0 \quad \text{in } \Omega \times Y_{0B} \times \cdots \times Y_{(j-1)B} \times J, \\ \rho_j = \rho_{j-1} \quad \text{on } \Omega \times Y_{0B} \times \cdots \times Y_{(j-2)B} \times \partial Y_{(j-1)B} \times J, \\ \rho_j = \rho_{\text{init}} \quad \text{in } \Omega \times Y_{0B} \times \cdots \times Y_{(j-1)B} \times \{0\}. \end{aligned} \tag{7.26}$$

8. Concluding remarks

Each equation in the final homogenized system of the $(N+1)$-scale problem contains interchange terms from the relatively smaller scales. This shows that the recursive homogenization procedure captures the microscale effects. Instead, if the entire reservoir was homogenized in a straightforward manner as was done in [1, 10], a single-porosity model with an average permeability would result. That would be inadequate here since the porous structures are quite distinct. Here, we retain the fine microscopic structures, yet we average their effects.

It is worth noting that instead of putting the external source term solely on the first level of fractures, we could have easily defined it on each level. If this is desired, then the above analysis still holds with the modification that in the final homogenized system of equations, the external source term would appear on each level with no modification from the homogenization.

Acknowledgment

This research is funded by an Oakland University Research Grant.

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