ON THE GLOBAL SOLVABILITY OF SOLUTIONS TO A QUASILINEAR WAVE EQUATION WITH LOCALIZED DAMPING AND SOURCE TERMS

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Received 18 July 2004 and in revised form 9 March 2005

We prove existence and uniform stability of strong solutions to a quasilinear wave equation with a locally distributed nonlinear dissipation with source term of power nonlinearity of the type $u'' - M(\int_\Omega |\nabla u|^2 dx)\Delta u + a(x)g(u') + f(u) = 0$, in $\Omega\times]0,+\infty[$, $u = 0$, on $\Gamma\times]0,+\infty[$, $u(x,0) = u_0(x)$, $u'(x,0) = u_1(x)$, in Ω.

1. Introduction

Let Ω be a bounded domain of $\mathbb{R}^N$ with a smooth boundary $\Gamma = \partial\Omega$. We consider the initial-boundary value problem

$$\begin{gathered} u'' - M\left(\int_\Omega |\nabla u|^2 dx\right)\Delta u + a(x)g(u') + f(u) = 0, \quad \text{in } \Omega\times]0,+\infty[, \\ u = 0, \quad \text{on } \Gamma\times]0,+\infty[, \\ u(x,0) = u_0(x), \quad u'(x,0) = u_1(x), \quad \text{in } \Omega, \end{gathered} \tag{1.1}$$

where $M(s)$ is a C^1-class function on $[0,+\infty[$ satisfying $M(s) \geq m_0 > 0$, for $s \geq 0$, with m_0 constant, a is a smooth nonnegative function but vanishes somewhere in $\overline{\Omega}$, $f(u)$ is a nonlinear term like $f(u) \sim -|u|^\alpha u$, and g is a real-valued function.

The problem (1.1), when $M(s) = 1$ and f is some type of nonlinear function, has been studied by Zuazua [10] and Nakao [9]. Recently, Cabanillas et al. have treated in [2, 3] a more delicate case where M is not a constant function ($f(u) = 0, -h_0 u$). Kouemou-Patcheu [6] investigated the case $M(s) = a_0 + bs$ with $a(x) = 1$ in Ω and $f(u) = 0$. We fix $x^0 \in \mathbb{R}^N$ and we set

$$m(x) = x - x^0, \qquad R = \sup\{|m(x)|;\, x \in \Omega\}, \qquad \Gamma_0 = \{x \in \Gamma;\, m(x)\cdot\nu(x) > 0\}, \tag{1.2}$$

where $\nu(x)$ denotes the outward unit normal at $x \in \Gamma$. Let $a = a(x)$ be a smooth nonnegative function such that

$$a(x) \geq a_0 > 0, \quad \text{a.e. in } \omega, \tag{1.3}$$

Journal of Applied Mathematics 2005:3 (2005) 219–233
DOI: 10.1155/JAM.2005.219

where ω is a neighborhood of Γ_0 and a_0 is a positive constant. By neighborhood of Γ_0, we actually mean the intersection of Ω and a neighborhood of Γ_0.

The goal of this work is to obtain global existence and decay estimates of the strong solutions of the quasilinear wave equation (1.1) when M is not a constant function, the function a satisfies (1.3), g is a C^1, odd, increasing function, and $f(u) \sim -|u|^{\alpha}u$.

2. Preliminaries and main result

Throughout this paper, the functions considered are all real valued and the notations for their norm are adopted as usual (e.g., Lions [7]).

We consider the following general hypotheses.

(A.1) Assumptions on M:

$$M \in C^1([0,+\infty[), \quad M(s) \geq m_0 > 0, \quad \forall s \geq 0, \tag{2.1}$$

$$|M'(s)| \leq \beta s^{\gamma/2}, \quad \forall s \geq 0 \tag{2.2}$$

for some constants $\beta \geq 0$, $\gamma \geq 0$.

(A.2) Assumptions on a:

$$a \in C^2(\Omega) \cap C(\overline{\Omega}), \quad |\Delta a(x)| \leq a_1 a(x), \quad a_1 > 0. \tag{2.3}$$

(A.3) Assumptions on f:

f is a C^1-class function on $\mathbb{R}$ and satisfies

$$|f(u)| \leq h_0|u|^{\alpha+1}, \quad |f'(u)| \leq h_0|u|^{\alpha}, \quad \forall u \in \mathbb{R}, \tag{2.4}$$

with some constant $h_0 > 0$ and

$$0 < \alpha < \frac{2}{(N-4)^+}, \tag{2.5}$$

where $(N-4)^+ = \max\{N-4, 0\}$.

(A.4) g is a C^1 odd increasing function and

$$\begin{gathered} C_1|s| \leq |g(s)| \leq C_2|s|^q \quad \text{if } |s| \geq 1 \text{ with } 1 \leq q \leq \frac{2}{(N-4)^+}, \\ C_3|s|^{p+1} \leq g(s)s \quad \text{if } |s| < 1,\ 1 \leq p < +\infty, \end{gathered} \tag{2.6}$$

where C_i, $i = 1,2,3$, are positive constants.

We have the following fundamental inequalities.

Lemma 2.1 (Sobolev-Poincaré inequality). *Let α be a number with $0 \leq \alpha < \infty$ $(N = 1,2)$ or $0 \leq \alpha \leq 4/(N-2)$ $(N \geq 3)$, then there is a constant $C_* > 0$ such that*

$$|u|_{\alpha+2} \leq C_*|\nabla u|_2 \quad \text{for } u \in H_0^1(\Omega). \tag{2.7}$$

Lemma 2.2 (Gagliardo-Nirenberg inequality). *Let $1 \leq r < q \leq +\infty$ and $p \leq q$. Then, the inequality*

$$|u|_{W^{k,q}} \leq C|u|_{W^{m,p}}^{\theta}|u|_r^{1-\theta} \quad \text{for } u \in W^{m,p} \cap L^r \tag{2.8}$$

holds with some $C > 0$ *and*

$$\theta = \left(\frac{k}{N} + \frac{1}{r} - \frac{1}{q}\right)\left(\frac{m}{N} + \frac{1}{r} - \frac{1}{p}\right)^{-1} \tag{2.9}$$

provided that $0 < \theta \leq 1$ *(assume that* $0 < \theta < 1$ *if* $q = +\infty$*).*

Lemma 2.3. *Let* $E : [0, +\infty[\to [0, +\infty[$ *be a nonincreasing function and assume that there are two constants* $p \geq 1$ *and* $A > 0$ *such that*

$$\int_S^{+\infty} E^{(p+1)/2}(t)\,dt \leq AE(S), \quad 0 \leq S < +\infty. \tag{2.10}$$

Then,

$$E(t) \leq \begin{cases} CE(0)e^{-\lambda t}, & \forall t \geq 0 \text{ if } p = 1, \\ CE(0)(1+t)^{-2/(p-1)}, & \forall t \geq 0 \text{ if } p > 1, \end{cases} \tag{2.11}$$

where C *and* λ *are positive constants independent of the initial energy* $E(0)$.

We will construct a stable set in $H_0^1 \cap H^2$. For this, we define the functionals

$$\begin{aligned} J(u) &= \frac{1}{2}\widetilde{M}(|\nabla u|^2) + \int_\Omega F(u)\,dx, \quad \text{for } u \in H_0^1, \\ I(u) &= M(|\nabla u|^2)|\nabla u|^2 + \int_\Omega f(u) \cdot u\,dx, \quad \text{for } u \in H_0^1, \\ E(u,v) &= \frac{1}{2}|v|^2 + J(u), \quad \text{for } (u,v) \in H_0^1 \times L^2, \end{aligned} \tag{2.12}$$

where

$$\widetilde{M}(s) = \int_0^s M(\xi)\,d\xi, \qquad F(\lambda) = \int_0^\lambda f(s)\,ds. \tag{2.13}$$

Lemma 2.4. *Let* $0 < \alpha < 4/(N-4)^+$. *Then, for any* $K > 0$, *there exists a number* $\varepsilon_0 = \varepsilon_0(K)$ *such that if* $|\Delta u| \leq K$ *and* $|\nabla u| \leq \varepsilon_0$,

$$J(u) \geq \frac{m_0}{4}|\nabla u|^2, \qquad I(u) \geq \frac{m_0}{2}|\nabla u|^2. \tag{2.14}$$

Proof. By the Gagliardo-Nirenberg inequality, we deduce that

$$|u|_{\alpha+2}^{\alpha+2} \leq C|u|_{2N/(N-2)}^{(\alpha+2)(1-\theta)}|\Delta u|^{(\alpha+2)\theta} \leq C|\nabla u|^{(\alpha+2)(1-\theta)}|\Delta u|^{(\alpha+2)\theta} \tag{2.15}$$

with

$$\theta = \left(\frac{N-2}{2N} - \frac{1}{\alpha+2}\right)^{+}\left(\frac{2}{N} + \frac{N-2}{2N} - \frac{1}{2}\right)^{-1} = \left(\frac{(N-2)\alpha-4}{2(\alpha+2)}\right)^{+} \leq 1. \tag{2.16}$$

Here, we note that

$$(\alpha+2)(1-\theta)-2 = \begin{cases} \alpha & \text{if } 0<\alpha<\dfrac{4}{N-2} \\ & (0<\alpha<+\infty, \text{ for } N=1,2), \\ \dfrac{(4-N)\alpha+4}{2} & \text{if } \dfrac{4}{N-2}<\alpha<\dfrac{4}{N-4} \\ & \left(\dfrac{4}{N-2}<\alpha<+\infty,\ N=3,4\right). \end{cases} \tag{2.17}$$

Hence, if $|\Delta u| \leq K$, we get

$$\begin{aligned} J(u) &\geq \frac{m_0}{2}|\nabla u|^2 - \frac{h_0}{\alpha+2}|u|_{\alpha+2}^{\alpha+2} \\ &\geq \frac{m_0}{2}|\nabla u|^2 - Ch_0|\nabla u|^{(\alpha+2)(1-\theta)}|\Delta u|^{(\alpha+2)\theta} \\ &\geq \left\{\frac{m_0}{2} - Ch_0K^{(\alpha+2)\theta}|\nabla u|^{(\alpha+2)(1-\theta)-2}\right\}|\nabla u|^2. \end{aligned} \tag{2.18}$$

Using (2.17), we can define $\varepsilon_0 = \varepsilon_0(K)$ by

$$CK^{(\alpha+2)\theta}\varepsilon_0^{(\alpha+2)(1-\theta)-2} \leq \frac{m_0}{4}. \tag{2.19}$$

Thus, we obtain

$$J(u) \geq \frac{m_0}{4}|\nabla u|^2 \tag{2.20}$$

if $|\nabla u| \leq \varepsilon_0$. In a completely analogous way, we can get (2.14) for $I(u)$. □

We define our stable set W_K by

$$W_K = \left\{(u,v) \in (H_0^1 \cap H^2) \times H_0^1 : |\Delta u| < K, |\nabla v| < K, \sqrt{4m_0^{-1}E(u_0,v_0)} < \varepsilon_0\right\} \tag{2.21}$$

for $K > 0$.

Remark 2.5. If we consider $f(u)\cdot u \geq 0$, then we need not take $\varepsilon_0(K)$, and W_K is replaced by

$$\widetilde{W}_K = \{(u,v) \in (H_0^1 \cap H^2) \times H_0^1 : |\Delta u| < K, |\nabla v| < K\}. \tag{2.22}$$

3. Statement of the results

In this section, we will state our main theorem.

Theorem 3.1 (local existence). *Let initial data* $\{u_0, u_1\}$ *belong to* $(H_0^1 \cap H^2) \times H_0^1$ *and let the assumptions (A.1)–(A.4) be fulfilled. Then there exists a unique local solution u of (1.1) belonging to*

$$C_w^0([0,T[;H_0^1 \cap H^2) \cap C_w^1([0,T[;H_0^1) \cap C^0([0,T[;H_0^1) \cap C^1([0,T[;L^2(\Omega)) \tag{3.1}$$

for some $T = T(|\Delta u_0|, |\nabla u_1|) > 0$.

Moreover, at least one of the following statements is valid:

(i) $T = +\infty$,

(ii) $|\nabla u'(t)|^2 + |\Delta u(t)|^2 \to \infty$ *as* $t \to T^-$,

(iii) $M(|\nabla u(t)|^2) \to 0$ *as* $t \to T^-$.

The proof of this theorem is well known.

Theorem 3.2 (global existence and decay property). *Suppose (A.1)–(A.4) hold. Then there exists an open set* S_0 *in* $(H_0^1 \cap H^2) \times H_0^1$, *which contains* $(0,0)$ *such that if* $(u_0, u_1) \in S_0$, *the problem (1.1) admits a unique global solution* $u(t)$ *on the class*

$$L^\infty([0,+\infty[;H_0^1 \cap H^2) \cap W^{1,\infty}([0,+\infty[;H_0^1) \cap W^{2,\infty}([0,+\infty[;L^2). \tag{3.2}$$

Moreover, the energy determined by the solution u has the decay states

$$\begin{aligned} E(u(t),u'(t)) &\le C_0 e^{-\lambda t} \quad \text{if } p = 1, \\ E(u(t),u'(t)) &\le \widetilde{C}_0 (1+t)^{-2/(p-1)} \quad \text{if } p > 1, \end{aligned} \tag{3.3}$$

where C_0, $\widetilde{C}_0$, *and* λ *are certain positive constants depending on* $|\nabla u_0|$, $|u_1|$, *and other quantities.*

Proof. We divide the proof into several lemmas. For the moment, we denote $E(u(t),u'(t))$ by $E(t)$.

Lemma 3.3. *Let* $u(t)$ *be a local solution to the problem (1.1) on* $[0,T[$, $T > 0$. *Then*

$$\forall 0 \le S \le T < +\infty, \quad E(S) - E(T) = \int_S^T \int_\Omega a(x) u' g(u') \, dx \, dt. \tag{3.4}$$

Multiplying the equation in (1.1) by $u'(t)$ and integrating on $[S,T[$, we get

$$\begin{aligned} -\int_S^T \int_\Omega a(x) u' g(u') \, dx \, dt &= \left[\frac{1}{2} |u'(t)|^2 + \frac{1}{2} \widetilde{M}\left(|\nabla u(t)|^2 \right) + F(u(t)) \right]_S^T \\ &= E(T) - E(S). \end{aligned} \tag{3.5}$$

It is easy to see the identity

$$E'(t) = -\int_\Omega a u' g(u') \, dx \le 0. \tag{3.6}$$

In particular, $E(t)$ is nonincreasing and

$$E(t) \leq E(0) \tag{3.7}$$

as long as the local solutions exist. □

Lemma 3.4. *Let $u(t)$ be a local solution to the problem (1.1) satisfying $(u(t),u'(t)) \in W_K$ on $[0,T[$ for some $K>0$. Then,*

$$E(t) \leq \begin{cases} CI_0 e^{-\lambda t}, & \text{on } [0,T[\quad \text{if } p=1, \\ \widehat{q}(1+t)^{-2/(p-1)}, & \phantom{\text{on } [0,T[} \quad \text{if } p>1, \end{cases} \tag{3.8}$$

where $I_0^2 = E(0)$, $\lambda = \lambda(K,I_0)$, and $\widehat{q} = \widehat{q}(K,I_0)$ denote certain positive constants continuously depending on K and I_0.

The proof of this lemma is based on the following identities given by the multiplier method. We omit to write the differential elements in the integrals, in order to simplify the expressions.

Lemma 3.5. *Let $q \in [W^{1,\infty}(\Omega)]^N$, $\beta \in \mathbb{R}$, and $\xi \in W^{1,\infty}(\Omega)$. Then*

$$\begin{aligned} &\int_S^T \int_\Gamma M\left(|\nabla u(t)|^2\right) q\cdot\nu \left|\frac{\partial u}{\partial \nu}\right|^2 E^\sigma \\ &\quad = (u', 2q\cdot\nabla u + \beta u)E^\sigma \Big|_S^T + \int_S^T \int_\Omega (\operatorname{div}(q)-\beta)\left[|u'|^2 - M\left(|\nabla u(t)|^2\right)|\nabla u(t)|^2 E^\sigma\right] \\ &\qquad + 2\int_S^T \int_\Omega M\left(|\nabla u(t)|^2\right)\frac{\partial q_k}{\partial x_i}\frac{\partial u}{\partial x_k}\frac{\partial u}{\partial x_i}E^\sigma - \sigma\int_S^T \int_\Omega u'(2q\cdot\nabla u + \beta u)E^{\sigma-1}E' \\ &\qquad + \int_S^T \int_\Omega ag(u')(2q\cdot\nabla u + \beta u)E^\sigma - \int_S^T \int_\Omega \operatorname{div}(q)F(u)E^\sigma + \beta\int_S^T \int_\Omega f(u)\cdot uE^\sigma, \end{aligned} \tag{3.9}$$

$$\begin{aligned} &(u', \xi u)E^\sigma\big|_S^T + \int_S^T \int_\Omega \xi\left[M(|\nabla u|^2)|\nabla u|^2 - |u'|^2\right]E^\sigma \\ &\quad - \sigma\int_S^T \int_\Omega u'u\xi E^{\sigma-1}E' + \int_S^T \int_\Omega M(|\nabla u(t)|^2)(\nabla u, u\nabla\xi)E^\sigma \\ &\quad + \int_S^T \int_\Omega ag(u')\xi uE^\sigma + \int_S^T \int_\Omega \xi f(u)uE^\sigma = 0. \end{aligned} \tag{3.10}$$

For the proof, see Lions [8] or Komornik [5].

Proof of Lemma 3.5. We proceed in several steps.
Step 1. Applying (3.9) with $q(x) = m(x)$, observing that $\operatorname{div} q = N$, we obtain

$$\begin{aligned}
&(u', 2m\cdot\nabla u+\beta u)E^{\sigma}|_S^T + (N-\beta)\int_S^T\int_\Omega |u'|^2E^{\sigma} \\
&\quad + (\beta - N + 2)\int_S^T\int_\Omega M(|\nabla u|^2)|\nabla u|^2E^{\sigma} - \sigma\int_S^T\int_\Omega u'(2m\cdot\nabla u+\beta u)E^{\sigma-1}E' \\
&\quad + \int_S^T\int_\Omega ag(u')(2m\cdot\nabla u+\beta u)E^{\sigma} + \beta\int_S^T\int_\Omega f(u)\cdot uE^{\sigma} - N\int_S^T\int_\Omega F(u)E^{\sigma} \\
&= \int_S^T\int_\Gamma M(|\nabla u|^2)\left|\frac{\partial u}{\partial\nu}\right|^2E^{\sigma} \le RM_0\int_S^T\int_{\Gamma_0}\left|\frac{\partial u}{\partial\nu}\right|^2E^{\sigma},
\end{aligned} \tag{3.11}$$

where

$$m_0 \le M(|\nabla u|^2) \le \max\left\{M(s),\ 0\le s\le \frac{4E(0)}{m_0}\right\} \equiv M_0. \tag{3.12}$$

Throughout the remaining part of this work, positive constants will be denoted by C and will change line to line. Here, we observe that under the assumption $(u(t),u'(t)) \in W_K$, the functionals $E(t)$, $e(t) = (1/2)(|u'(t)|^2 + |\nabla u(t)|^2)$ and $|u'(t)|^2 + I(u(t))$ are all equivalent, by Lemma 2.4.

We take $\beta \in]N-2,N[$ and $\theta_0 = \min\{2(N-\beta),\beta-N+2\}$, we deduce that

$$\begin{aligned}
\theta_0\int_S^T E^{\sigma+1} \le RM_0&\int_S^T\int_{\Gamma_0}\left|\frac{\partial u}{\partial\nu}\right|^2E^{\sigma} + \left|(u',2m\cdot\nabla u+\beta u)|_S^T\right| \\
&+\left|\int_S^T\int_\Omega ag(u')(2m\cdot\nabla u+\beta u)E^{\sigma}\right| \\
&+\sigma\left|\int_S^T\int_\Omega u'(2m\cdot\nabla u+\beta u)E^{\sigma-1}E'\right| \\
&+\beta\left|\int_S^T\int_\Omega f(u)\cdot uE^{\sigma}\right| + N\left|\int_S^T\int_\Omega F(u)E^{\sigma}\right|.
\end{aligned} \tag{3.13}$$

Since the energy is nonincreasing, using the result of Komornik [5], we find that

$$\left|(u',2m\cdot\nabla u+\beta u)|_S^T E^{\sigma}\right| \le CE(S), \tag{3.14}$$

$$\left|\sigma\int_S^T\int_\Omega u'(2m\cdot\nabla u+\beta u)E^{\sigma-1}E'\right| \le CE(S). \tag{3.15}$$

By the Hölder inequality, we have

$$\begin{aligned}\left|\int_S^T\int_{\Omega_1} ag(u')(2m\cdot\nabla u+\beta u)E^\sigma\right| &\le \int_S^T E^\sigma\left(\int_{\Omega_1} a^2g^2(u')\right)^{1/2}\left(\int_{\Omega_1}|2m\cdot\nabla u+\beta u|\right)^{1/2}\\ &\le \int_S^T E^\sigma\left(\int_{\Omega_1} a^2\left(u'g(u')\right)^{2/(p+1)}\right)^{1/2}E^{1/2}\\ &\le C\int_S^T E^{\sigma+1/2}|E'|^{1/(p+1)},\end{aligned}\tag{3.16}$$

$$\begin{aligned}\left|\int_S^T\int_{\Omega_2} ag(u')(2m\cdot\nabla u+\beta u)E^\sigma\right| &\le C\int_S^T E^\sigma\left\{\int_{\Omega_2}|ag(u')||\nabla u|+\int_{\Omega_2}|ag(u')||u|\right\}\\ &\le C\int_S^T E^\sigma|ag(u')|_{1+q^{-1}}\{|\nabla u|_{q+1}+|\nabla u|\}.\end{aligned}\tag{3.17}$$

We observe here, from Lemma 2.2, that

$$|\nabla u|_{q+1}\le C|\nabla u|^{1/(q+1)}|\Delta u|^{q/(q+1)}\le CK^{q/(q+1)}E^{1/2(q+1)},\tag{3.18}$$

$$\begin{aligned}|ag(u')|_{1+q^{-1}} &= \left[\int_{\Omega_2}|ag(u')|^{(q+1)/q}\right]^{q/(q+1)}\\ &\le C\left[\int_{\Omega_2} au'g(u')\right]^{q/(q+1)}\le C|E'|^{q/(q+1)}.\end{aligned}\tag{3.19}$$

From (3.17), (3.18), and (3.19), we have

$$\left|\int_S^T\int_{\Omega_2} ag(u')(2m\cdot\nabla u+\beta u)E^\sigma\right|\le C\int_S^T|E|^{\sigma+1/2(q+1)}|E'|^{q/(q+1)},\tag{3.20}$$

where we set for each $t\ge 0$,

$$\Omega_1=\Omega_1(t)=\{x\in\Omega:|u'(x,t)|\le 1\},\quad \Omega_2=\Omega\setminus\Omega_1.\tag{3.21}$$

Thus, from (3.16) and (3.20), we get

$$\left|\int_S^T\int_{\Omega} ag(u')(2m\cdot\nabla u+\beta u)E^\sigma\right|\le C\int_S^T\left[E^{\sigma+1/2}|E'|^{1/(p+1)}+|E|^{\sigma+1/2(q+1)}|E'|^{q/(q+1)}\right].\tag{3.22}$$

Now, using the Young inequality, we obtain

$$\left|\int_S^T\int_{\Omega} ag(u')(2m\cdot\nabla u+\beta u)E^\sigma\right|\le \varepsilon\int_S^T E^{\sigma+1}+E(S),\quad \varepsilon>0.\tag{3.23}$$

It follows from (3.14), (3.15), and (3.23) that

$$\frac{\theta_0}{2}\int_S^T E^{\sigma+1} \le CE(S) + \int_S^T\int_\Omega \Big(\frac{\beta^2}{\theta_0}\,|f(u)|^2 + NF(u)\Big)E^\sigma + RM_0\int_S^T\int_{\Gamma_0}\Big|\frac{\partial u}{\partial \nu}\Big|^2 E^\sigma. \quad (3.24)$$

To estimate the last term in (3.24), we utilize (3.10) with $\xi = \eta$, where $\eta \in W^{1,\infty}(\Omega)$ is a function (constructed by Zuazua in [10]) which satisfies

$$\begin{gathered} 0 \le \eta \le 1, \\ \eta = 1, \quad \text{in } \widehat{\omega}, \qquad \frac{|\nabla \eta|^2}{\eta} \in L^\infty(\omega), \\ \eta = 0, \quad \text{in } \Omega\setminus\omega, \end{gathered} \quad (3.25)$$

and $\widehat{\omega}$ is an open set in Ω, with $\Gamma_0 \subseteq \widehat{\omega} \subsetneq \omega$.

First, we have from (3.10)

$$\begin{aligned} \int_S^T\int_\Omega \eta M(|\nabla u|^2)|\nabla u|^2E^\sigma = (-u',\eta u)E^\sigma|_S^T &- \int_S^T\int_\Omega a g(u')u\eta E^\sigma \\ &- \int_S^T\int_\Omega M(|\nabla u|^2)\nabla u\cdot u\nabla\eta E^\sigma \\ &+ \int_S^T\int_\Omega \eta|u'|^2E^\sigma - \int_S^T\int_\Omega \eta f(u)uE^\sigma \\ &+ \sigma\int_S^T\int_\Omega u'u\xi E^{\sigma-1}E'. \end{aligned} \quad (3.26)$$

Simple calculations, using the Young inequalities, show that

$$\Big| -(u',\eta u)E^\sigma|_S^T + \sigma\int_S^T\int_\Omega u'u\xi E^{\sigma-1}E'\Big| \le CE(S), \quad (3.27)$$

$$\Big|\int_S^T (ag(u'),\eta u)E^\sigma\Big| \le CE(S) + \frac{\varepsilon}{2}\int_S^T E^{\sigma+1}, \quad \varepsilon > 0, \quad (3.28)$$

$$\Big| -\int_S^T\int_\Omega M(|\nabla u|^2)\nabla u\cdot u\nabla\eta\, E^\sigma\Big| \le C\int_S^T\int_\omega |u|^2E^\sigma + \frac{1}{2}\int_S^T\int_\Omega \eta M(|\nabla u|^2)|\nabla u|^2E^\sigma. \quad (3.29)$$

From (3.26)–(3.29), we obtain

$$\begin{aligned} \frac{1}{2}\int_S^T\int_\Omega \eta M(|\nabla u|^2)|\nabla u|^2E^\sigma \le C\Big[E(S) &+ \int_S^T\int_\omega (|u'|^2 + |u|^2E^\sigma)\Big] \\ &+ \frac{C_*}{m_0^2}\int_S^T\int_\Omega |f(u)|^2E^\sigma + \varepsilon\int_S^T E^{\sigma+1}. \end{aligned} \quad (3.30)$$

Step 2. We take a vector field $h \in [W^{1,\infty}(\Omega)]^N$ such that

$$h = \nu, \quad \text{on } \Gamma_0, \qquad h\cdot\nu \ge 0, \quad \text{on } \Gamma, \qquad h = 0, \quad \text{on } \Omega\setminus\widehat{\omega}. \quad (3.31)$$

Choosing $\beta = 0$ and $q = h$ in (3.9), we get

$$\left| \int_S^T \int_\Omega a g(u')h \cdot |\nabla u| E^\sigma \right| \le C \int_S^T \left[E^{\sigma+1/2} |E'|^{1/(p+1)} + E^{\sigma+1/2(q+1)} |E'|^{q/(q+1)} \right],$$

$$m_0 \int_S^T \int_{\Gamma_0} \left| \frac{\partial u}{\partial \nu} \right|^2 E^\sigma \le CE(S) + 3\alpha_0 \int_S^T \int_{\hat{\omega}} \left(|u'|^2 + M(|\nabla u|^2) |\nabla u|^2 \right) E^\sigma + \alpha_0 \int_S^T \int_\Omega |F(u)| E^\sigma + \varepsilon' \int_S^T E^{\sigma+1}, \quad \varepsilon' > 0, \tag{3.32}$$

where $\sum_{i,j=1}^{N} |\partial h_j / \partial x_i| \le \alpha_0$, for all $x \in \overline{\Omega}$.

Combining (3.30) and (3.32), we have

$$m_0 \int_S^T \int_{\Gamma_0} \left| \frac{\partial u}{\partial \nu} \right|^2 E^\sigma \le C \left[E(S) + \int_S^T \int_\omega \left(|u'|^2 + |u|^2 \right) E^\sigma \right] + \frac{\sigma C_*^2 \alpha_0}{m_0^2} \int_S^T \int_\Omega |f(u)|^2 E^\sigma + \alpha_0 \int_S^T \int_\Omega |F(u)| E^\sigma + \varepsilon \int_S^T E^{\sigma+1}. \tag{3.33}$$

We conclude from (3.24) and (3.33) that

$$\frac{\theta_0}{2} \int_S^T E^{\sigma+1} \le \left(\frac{\sigma C_*^2 \alpha_0 R M_0}{m_0^3} + \frac{\beta^2}{\theta_0} \right) \int_S^T \int_\Omega |f(u)|^2 + \left(N + \frac{\alpha_0 R M_0}{m_0} \right) \int_S^T \int_\Omega |F(u)| + C \left[E(S) + \int_S^T \int_\omega \left(|u'|^2 + |u|^2 \right) \right]. \tag{3.34}$$

Now, in order to absorb the last term into the right-hand side of (3.34), we adapt a method introduced in Conrad and Rao [4]. To this end, we consider $z(t) \in H_0^1(\Omega)$, solution of

$$-\Delta z = \chi(\omega) u, \quad \text{in } \Omega, \qquad z = 0, \quad \text{on } \Gamma, \tag{3.35}$$

where $\chi(\omega)$ is the characteristic function of ω. It is easy to verify that z' is solution of the problem

$$-\Delta z' = \chi(\omega) u', \quad \text{in } \Omega, \qquad z' = 0, \quad \text{on } \Gamma. \tag{3.36}$$

A simple computation gives

$$|z| \le C |u|_{L^2(\omega)}, \qquad |z'| \le C |u'|_{L^2(\omega)}, \qquad (\nabla z, \nabla u) = |u|_{L^2(\omega)}^2. \tag{3.37}$$

Next, we "multiply" the equation in (1.1) by zE^σ, integrate by parts on $\Omega\times]S,T[$, and use (3.37). Thus, we find

$$\int_S^T\int_\omega M(|\nabla u|^2)|\nabla u|^2E^\sigma = -(u',z)E^\sigma|_S^T + \int_S^T (u',z')E^\sigma + \sigma\int_S^T E^{\sigma-1}E'(u',z') - \int_S^T (ag(u'),z)E^\sigma - \int_S^T (f(u),z)E^\sigma. \tag{3.38}$$

Here, we note that

$$\left| -(u',z)E^\sigma|_S^T + \sigma\int_S^T E^{\sigma-1}E'(u',z')\right| \leq CE(S), \tag{3.39}$$

$$\left|\int_S^T (u',z)E^\sigma\right| \leq C\int_S^T\int_\omega |u'|^2E^\sigma + \varepsilon\int_S^T E^{\sigma+1}, \quad \varepsilon>0, \tag{3.40}$$

$$\left|\int_S^T (ag(u'),z)E^\sigma\right| \leq CE(S) + \varepsilon'\int_S^T E^{\sigma+1}, \quad \varepsilon'>0, \tag{3.41}$$

$$\left| -\int_S^T (f(u),z)E^\sigma\right| \leq \frac{1}{2m_0}\int_S^T\int_\Omega |f(u)|^2E^\sigma + \frac{m_0}{2}\int_S^T\int_\omega |u|^2E^\sigma. \tag{3.42}$$

Using (3.39)–(3.41), we have in (3.38)

$$\int_S^T\int_\omega |u|^2E^\sigma \leq C\left(E(S) + \int_S^T\int_\omega |u'|^2E^\sigma\right) + \frac{1}{m_0^2}\int_S^T\int_\Omega |f(u)|^2E^\sigma + \varepsilon\int_S^T E^{\sigma+1}, \quad \varepsilon>0. \tag{3.43}$$

Then inserting (3.43) into (3.34) gives

$$\int_S^T E^{\sigma+1} \leq C\left(E(S) + \int_S^T\int_\omega |u'|^2E^\sigma\right) + \delta_0\int_S^T\int_\Omega |f(u)|^2E^\sigma + \delta_1\int_S^T |F(u)|E^\sigma, \tag{3.44}$$

where

$$\delta_0 = \frac{8}{\theta_0}\left[\frac{\sigma C_*\alpha_0RM_0}{m_0^3} + \frac{\beta^2}{\theta_0} + \frac{1}{m_0^2}\right], \qquad \delta_1 = \frac{8}{\theta_0}\left[N + \frac{\alpha_0RM_0}{m_0}\right]. \tag{3.45}$$

Now, we observe that

$$\delta_0\int_\Omega |f(u)|^2 \leq \delta_0h_0|u|_{2(\alpha+1)}^{2(\alpha+1)} \leq C\delta_0h_0|u|_{2N/(N-2)}^{2(1-\theta_1)(\alpha+1)}|\Delta u|^{2\theta_1(\alpha+1)} \leq C\delta_0h_0\varepsilon_0^{2(1-\theta_1)(\alpha+1)-2}K^{2\theta_1(\alpha+1)}|\nabla u|^2 \tag{3.46}$$

if $N\geq 3$ and $\alpha>2/(N-2)$, with $\theta_1 = ((\alpha+1)(N-2)-N)/2(\alpha+1)$.

Further, we have that since $\alpha\leq 2/(N-4)^+$,

$$(1-\theta_1)(\alpha+1)\geq 1. \tag{3.47}$$

Thus,

$$\delta_0h_0|u|_{2(\alpha+1)}^{2(\alpha+1)} \leq C\delta_0h_0\varepsilon_0^{2(1-\theta_1)(\alpha+1)-2}K^{2\theta_1(\alpha+1)}|\nabla u|^2. \tag{3.48}$$

When $N \le 2$ or $\alpha \le 2/(N-2)^+$, we see that (3.48) holds with $\theta_1 = 0$. Thus, under a little more stronger assumption than (2.19),

$$Ch_0\left(\varepsilon_0^{(\alpha+2)(1-\theta_1)-2}K^{(\alpha+2)\theta} + \varepsilon_0^{2(1-\theta_1)(\alpha+1)-2}K^{2\theta_1(\alpha+1)}\right) \le \frac{m_0}{4}, \tag{3.49}$$

we obtain from (3.44) that

$$\int_S^T E^{\sigma+1}dt \le C\left(E(S) + \int_S^T \int_\omega |u'|^2 E^\sigma\right). \tag{3.50}$$

In order to absorb the second term in (3.50), we consider two cases.

(i) The case $p = 1$. We take $\sigma = 0$, hence by (2.6),

$$\int_S^T E \le CE(S) + \frac{C}{a_0}\int_S^T \int_\omega a|u'|^2 \le CE(S) + \int_S^T \int_\omega a|g(u')|u' \le CE(S). \tag{3.51}$$

Applying Lemma 2.3, we obtain

$$E(t) \le CI_0 e^{-\lambda t}, \quad \lambda = \lambda(K, I_0). \tag{3.52}$$

(ii) When $p > 1$, we take $\sigma = (p-1)/2$. It follows from (2.6) that

$$\begin{aligned}\int_S^T E^{(p+1)/2} &\le CE(S) + \frac{C}{a_0}\int_S^T \int_\omega a|u'|^2 E^{(p-1)/2} \\ &\le CE(S) + C\int_S^T \int_\Omega a|u'|^2 E^{(p-1)/2} \\ &\le C\left[E(S) + \int_S^T E^{(p-1)/2}\left(\int_{\Omega_1} a|u'|^2 + \int_{\Omega_2} a|u'|^2\right)\right] \\ &\le C\left[E(S) + \int_S^T \left(E^{(p-1)/2}|E'|^{2/(p+1)} + E^{(p-1)/2}|E'|\right)\right] \\ &\le C[E(S) + E^{(p+1)/2}(S)] + \varepsilon\int_S^T E^{(p+1)/2}, \quad \varepsilon > 0.\end{aligned} \tag{3.53}$$

Hence

$$\int_S^T E^{(p+1)/2} \le CE(S). \tag{3.54}$$

Using Lemma 2.3 again, we conclude that

$$E(t) \le q(K, I_0)(1+t)^{-2/(p-1)}, \quad t \ge 0, \tag{3.55}$$

as long as the local solutions exist, where $q(K, I_0)$ denotes a certain constant continuously depending on K and I_0. □

We are now in a position to obtain H^2 a priori bounds. Set

$$E_1(t) = M\left(|\nabla u(t)|^2\right)|\Delta u(t)|^2 + |\nabla u'(t)|^2. \tag{3.56}$$

Then we have the following estimate, which is the heart of this paper.

LEMMA 3.6. *Assume that $u(t)$ is a solution satisfying $(u(t),u'(t)) \in W_K$ on $[0,T[$, for some $K > 0$. Then,*

$$|\nabla u'(t)|^2 + |\Delta u(t)|^2 \le Q^2(I_0,I_1,K), \tag{3.57}$$

with

$$\lim_{I_0\to 0} Q^2(I_0,I_1,K) = \frac{I_1^2}{\min\{1,m_0\}}, \tag{3.58}$$

and where

$$I_1^2 = |\nabla u_1|^2 + M\Big(|\nabla u_0|^2\Big)|\Delta u_0|^2. \tag{3.59}$$

Proof. For $E_1(t)$ with respect to t, we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}E_1(t) + (ag'(u')\nabla u'(t),\nabla u'(t)) \\
&\quad = -(f'(u)\nabla u,\nabla u') + M'\Big(|\nabla u(t)|^2\Big)(\nabla u'(t),\nabla u(t))\,|\Delta u(t)|^2 - (g(u')\nabla a,\nabla u').
\end{aligned} \tag{3.60}$$

Using the assumptions on a, g, and f, it follows from (3.60) that

$$\begin{aligned}
\frac{d}{dt}E_1(t) &\le C\Big\{\int_\Omega (|u|^\alpha|\nabla u||\nabla u'|) + |\nabla u|_2^{\gamma+1}|\nabla u'|_2|\Delta u|_2^2 + \sum_{i=1}^n \int_\Omega \frac{\partial a}{\partial x_i}\frac{\partial}{\partial x_i}\int_0^{u'} g(\xi)d\xi\Big\} \\
&\le C\Big\{\int_\Omega (|u|^\alpha|\nabla u||\nabla u'|) + |\nabla u|_2^{\gamma+1}|\nabla u'|_2|\Delta u|_2^2 + \int_\Omega \Big|\int_0^{u'} g\Big|\,|\Delta a|\Big\} \\
&\le C\Big\{\Big(\int_\Omega |u|^{2\alpha}|\nabla u|^2\Big)^{1/2}\Big(\int_\Omega |\nabla u'|^2\Big)^{1/2} + E(t)^{(\gamma+1)/2}K^3 + \int_\Omega ag(u')u'\Big\}.
\end{aligned} \tag{3.61}$$

Further, we observe that

$$\begin{aligned}
\Big(\int_\Omega |u|^{2\alpha}|\nabla u|^2\Big)^{1/2} &\le |u|_{N\alpha}^\alpha|\nabla u|_{2N/(N-2)} \\
&\le C|u|_{2N/(N-2)}^{\alpha(1-\theta_0)}|\Delta u|_2^{\alpha\theta_0}|\Delta u|_2 \quad \text{(by Lemma 2.2)} \\
&\le C|\nabla u|_2^{\alpha(1-\theta_0)}|\Delta u|_2^{\alpha\theta_0+1} \\
&\le CE(t)^{\alpha(1-\theta_0)/2}K^{\alpha\theta_0+1},
\end{aligned} \tag{3.62}$$

with

$$\theta_0 = \Big(\frac{N-2}{2} - \frac{1}{\alpha}\Big)^+ = \frac{((N-2)\alpha-2)^+}{2\alpha} \le 1. \tag{3.63}$$

Then, it follows from (3.61) and (3.62) that

$$\frac{d}{dt}E_1(t) \le C\{E(t)^{\alpha(1-\theta_0)/2}K^{\alpha\theta_0+2} + E(t)^{(\gamma+1)/2}K^3 - E'(t)\}. \tag{3.64}$$

Thus, integrating (3.64) over $[0,t[$, we obtain $\int_\Omega |\int_0^{u'} g||\Delta a|$,

$$
\begin{aligned}
&|\Delta u(t)|^2 + |\nabla u'(t)|^2 \\
&\quad \le \frac{1}{\min\{1,m_0\}}\left\{I_1^2 + CK^{\alpha\theta_0+2}\int_0^{+\infty} E(t)^{\alpha(1-\theta_0)/2}dt + CK^3\int_0^{+\infty} E(t)^{(\gamma+1)/2}dt + CI_0^2\right\} \\
&\quad \le \frac{1}{\min\{1,m_0\}}\left\{I_1^2 + CI_0^2 + CI_0^{(\gamma+1)/2}K^3 + CI_0^{\alpha(1-\theta_0)/2}K^{\alpha\theta_0+2}\right\} \\
&\quad \equiv Q^2(I_0,I_1,K), \quad \text{on } [0,T[.
\end{aligned}
\tag{3.65}
$$

Define

$$
\begin{gathered}
S_K = \{(u_0,u_1) \in (H_0^1 \cap H^2) \times H_0^1 : Q^2(I_0,I_1,K) < K\}, \\
S_0 = \bigcup_{K>0} S_K.
\end{gathered}
\tag{3.66}
$$

Since

$$\lim_{I_0 \to 0} Q^2(I_0,I_1,K) = \frac{I_1^2}{\min\{1,m_0\}}, \tag{3.67}$$

the S_K is not empty if $I_1 < \min\{1,m_0\}K^2$ and I_0 is sufficiently small.

If $(u_0,u_1) \in S_K$ for some $K > 0$, then the corresponding local solution $u(t)$ exists on some interval $[0,T[$ and satisfies

$$E_1(u(t)) \le \min\{1,m_0\}K^2 \quad \text{on } 0 \le t < T. \tag{3.68}$$

From (3.43), we have $(u(t),u'(t)) \in W_K$.

Hence, it follows from Lemma 3.6 that

$$|\nabla u'(t)|^2 + |\Delta u(t)|^2 \le Q^2(I_0,I_1,K) < K^2, \quad \text{on } [0,T[. \tag{3.69}$$

Next, we affirm that

$$(u(t),u'(t)) \in S_K, \quad \text{on } [0,T[. \tag{3.70}$$

In fact, suppose that there is a number $t^* \in [0,T[$ such that $(u(t),u'(t)) \in S_K$ on $[0,T[$ and $(u(t^*),u'(t^*)) \notin S_K$, then it follows that

$$Q^2(E(u(t^*)),E_1(u(t^*)),K) \ge K^2. \tag{3.71}$$

Note that $Q^2(I_0,I_1,K)$ is the increasing function with respect to I_0 and I_1. Hence, we see from (3.48), the energy identity (3.4), and Lemma 3.6 that

$$I_1^2 + CI_0^2 + C_1(I_0)K^3 + C_2(I_0)K^{\alpha\theta_0+2} \ge \min\{1,m_0\}K^2. \tag{3.72}$$

But, then, since

$$\lim_{I_0 \to 0} Q^2(I_0,I_1,K) = \frac{I_1^2}{\min\{1,m_0\}}, \tag{3.73}$$

we may take I_1 so that

$$\frac{2I_1^2}{\min\{1, m_0\}} < K^2 \tag{3.74}$$

with sufficiently small I_0. This contradicts (3.72).

Now, since we can repeat the continuation procedure indefinitely, we conclude that $u(t)$ can be continued globally on $[0, +\infty[$ and $(u(t), u'(t)) \in S_0$, for all $t \geq 0$.

The uniqueness follows from a similar argument as in [1]. □

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