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Research Article

Generalized $S(C, A, B)$-Pairs for Uncertain Linear Infinite-Dimensional Systems

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We introduce the concept of generalized $S(C, A, B)$-pairs which is related to generalized $S(A, B)$-invariant subspaces and generalized $S(C, A)$-invariant subspaces for infinite-dimensional systems. As an application the parameter-insensitive disturbance-rejection problem with dynamic compensator is formulated and its solvability conditions are presented. Further, an illustrative example is also examined.

1. Introduction

In the framework of the so-called geometric approach, many control problems with state feedback and/or incomplete-state feedback (e.g., controllability and observability problems, decoupling problems, and disturbance-rejection problems, etc.) have been studied for finite-dimensional systems (see, e.g., [1, 2]). Further, the concept of (C, A, B)-pairs was first introduced by Schumacher [3], and this concept has been used successfully to design dynamic compensators. After that Curtain extended the geometric concepts to infinite-dimensional systems and various control problems have been studied (see, e.g., [4–11]). On the other hand, from the practical viewpoint, Ghosh [12] and Otsuka [13] studied the concepts of simultaneous (C, A, B)-pairs and of generalized (C, A, B)-pairs, respectively, for finite-dimensional systems, and the parameter-insensitive disturbance-rejection problems for uncertain linear systems were studied. Then, Otsuka and Inaba [14–16] extended the concepts of simultaneous invariant subspaces and simultaneous (C, A, B)-pairs to infinite-dimensional systems. Further, Otsuka and Hinata [17] studied the concept of generalized invariant subspaces for infinite-dimensional systems.

The objective of this paper is to investigate the concept of generalized $S(C, A, B)$-pairs for infinite-dimensional systems and to study the parameter-insensitive disturbance-rejection problem with dynamic compensator.

The paper is organized as follows. Section 2 gives the concept of generalized $S(C, A, B)$-pairs and its properties. In Section 3, the parameter-insensitive disturbance-rejection problem with dynamic compensator is formulated and its solvability conditions are presented. Section 4 gives an example to illustrate our results. Finally, some concluding remarks are given in Section 5.

2. Generalized $S(C, A, B)$-pairs

First, we give some notations used throughout this investigation. Let $\mathbf{B}(\mathcal{X}; \mathcal{Y})$ denote the set of all bounded linear operators from a Hilbert space $\mathcal{X}$ into another Hilbert space $\mathcal{Y}$; for notational simplicity, we write $\mathbf{B}(\mathcal{X})$ for $\mathbf{B}(\mathcal{X}; \mathcal{X})$. For a linear operator A the domain, the image, the kernel, and the C_0-semigroup generated by A are denoted by $\mathrm{D}(A)$, $\mathrm{Im}A$, $\mathrm{Ker}\, A$, and $\{S_A(t); t \geq 0\}$, respectively. Further, the dimension and the orthogonal complement of a closed subspace $\mathcal{V}$ are denoted by $\dim(\mathcal{V})$ and $(\mathcal{V})^{\perp}$, respectively.

Next, consider the following linear systems defined in a Hilbert space $\mathcal{X}$:

$$S(\alpha, \beta, \gamma) : \begin{cases} \dfrac{d}{dt} x(t) = A(\alpha) x(t) + B(\beta) u(t), \\ y(t) = C(\gamma) x(t), \end{cases} \tag{2.1}$$

where $x(t) \in \mathcal{X}$, $u(t) \in \mathcal{U} := \mathbf{R}^m$, $y(t) \in \mathcal{Y} := \mathbf{R}^{\ell}$ are the state, the input, and the measurement output, respectively. Operators $A(\alpha)$, $B(\beta)$, and $C(\gamma)$ are unknown in the sense that they are represented as the forms:

$$\begin{aligned} A(\alpha) &= A_0 + \alpha_1 A_1 + \cdots + \alpha_p A_p := A_0 + \Delta A(\alpha), \\ B(\beta) &= B_0 + \beta_1 B_1 + \cdots + \beta_q B_q := B_0 + \Delta B(\beta), \\ C(\gamma) &= C_0 + \gamma_1 C_1 + \cdots + \gamma_r C_r := C_0 + \Delta C(\gamma), \end{aligned} \tag{2.2}$$

where $\alpha := (\alpha_1, \ldots, \alpha_p) \in \mathbf{R}^p$, $\beta := (\beta_1, \ldots, \beta_q) \in \mathbf{R}^q$, $\gamma := (\gamma_1, \ldots, \gamma_r) \in \mathbf{R}^r$, A_0 is the infinitesimal generator of a C_0-semigroup $\{S_{A_0}(t); t \geq 0\}$ on $\mathcal{X}$, $A_i \in \mathbf{B}(\mathcal{X})$ $(i = 1, \ldots, p)$, $B_i \in \mathbf{B}(\mathbf{R}^m; \mathcal{X})$ $(i = 0, \ldots, q)$, and $C_i \in \mathbf{B}(\mathcal{X}; \mathbf{R}^{\ell})$ $(i = 0, \ldots, r)$. Here, in the system $S(\alpha, \beta, \gamma)$ (A_0, B_0, C_0) and $(\Delta A(\alpha), \Delta B(\beta), \Delta C(\gamma))$ mean the nominal system model and a specific uncertain perturbation, respectively.

Since A_i $(i = 1, \ldots, p)$ are bounded linear operators, we remark that $A(\alpha)$ always generates a C_0-semigroup and has the domain $\mathrm{D}(A(\alpha)) = \mathrm{D}(A_0)$ for all $\alpha \in \mathbf{R}^p$. Further, from the practical viewpoint it is assumed that the dimensions of input and output are finite.

Now, introduce a compensator (K, L, M, N) defined in a Hilbert space $\mathcal{W}$ of the form :

$$\Sigma : \begin{cases} \dfrac{d}{dt} w(t) = N w(t) + M y(t), \\ u(t) = L w(t) + K y(t), \end{cases} \tag{2.3}$$

where N is the infinitesimal generator of a C_0-semigroup $\{S_N(t); t \geq 0\}$ on a Hilbert space $\mathcal{W}$ with the domain $D(N) = \mathcal{W}$, $M \in \mathbf{B}(\mathbf{R}^\ell; \mathcal{W})$, $L \in \mathbf{B}(\mathcal{W}; \mathbf{R}^m)$, and $K \in \mathbf{B}(\mathbf{R}^\ell; \mathbf{R}^m)$.

If a compensator of the form Σ is applied to the system $S(\alpha, \beta, \gamma)$, the resulting closed-loop system $S_{\text{cl}}(\alpha, \beta, \gamma)$ with the extended state space $\mathcal{X}^e := \mathcal{X} \oplus \mathcal{W}$ is easily seen to be

$$\frac{d}{dt}\begin{bmatrix} x(t) \\ w(t) \end{bmatrix} = \begin{bmatrix} A(\alpha) + B(\beta)KC(\gamma) & B(\beta)L \\ MC(\gamma) & N \end{bmatrix}\begin{bmatrix} x(t) \\ w(t) \end{bmatrix}, \tag{2.4}$$

where $\mathcal{X} \oplus \mathcal{W}$ means the direct sum of $\mathcal{X}$ and $\mathcal{W}$. For the closed-loop system $S_{\text{cl}}(\alpha, \beta, \gamma)$, define

$$x^e(t) := \begin{bmatrix} x(t) \\ w(t) \end{bmatrix}, \qquad A^e_{\alpha\beta\gamma} := \begin{bmatrix} A(\alpha) + B(\beta)KC(\gamma) & B(\beta)L \\ MC(\gamma) & N \end{bmatrix} \tag{2.5}$$

with domain $\mathrm{D}(A^e_{\alpha\beta\gamma})(= \mathrm{D}(A_0) \oplus \mathcal{W})$.

For the system $S(\alpha, \beta, \gamma)$, we give the following invariant subspaces.

Definition 2.1. Let $\mathcal{V}$ be a closed subspace of $\mathcal{X}$.

(i) $\mathcal{V}$ is said to be a generalized (A, B)-invariant if there exists an $F \in \mathbf{B}(\mathcal{X}; \mathbf{R}^m)$ such that

$$(A(\alpha) + B(\beta)F)(\mathcal{V} \cap \mathrm{D}(A_0)) \subset \mathcal{V}, \quad \forall \alpha, \beta. \tag{2.6}$$

Also $\mathbf{F}(\mathcal{V}) := \{F \in \mathbf{B}(\mathcal{X}; \mathbf{R}^m) \mid (A(\alpha) + B(\beta)F)(\mathcal{V} \cap \mathrm{D}(A_0)) \subset \mathcal{V}$ for all $\alpha, \beta\}$.

(ii) $\mathcal{V}$ is said to be a generalized $S(A, B)$-invariant if there exists an $F \in \mathbf{B}(\mathcal{X}; \mathbf{R}^m)$ such that

$$S_{A(\alpha)+B(\beta)F}(t)\mathcal{V} \subset \mathcal{V}, \quad \forall t \geq 0 \text{ and all } \alpha, \beta. \tag{2.7}$$

Also $\mathcal{V}(A, B; \Lambda) := \{\mathcal{V} \mid \mathcal{V}$ is a generalized $S(A, B)$-invariant and is contained in a given closed subspace $\Lambda.\}$. $\mathbf{F}_s(\mathcal{V}) := \{F \in \mathbf{B}(\mathcal{X}; \mathbf{R}^m) \mid S_{A(\alpha)+B(\beta)F}(t)\mathcal{V} \subset \mathcal{V}$ forall $t \geq 0$ and all $\alpha, \beta\}$.

(iii) $\mathcal{V}$ is said to be a generalized (C, A)-invariant if there exists a $G \in \mathbf{B}(\mathbf{R}^\ell; \mathcal{X})$ such that

$$(A(\alpha) + GC(\gamma))(\mathcal{V} \cap \mathrm{D}(A_0)) \subset \mathcal{V}, \quad \forall \alpha, \gamma. \tag{2.8}$$

Also $\mathbf{G}(\mathcal{V}) := \{G \in \mathbf{B}(\mathbf{R}^\ell; \mathcal{X}) \mid (A(\alpha) + GC(\gamma))(\mathcal{V} \cap \mathrm{D}(A_0)) \subset \mathcal{V}$ for all $\alpha, \gamma\}$.

(iv) $\mathcal{V}$ is said to be a generalized $S(C, A)$-invariant if there exists a $G \in \mathbf{B}(\mathbf{R}^\ell; \mathcal{X})$ such that

$$S_{A(\alpha)+GC(\gamma)}(t)\mathcal{V} \subset \mathcal{V}, \quad \forall t \geq 0 \text{ and all } \alpha, \gamma. \tag{2.9}$$

Also $\mathcal{V}(\varepsilon; C, A) := \{\mathcal{V} \mid \mathcal{V}$ is a generalized $S(C, A)$-invariant and contains a given closed subspace $\varepsilon\}$. $\mathbf{G}_s(\mathcal{V}) := \{G \in \mathbf{B}(\mathbf{R}^\ell; \mathcal{X}) \mid S_{A(\alpha)+GC(\gamma)}(t)\mathcal{V} \subset \mathcal{V}$ for all $t \geq 0$ and all $\alpha, \gamma\}$.

Remark 2.2. (i) For the system $S(\alpha,\beta,\gamma)$ a generalized $S(A,B)$-invariant subspace $\mathcal{V}$ has the property that if an arbitrary initial state $x(0) \in \mathcal{V}$, then there exists a state feedback $u(t) = Fx(t)$ which is independent of α and β such that the state trajectory $x(t) \in \mathcal{V}$ for all $t \geq 0$.

(ii) If A_0 is a bounded linear operator on $\mathcal{X}$ (i.e., $A_0 \in \mathbf{B}(\mathcal{X})$), then the statements (i), (ii) and (iii), (iv) in Definition 2.1 are equivalent, respectively. Further, in this case $\mathbf{F}_s(\mathcal{V}) = \mathbf{F}(\mathcal{V})$ and $\mathbf{G}_s(\mathcal{V}) = \mathbf{G}(\mathcal{V})$.

Theorem 2.3 (see [17, 18]). *Suppose that p_i, q_i $(i = 1,\dots,p)$, r_i, s_i $(i = 1,\dots,q)$ are arbitrary fixed real numbers such that $p_i < q_i$ $(i = 1,\dots,p)$ and $r_i < s_i$ $(i = 1,\dots,q)$. Then, the following three statements are equivalent.*

(i) *$\mathcal{V}$ is a generalized $S(A,B)$-invariant.*

(ii) *There exists an $F \in \mathbf{B}(\mathcal{X};\mathbf{R}^m)$ such that $S_{A_0+B_0F}(t)\mathcal{V} \subset \mathcal{V}$ $(t \geq 0)$ and $B_iF\mathcal{V} \subset \mathcal{V}(i = 1,\dots,q)$, and $A_i\mathcal{V} \subset \mathcal{V}$ $(i = 1,\dots,p)$.*

(iii) *There exists an $F \in \mathbf{B}(\mathcal{X};\mathbf{R}^m)$ such that*

$$S_{A(\alpha)+B(\beta)F}(t)\mathcal{V} \subset \mathcal{V}, \quad (t \geq 0) \tag{2.10}$$

for all $\alpha_i \in [p_i, q_i]$ $(i = 1,\dots,p)$ and $\beta_i \in [r_i, s_i]$ $(i = 1,\dots,q)$.

(iv) *There exists an $F \in \mathbf{B}(\mathcal{X};\mathbf{R}^m)$ such that*

$$S_{A(\alpha)+B(\beta)F}(t)\mathcal{V} \subset \mathcal{V}, \quad (t \geq 0) \tag{2.11}$$

for all $\alpha_i \in \{p_i, q_i\}$ $(i = 1,\dots,p)$ and $\beta_i \in \{r_i, s_i\}$ $(i = 1,\dots,q)$.

The following theorem is the dual version of Theorem 2.3.

Theorem 2.4 (see [17, 18]). *Suppose that p_i, q_i $(i = 1,\dots,p)$, t_i, u_i $(i = 1,\dots,r)$ are arbitrary fixed real numbers such that $p_i < q_i$ $(i = 1,\dots,p)$ and $t_i < u_i$ $(i = 1,\dots,r)$. Then, the following three statements are equivalent.*

(i) *$\mathcal{V}$ is a generalized $S(C,A)$-invariant.*

(ii) *There exists a $G \in \mathbf{B}(\mathbf{R}^\ell;\mathcal{X})$ such that $S_{A_0+GC_0}(t)\mathcal{V} \subset \mathcal{V}$ $(t \geq 0)$ and $GC_i\mathcal{V} \subset \mathcal{V}$ $(i = 1,\dots,r)$, and $A_i\mathcal{V} \subset \mathcal{V}$ $(i = 1,\dots,p)$.*

(iii) *There exists a $G \in \mathbf{B}(\mathbf{R}^\ell;\mathcal{X})$ such that*

$$S_{A(\alpha)+GC(\gamma)}(t)\mathcal{V} \subset \mathcal{V}, \quad (t \geq 0) \tag{2.12}$$

for all $\alpha_i \in [p_i, q_i]$ $(i = 1,\dots,p)$ and $\gamma_i \in [t_i, u_i]$ $(i = 1,\dots,r)$.

(iv) *There exists a $G \in \mathbf{B}(\mathbf{R}^\ell;\mathcal{X})$ such that*

$$S_{A(\alpha)+GC(\gamma)}(t)\mathcal{V} \subset \mathcal{V}, \quad (t \geq 0) \tag{2.13}$$

for all $\alpha_i \in \{p_i, q_i\}(i = 1,\dots,p)$ and $\gamma_i \in \{t_i, u_i\}(i = 1,\dots,r)$.

For finite-dimensional systems, Schumacher [3] first introduced the concept of (C, A, B)-pair. The following definition is a generalized and infinite-dimensional version of (C, A, B)-pair.

Definition 2.5. Let $\mathcal{V}_1$ and $\mathcal{V}_2$ be closed subspaces of $\mathcal{X}$. A pair $(\mathcal{V}_1, \mathcal{V}_2)$ of subspaces is said to be a generalized $S(C, A, B)$-pair if the following three conditions hold.

(i) $\mathcal{V}_1$ is a generalized $S(C, A)$-invariant.

(ii) $\mathcal{V}_2$ is a generalized $S(A, B)$-invariant.

(iii) $\mathcal{V}_1 \subset \mathcal{V}_2$.

For closed-loop system $S_{\mathrm{cl}}(\alpha, \beta, \gamma)$, we give the following definition.

Definition 2.6. Let $\mathcal{V}^e$ be a closed subspace of $\mathcal{X}^e$.

(i) $\mathcal{V}^e$ is said to be a generalized A^e-invariant if $A^e_{\alpha,\beta,\gamma}(\mathcal{V}^e \cap \mathrm{D}(A^e_{\alpha,\beta,\gamma})) \subset \mathcal{V}^e$ for all α, β, γ.

(ii) $\mathcal{V}^e$ is said to be a generalized $S_{A^e}(t)$-invariant if $S_{A^e_{\alpha,\beta,\gamma}}(t)\mathcal{V}^e \subset \mathcal{V}^e$ for all $t \geq 0$ and all α, β, γ.

The following lemma was shown by Zwart.

Lemma 2.7 (see [11]). *Let $\mathcal{V}^e$ be a closed subspace of $\mathcal{X}^e$ and the following three subspaces are introduced:*

$$
\begin{aligned}
S_{orth} &:= \left\{ x \in \mathcal{X} \middle| \begin{bmatrix} x \\ w \end{bmatrix} \in [\mathcal{V}^e]^{\perp} \text{ for some } w \in \mathcal{W} \right\} \\
&= P_{\mathcal{X}}\left([\mathcal{V}^e]^{\perp}\right), \\
S_1 &:= [S_{orth}]^{\perp}, \\
S_2 &:= \left\{ x \in \mathcal{X} \middle| \begin{bmatrix} x \\ w \end{bmatrix} \in [\mathcal{V}^e] \text{ for some } w \in \mathcal{W} \right\} \\
&= P_{\mathcal{X}}([\mathcal{V}^e]),
\end{aligned}
\tag{2.14}
$$

where $P_{\mathcal{X}}$ is the projection operator from $\mathcal{X}^e$ onto $\mathcal{X}$ along $\mathcal{W}$. Then, the following statements hold.

(i) $S_1 = \{x \in \mathcal{X} | \left[\begin{smallmatrix} x \\ 0 \end{smallmatrix}\right] \in \mathcal{V}^e\}$.

(ii) $S_1 \subset S_2$.

(iii) *If $\dim(\mathcal{W}) < \infty$, then S_2 is a closed subspace of $\mathcal{X}$ and $\dim(S_2 \cap S_1^{\perp}) < \infty$.*

Lemma 2.8 (see [4, 6, 11]). *Suppose that A is the infinitesimal generator of a C_0-semigroup $\{S_A(t); t \geq 0\}$ on $\mathcal{X}$, and $\mathcal{V}$ is a closed subspace of $\mathcal{X}$ and $Q_1 \in \mathbf{B}(\mathcal{X})$. Then, the following statements hold.*

(i) *If $S_A(t)\mathcal{V} \subset \mathcal{V}$ for all $t \geq 0$, then $A(\mathcal{V} \cap D(A)) \subset \mathcal{V}$.*

(ii) *If $\mathcal{V} \subset D(A)$ and $A\mathcal{V} \subset \mathcal{V}$, then $S_A(t)\mathcal{V} \subset \mathcal{V}$ for all $t \geq 0$.*

(iii) *If* $S_{A+Q_1}(t)\mathcal{V} \subset \mathcal{V}$ *for all* $t \geq 0$, *then* $\overline{\mathcal{V} \cap D(A)} = \mathcal{V}$.

(iv) *If there exists a* $Q_2 \in \mathbf{B}(\mathcal{X})$ *such that* $S_{A+Q_2}(t)\mathcal{V} \subset \mathcal{V}$ *for all* $t \geq 0$ *and* $(Q_1 - Q_2)(\mathcal{V} \cap D(A)) \subset \mathcal{V}$, *then* $S_{A+Q_1}(t)\mathcal{V} \subset \mathcal{V}$ *for all* $t \geq 0$.

(v) *If there exists a* $Q_2 \in \mathbf{B}(\mathcal{X})$ *such that* $S_{A+Q_2}(t)\mathcal{V} \subset \mathcal{V}$ *for all* $t \geq 0$ *and* $(Q_1 - Q_2)(\mathcal{V} \cap D(A)) = \{\mathbf{0}\}$, *then* $S_{A+Q_1}(t)x = S_{A+Q_2}(t)x$ *for all* $t \geq 0$ *and all* $x \in \mathcal{V}$.

The following two lemmas are extensions of the results of Otsuka [13] to infinite-dimensional systems.

Lemma 2.9. *If a pair* $(\mathcal{V}_1, \mathcal{V}_2)$ *of subspaces of* $\mathcal{X}$ *is a generalized* $S(C, A, B)$*-pair such that*

$$\sum_{i=1}^{q} \mathrm{Im} B_i \subset \mathcal{V}_1 \subset \mathcal{V}_2 \subset \bigcap_{i=1}^{r} \mathrm{Ker}\, C_i, \qquad A_i \mathcal{V}_2 \subset \mathcal{V}_1, \quad (i = 1, \ldots, p), \tag{2.15}$$

then there exist $G \in \mathbf{G}_s(\mathcal{V}_1)$, $G(\beta) \in \mathbf{B}(\mathbf{R}^{\ell}; \mathcal{X})$, $F(\gamma) \in \mathbf{F}_s(\mathcal{V}_2)$, $F_0 \in \mathbf{B}(\mathcal{X}; \mathbf{R}^m)$ *and* $K \in \mathbf{B}(\mathbf{R}^{\ell}; \mathbf{R}^m)$ *such that*

$$G = B(\beta)K + G(\beta), \qquad \mathrm{Im} G(\beta) \subset \mathcal{V}_2, F(\gamma) = KC(\gamma) + F_0, \qquad \mathrm{Ker}\, F_0 \supset \mathcal{V}_1 \tag{2.16}$$

for all $(\beta, \gamma) \in \mathbf{R}^q \times \mathbf{R}^r$.

Proof. Suppose that a pair $(\mathcal{V}_1, \mathcal{V}_2)$ is a generalized $S(C, A, B)$-pair satisfying the stated above conditions. Since $\sum_{i=1}^{q} \mathrm{Im} B_i \subset \mathcal{V}_2$, we remark that $\mathcal{V}_2 + \mathrm{Im} B(\beta) = \mathcal{V}_2 + \mathrm{Im} B_0$.

Claim 1. $\widehat{G}C(\gamma)\mathcal{V}_1 \subset \mathcal{V}_2 + \mathrm{Im} B_0$ for all $\widehat{G} \in \mathbf{G}_s(\mathcal{V}_1)$ and $\gamma \in \mathbf{R}^r$.

To prove Claim 1, choose an arbitrary element $x \in \mathcal{V}_1$. Then, by Lemma 2.8(iii) there exists an $x_n \in \mathcal{V}_1 \cap D(A_o)$ such that

$$\lim_{n \to \infty} x_n = x. \tag{2.17}$$

Now, noticing that $(A(\alpha) + \widehat{G}C(\gamma))(\mathcal{V}_1 \cap D(A_0)) \subset \mathcal{V}_1$ and $\mathcal{V}_1 \subset \mathcal{V}_2$ we have

$$\begin{aligned} \widehat{G}C(\gamma)x_n &= \left(A(\alpha) + \widehat{G}C(\gamma)\right)x_n - A(\alpha)x_n \\ &\in \mathcal{V}_1 + \mathcal{V}_2 + \mathrm{Im} B(\beta) \\ &= \mathcal{V}_2 + \mathrm{Im} B(\beta), \\ &= \mathcal{V}_2 + \mathrm{Im} B_0. \end{aligned} \tag{2.18}$$

Since $\dim B(\beta) < \infty$, that $\mathcal{V}_2 + \mathrm{Im} B_0$ is a closed subspace. Further, noticing that $\widehat{G}C(\gamma)$ are bounded operators,

$$\widehat{G}C(\gamma)x = \lim_{n \to \infty} \widehat{G}C(\gamma)x_n \in \mathcal{V}_2 + \mathrm{Im} B_0, \tag{2.19}$$

which proves Claim 1.

Next, Claims 2 and 3 hold as follows.

Claim 2. There exists a $G \in \mathbf{G}_s(\mathcal{V}_1)$ such that $\mathrm{Im}G \subset \mathcal{V}_2 + \mathrm{Im}B_0$.

To prove Claim 2, choose a $\widehat{G} \in \mathbf{G}_\mathbf{s}(\mathcal{V}_1)$ and $x \ \ (= y + z) \in \mathbf{R}^\ell$ such that $y \in \sum_{i=0}^{r} C_i\mathcal{V}_1$ and $z \in \phi$ with $\sum_{i=0}^{r} C_i\mathcal{V}_1 \oplus \phi = \mathbf{R}^\ell$. Define a linear map $G \in \mathbf{R}^{n\times\ell}$ by $Gx := \widehat{G}y$. Then, for some $x_i \in \mathcal{V}_1$

$$\begin{aligned} Gx = \widehat{G}y &= \sum_{i=0}^{r} \widehat{G}C_i x_i \\ &\in \mathcal{V}_2 + \mathrm{Im}B_0, \quad (\text{by Claim } 1), \end{aligned} \tag{2.20}$$

which proves Claim 2.

Claim 3. There exists a $K \in \mathbf{B}(\mathbf{R}^\ell;\mathbf{R}^m)$ and $G(\beta) \in \mathbf{B}(\mathbf{R}^\ell;\mathcal{X})$ such that $G = B(\beta)K + G(\beta)$, $\mathrm{Im}G(\beta) \subset \mathcal{V}_2$ for all $\beta \in \mathbf{R}^q$.

To prove Claim 3, let $\{y_1, \ldots, y_\ell\}$ be a basis of $\mathbf{R}^\ell$. Then, it follows from Claim 2 that there exists an $x_i \in \mathcal{V}_2$ and $u_i \in \mathbf{R}^m$ such that

$$\begin{aligned} Gy_i &= x_i + B_0 u_i \\ &= x_i - \sum_{i=1}^{q} \beta_i B_i u_i + B_0 u_i + \sum_{i=1}^{q} \beta_i B_i u_i \\ &= \left(x_i - \sum_{i=1}^{q} \beta_i B_i u_i \right) + B(\beta) u_i. \end{aligned} \tag{2.21}$$

Define linear maps $K \in \mathbf{B}(\mathbf{R}^\ell;\mathbf{R}^m)$ and $G(\beta) \in \mathbf{B}(\mathbf{R}^\ell;\mathcal{X})$ by

$$K y_i := u_i, \qquad G(\beta) y_i := x_i - \sum_{i=1}^{q} \beta_i B_i u_i, \ \text{respectively.} \tag{2.22}$$

Then,

$$G y_i = G(\beta) y_i + B(\beta) K y_i, \qquad G(\beta) y_i \in \mathcal{V}_2, \tag{2.23}$$

which proves Claim 3.

Claim 4. There exists an $F^\dagger \in \mathbf{B}(\mathcal{X};\mathbf{R}^m)$ such that $\mathrm{Im}(GC(\gamma) + B(\beta)F^\dagger)|_{\mathcal{V}_2} \subset \mathcal{V}_2$.

In fact, it follows from Claim 2 and hypotheses of this lemma that there exists a $G \in \mathbf{G}_s(\mathcal{V}_1)$ such that

$$\mathrm{Im}GC_0|_{\mathcal{V}_2} \subset \mathrm{Im}G \subset \mathcal{V}_2 + \mathrm{Im}B_0. \tag{2.24}$$

Let $y \in \mathcal{V}_2$ be an arbitrary element. Then, there exist $x \in \mathcal{V}_2$ and $u \in \mathbf{R}^m$ such that $GC_0 y = x + B_0 u$. Define $F^\dagger \in \mathbf{B}(\mathcal{X}; \mathbf{R}^m)$ such that

$$F^\dagger y = -u, \tag{2.25}$$

Hence, $GC_0 y = x + B_0(-F^\dagger y)$ which implies $(GC_0 + B_0 F^\dagger)y = x \in \mathcal{V}_2$. Then, we can easily obtain

$$\operatorname{Im}\Big(GC(\gamma) + B(\beta)F^\dagger\Big)|_{\mathcal{V}_2} \subset \mathcal{V}_2, \tag{2.26}$$

which proves Claim 4.

Now, choose $F^* \in \mathbf{F}_s(\mathcal{V}_2)$ and define $F_0 \in \mathbf{B}(\mathcal{X}; \mathbf{R}^m)$ such that

$$\begin{aligned} &F_0 = F^* + F^\dagger, \quad \text{on } \mathcal{V}_1^\perp, \\ &F_0 = 0, \quad \text{on } \mathcal{V}_1. \end{aligned} \tag{2.27}$$

Then, the following claim holds.

Claim 5. One has $(A(\alpha) + GC(\gamma) + B(\beta)F_0)(\mathcal{V}_2 \cap D(A_0)) \subset \mathcal{V}_2$ for all $(\alpha, \beta, \gamma) \in \mathbf{R}^p \times \mathbf{R}^q \times \mathbf{R}^r$ and $\mathcal{V}_1 \subset \operatorname{Ker} F_0$.

In fact, at first, $\mathcal{V}_1 \subset \operatorname{Ker} F_0$ is obvious. Therefore, we prove the first one. Since $F^* \in \mathbf{F}_s(\mathcal{V}_2)$ implies $S_{A(\alpha)+B(\beta)F^*}(t)\mathcal{V}_2 \subset \mathcal{V}_2$, it follows from Lemma 2.8(v) that it suffices to show $(B(\beta)F_0 + GC(\gamma) - B(\beta)F^*)\mathcal{V}_2 \subset \mathcal{V}_2$ in order to prove $S_{A(\alpha)+B(\beta)F_0+GC(\gamma)}(t)\mathcal{V}_2 \subset \mathcal{V}_2$.

Now, let $x \in \mathcal{V}_2$ be an arbitrary element. Then, there exists $y \in \mathcal{V}_1$ and $z \in \mathcal{V}_1^\perp \cap \mathcal{V}_2$ such that $x = y + z$. Then, we have

$$\begin{aligned} (B(\beta)F_0 + GC(\gamma) - B(\beta)F^*)x &= (B(\beta)F_0 + GC(\gamma) - B(\beta)F^*)y \\ &\quad + (B(\beta)F_0 + GC(\gamma) - B(\beta)F^*)z \\ &= (GC(\gamma) - B(\beta)F^*)y + \Big(B(\beta)F^\dagger + GC(\gamma)\Big)z. \end{aligned} \tag{2.28}$$

Now, it follows from Lemma 2.8(iii) that there exists a y_n such that $y_n \to y$.

Hence,

$$(GC(\gamma) - B(\beta)F^*)y_n = (A(\alpha) + GC(\gamma))y_n - (A(\alpha) + B(\beta)F^*)y_n \in \mathcal{V}_2. \tag{2.29}$$

Noticing that $(GC(\gamma) - B(\beta)F^*)$ is bounded linear operator, it follows from (2.28) and Claim 4 that

$$(B(\beta)F_0 + GC(\gamma) - B(\beta)F^*)x \in \mathcal{V}_2, \tag{2.30}$$

which proves Claim 5.

Finally, define a bounded linear operator $F(\gamma) \in \mathbf{B}(\mathcal{X}; \mathbf{R}^m)$ such that $F(\gamma) := KC(\gamma) + F_0$. Then, the following claim holds.

Claim 6 ($F(\gamma) \in \mathbf{F}(\mathcal{V}_2)$). In fact,

$$
\begin{aligned}
&(A(\alpha) + B(\beta)F(\gamma))(\mathcal{V}_2 \cap D(A_0)) \\
&\quad = (A(\alpha) + B(\beta)KC(\gamma) + B(\beta)F_0)(\mathcal{V}_2 \cap D(A_0)) \\
&\quad = \{A(\alpha) + (B(\beta)K + G(\beta))C(\gamma) - G(\beta)C(\gamma) + B(\beta)F_0\}(\mathcal{V}_2 \cap D(A_0)) \\
&\quad = \{A(\alpha) + GC(\gamma) + B(\beta)F_0 - G(\beta)C(\gamma)\}(\mathcal{V}_2 \cap D(A_0)), \quad \text{(by Claim 3)} \\
&\quad \subset (A(\alpha) + GC(\gamma) + B(\beta)F_0)(\mathcal{V}_2 \cap D(A_0)) + \operatorname{Im} G(\beta) \\
&\quad \subset \mathcal{V}_2, \quad \text{(by Claim 3 and Claim 5)},
\end{aligned}
\tag{2.31}
$$

which proves Claim 6. This completes the proof of Lemma 2.9. □

Lemma 2.10. *If a pair $(\mathcal{V}_1, \mathcal{V}_2)$ of subspaces of $\mathcal{X}$ is a generalized $S(C, A, B)$-pair such that*

$$
\sum_{i=1}^{q} \operatorname{Im} B_i \subset \mathcal{V}_1 \subset \mathcal{V}_2 \subset \bigcap_{i=1}^{r} \operatorname{Ker} C_i, \qquad A_i \mathcal{V}_2 \subset \mathcal{V}_1, \quad (i = 1, \ldots, p), \ \mathcal{V}_2 \subset D(A_0), \tag{2.32}
$$

then there exist a compensator (K, L, M, N) on $\mathcal{W} := (\mathcal{V}_2 \cap \mathcal{V}_1^{\perp})$ and a subspace $\mathcal{V}^e$ of $\mathcal{X}^e$ such that $\mathcal{V}_1 = S_1$, $\mathcal{V}_2 = S_2$, and $\mathcal{V}^e$ is generalized $S_{A^e}(t)$-invariant, where S_1 and S_2 are given in Lemma 2.7.

Proof. Suppose that there exists a pair $(\mathcal{V}_1, \mathcal{V}_2)$ of subspaces satisfying the stated above conditions. Since $\mathcal{V}_1$ and $\mathcal{V}_2$ are closed subspaces and $\mathcal{V}_1 \subset \mathcal{V}_2$, we have $\mathcal{V}_2 = \mathcal{V}_1 \oplus (\mathcal{V}_2 \cap \mathcal{V}_1^{\perp})$. Define $\mathcal{W} := (\mathcal{V}_2 \cap \mathcal{V}_1^{\perp})$ and $\mathcal{X}^e := \mathcal{X} \oplus \mathcal{W}$. Let $R : \mathcal{V}_2 \to \mathcal{W}$ be a bounded linear operator such that $\operatorname{Ker} R = \mathcal{V}_1$ and $\operatorname{Im} R = \mathcal{W}$. Then, there exists a $R^{\dagger} \in \mathbf{B}(\mathcal{W}; \mathcal{V}_2)$ such that $RR^{\dagger} = I$, which implies $R^{\dagger} R x = 0 \Leftrightarrow x \in \mathcal{V}_1$ (see [2, p.95]). Further, define

$$
\mathcal{V}^e := \left\{ \begin{bmatrix} x \\ Rx \end{bmatrix} \mid x \in \mathcal{V}_2 \right\}. \tag{2.33}
$$

Then, it follows from Lemma 2.7 that $\mathcal{V}_1 = S_1$ and $\mathcal{V}_2 = S_2$. Further, it follows from Lemma 2.9 that there exist $G \in \mathbf{G}_s(\mathcal{V}_1)$, $G(\beta) \in \mathbf{B}(\mathbf{R}^{\ell}; \mathcal{X})$, $F(\gamma) \in \mathbf{F}_s(\mathcal{V}_2)$, $F_0 \in \mathbf{B}(\mathcal{X}; \mathbf{R}^m)$, and $K \in \mathbf{B}(\mathbf{R}^{\ell}; \mathbf{R}^m)$ such that

$$
G = B(\beta)K + G(\beta), \qquad \operatorname{Im} G(\beta) \subset \mathcal{V}_2, \qquad F(\gamma) = KC(\gamma) + F_0, \qquad \operatorname{Ker} F_0 \supset \mathcal{V}_1 \tag{2.34}
$$

for all $(\beta, \gamma) \in \mathbf{R}^q \times \mathbf{R}^r$.

Define $L \in \mathbf{B}(\mathcal{W}; \mathbf{R}^m)$ and $M \in \mathbf{B}(\mathbf{R}^p; \mathcal{W})$ such that $L := F_0 R^{\dagger}$, $M := -RG(0)$.

Noticing that $\mathcal{V}_1 \subset \mathcal{V}_2 \subset \mathrm{D}(A_0)$, it is easily shown that

$$
(A(\alpha) + B(\beta)F_0 + GC(\gamma))\mathcal{V}_i \subset \mathcal{V}_i, \quad (i = 1, 2), \ \forall \alpha, \beta, \gamma. \tag{2.35}
$$

Hence, since $(A_0 + B_0F_0 + GC_0)\mathcal{V}_2 \subset \mathcal{V}_2$ and $\mathcal{V}_1 = \operatorname{Ker} R$, we have $\operatorname{Ker}R \subset \operatorname{Ker}R(A_0 + B_0F_0 + GC_0)|_{\mathcal{V}_2}$ which is equivalent to that there exists an $N \in \mathbf{B}(\mathcal{W})$ such that

$$NR = R(A_0 + B_0F_0 + GC_0)|_{\mathcal{V}_2} = R(A_0 + B_0F(0) + G(0)C_0)|_{\mathcal{V}_2}. \tag{2.36}$$

Then, the following two Claims hold.

Claim 1. $R(A(\alpha) + B(\beta)F(\gamma))\mathcal{V}_2 = R(A_0 + B_0F(0))\mathcal{V}_2$ for all α, β, γ.

In fact, let x be an arbitrary element of $\mathcal{V}_2$:

$$\begin{aligned}
&R(A(\alpha) + B(\beta)F(\gamma))x - R(A_0 + B_0F(0))x \\
&\quad = R(\alpha_1A_1 + \cdots + \alpha_pA_p)x + RB(\beta)(KC(\gamma) + F_0)x - RB_0(KC_0 + F_0)x \\
&\quad = RB(\beta)KC(\gamma)x - RB_0KC_0x + R(B(\beta) - B_0)F_0x \\
&\quad = R\{B(\beta)KC_0 - B_0KC_0\}x \\
&\quad = 0
\end{aligned} \tag{2.37}$$

Claim 2. $(MC(\gamma) + NR)\mathcal{V}_2 = R(A(\alpha) + B(\beta)F(\gamma))\mathcal{V}_2$ for all α, β, γ.

In fact,

$$\begin{aligned}
(MC(\gamma) + NR)\mathcal{V}_2 &= (MC_0 + NR)\mathcal{V}_2 \\
&= (-RG(0)C_0 + R(A_0 + B_0F(0) + G(0)C_0))\mathcal{V}_2 \\
&= R(A_0 + B_0F(0))\mathcal{V}_2 \\
&= R(A(\alpha) + B(\beta)F(\gamma))\mathcal{V}_2, \quad (\text{by Claim 1}).
\end{aligned} \tag{2.38}$$

Finally, we have the following claim.

Claim 3. $A^e(\alpha, \beta, \gamma)\mathcal{V}^e \subset \mathcal{V}^e$ for all α, β, γ.

Choose an arbitrary element $\begin{bmatrix} x \\ Rx \end{bmatrix}$ of $\mathcal{V}^e$ $(x \in \mathcal{V}_2)$.

Since $R^\dagger Rx - x \in \operatorname{Ker} R = \mathcal{V}_1$, we have $LRx = F_0x$. Then,

$$\begin{aligned}
\begin{bmatrix} A(\alpha) + B(\beta)KC(\gamma) & B(\beta)L \\ MC(\gamma) & N \end{bmatrix}\begin{bmatrix} x \\ Rx \end{bmatrix} &= \begin{bmatrix} (A(\alpha) + B(\beta)KC(\gamma))x + B(\beta)LRx \\ (MC(\gamma) + NR)x \end{bmatrix} \\
&= \begin{bmatrix} (A(\alpha) + B(\beta)(KC(\gamma) + F_0))x \\ R(A(\alpha) + B(\beta)F(\gamma))x \end{bmatrix}, \quad (\text{by Claim 2}) \\
&= \begin{bmatrix} (A(\alpha) + B(\beta)F(\gamma))x \\ R(A(\alpha) + B(\beta)F(\gamma))x \end{bmatrix} \\
&\in \mathcal{V}^e, \quad \forall(\alpha, \beta, \gamma),
\end{aligned} \tag{2.39}$$

which proves that $\mathcal{V}^e$ is generalized A^e-invariant.

Since $\mathcal{V}_e \subset D(A^e_{\alpha,\beta,\gamma})$, it follows from Lemma 2.8(ii) that $\mathcal{V}^e$ is generalized $S_{A^e}(t)$-invariant. This completes the proof of this lemma. □

3. Parameter-Insensitive Disturbance-Rejection by Dynamic Compensator

In this section, the infinite-dimensional version of parameter insensitive disturbance-rejection problem for uncertain linear systems which was investigated by Otsuka [13] is studied.

Consider the following uncertain linear system $S(\alpha,\beta,\gamma,\delta,\sigma)$ defined in a Hilbert space $\mathcal{X}$:

$$\begin{aligned} \frac{d}{dt}x(t) &= A(\alpha)x(t) + B(\beta)u(t) + E(\sigma)\xi(t), \\ y(t) &= C(\gamma)x(t), \\ z(t) &= D(\delta)x(t), \end{aligned} \tag{3.1}$$

where $x(t) \in \mathcal{X}, u(t) \in \mathcal{U} := \mathbf{R}^m, y(t) \in \mathcal{Y} := \mathbf{R}^\ell, z(t) \in \mathcal{Z} := \mathbf{R}^\mu$, and $\xi(t) \in L_1^{loc}((0,\infty);Q)$ are the state, the input, the measurement output, the controlled output, and the disturbance which is a Hilbert space Q valued locally integrable function, respectively. It is assumed that coefficient operators have the following unknown parameters:

$$\begin{aligned} A(\alpha) &= A_0 + \alpha_1 A_1 + \cdots + \alpha_p A_p := A_0 + \Delta A(\alpha), \\ B(\beta) &= B_0 + \beta_1 B_1 + \cdots + \beta_q B_q := B_0 + \Delta B(\beta), \\ C(\gamma) &= C_0 + \gamma_1 C_1 + \cdots + \gamma_r C_r := C_0 + \Delta C(\gamma), \\ D(\delta) &= D_0 + \delta_1 D_1 + \cdots + \delta_s D_s := D_0 + \Delta D(\delta), \\ E(\sigma) &= E_0 + \sigma_1 E_1 + \cdots + \sigma_t E_t := E_0 + \Delta E(\sigma), \end{aligned} \tag{3.2}$$

where A_i, B_i, C_i are the same as system $S(\alpha,\beta,\gamma)$ in Section 2, $D_i \in \mathbf{B}(\mathcal{X};\mathbf{R}^\mu)$, $E_i \in \mathbf{B}(Q;\mathcal{X})$, and $\alpha := (\alpha_1,\ldots,\alpha_p), \beta := (\beta_1,\ldots,\beta_q), \gamma := (\gamma_1,\ldots,\gamma_r), \delta := (\delta_1,\ldots,\delta_s), \sigma := (\sigma_1,\ldots,\sigma_t)$. Further, from the practical viewpoint, we assume that uncertain parameters satisfy

$$\begin{gathered} \alpha_i \in [p_i, q_i] \quad (i = 1,\ldots,p), \qquad \beta_i \in [r_i, s_i] \quad (i = 1,\ldots,q), \qquad \gamma_i \in [t_i, u_i] \quad (i = 1,\ldots,r), \\ \delta_i \in [h_i, j_i] \quad (i = 1,\ldots,s), \qquad \sigma_i \in [k_i, l_i] \quad (i = 1,\ldots,t) \end{gathered} \tag{3.3}$$

where $p_i, q_i\ (i = 1,\ldots,p)$, $r_i, s_i\ (i = 1,\ldots,q) t_i, u_i\ (i = 1,\ldots,r) h_i, j_i\ (i = 1,\ldots,s)$, and $k_i, l_i\ (i = 1,\ldots,t)$ are given real numbers.

In system $S(\alpha,\beta,\gamma,\delta,\sigma), (A_0, B_0, C_0, D_0, E_0)$ and $(\Delta A(\alpha), \Delta B(\beta), \Delta C(\gamma), \Delta D(\delta), \Delta E(\sigma))$ represent the nominal system model and a specific uncertain perturbation, respectively.

If a compensator of the form Σ is applied to system $S(\alpha,\beta,\gamma,\delta,\sigma)$, the resulting closed-loop system with the extended state space $\mathcal{X}^e := \mathcal{X} \oplus \mathcal{W}$ is easily obtained as

$$\begin{aligned} \frac{d}{dt}\begin{bmatrix} x(t) \\ w(t) \end{bmatrix} &= \begin{bmatrix} A(\alpha)+B(\beta)KC(\gamma) & B(\beta)L \\ MC(\gamma) & N \end{bmatrix} \cdot \begin{bmatrix} x(t) \\ w(t) \end{bmatrix} + \begin{bmatrix} E(\sigma) \\ 0 \end{bmatrix}\xi(t), \\ z(t) &= \begin{bmatrix} D(\delta) & 0 \end{bmatrix}\begin{bmatrix} x(t) \\ w(t) \end{bmatrix}. \end{aligned} \tag{3.4}$$

For convenience, we set

$$\begin{gathered} x^e(t) := \begin{bmatrix} x(t) \\ w(t) \end{bmatrix}, \qquad A^e_{\alpha,\beta,\gamma} := \begin{bmatrix} A(\alpha)+B(\beta)KC(\gamma) & B(\beta)L \\ MC(\gamma) & N \end{bmatrix}, \qquad E^e(\sigma) := \begin{bmatrix} E(\sigma) \\ 0 \end{bmatrix}, \\ D^e(\delta) := \begin{bmatrix} D(\delta) & 0 \end{bmatrix}. \end{gathered} \tag{3.5}$$

Then, our disturbance-rejection problem with dynamic compensator is to find a compensator (K,L,M,N) of Σ such that

$$D^e(\delta)\int_0^t S_{A^e_{\alpha,\beta,\gamma}}(t-\tau)E^e(\sigma)\xi(\tau)d\tau = 0 \tag{3.6}$$

for all $\xi(\cdot) \in L_1^{loc}(0,\infty;\mathcal{Q})$ which is a set of all locally square integrable functions on $(0,1)$, all $t \geq 0$, and all parameters $\alpha,\beta,\gamma,\delta,\sigma$.

This problem can be formulated as follows.

Parameter Insensitive Disturbance-Rejection Problem with Dynamic Compensator (PIDRPDC)

Given A_i, B_i, C_i, D_i, E_i, find (if possible) a compensator (K,L,M,N) of (2.3) such that

$$< S_{A^e_{\alpha,\beta,\gamma}}(\cdot) \mid \operatorname{Im}E^e(\sigma) >:= \overline{\mathcal{L}\left(\bigcup_{t\geq 0} S_{A^e_{\alpha,\beta,\gamma}}(t)(\operatorname{Im}E^e(\sigma))\right)} \subset \operatorname{Ker} D^e(\delta) \tag{3.7}$$

for all parameters $\alpha,\beta,\gamma,\delta,\sigma$, where $\mathcal{L}(\Omega)$ and the over bar indicate the linear subspace generated by the set Ω and the closure in $\mathcal{X}^e$, respectively.

The following results are extensions of the results of Otsuka [13] to infinite-dimensional systems.

Theorem 3.1. *If there exists a generalized $S(C,A,B)$-pair $(\mathcal{V}_1,\mathcal{V}_2)$ such that*

$$\begin{gathered} \left\{\sum_{i=1}^{q}\operatorname{Im}B_i + \sum_{i=0}^{t}\operatorname{Im}E_i\right\} \subset \mathcal{V}_1 \subset \mathcal{V}_2 \subset \left\{\bigcap_{i=1}^{r}\operatorname{Ker}C_i \cap \bigcap_{i=0}^{s}\operatorname{Ker}D_i\right\}, \\ A_i\mathcal{V}_2 \subset \mathcal{V}_1 \quad (i=1,\ldots,p), \qquad \mathcal{V}_2 \subset D(A_0), \end{gathered} \tag{3.8}$$

then the PIDRPDC is solvable.

Proof. Suppose that the stated above conditions are satisfied. Then, it follows from Lemma 2.10 that there exist a compensator (K, L, M, N) on $\mathcal{W} := (\mathcal{V}_2 \cap \mathcal{V}_1^{\perp})$ and a subspace $\mathcal{V}^e$ of $\mathcal{X}^e$ such that $\mathcal{V}_1 = S_1, \mathcal{V}_2 = S_2$ and $\mathcal{V}^e$ is generalized $S_{A^e}(t)$-invariant. Further, it can be easily shown that $\mathrm{Im}E^e(\sigma) \subset \mathcal{V}^e \subset \mathrm{Ker}\, D^e(\delta)$. Then,

$$< S_{A^e_{\alpha,\beta,\gamma}}(\cdot) \mid \mathrm{Im}E^e(\sigma) > \subset < S_{A^e_{\alpha,\beta,\gamma}}(\cdot) \mid \mathcal{V}^e >= \mathcal{V}^e \subset \mathrm{Ker}\, D^e(\delta) \tag{3.9}$$

for all parameters $\alpha, \beta, \gamma, \delta, \sigma$ which implies the PIDRPDC is solvable. □

Corollary 3.2. *Assume that* $\mathcal{V}(\sum_{i=0}^{t} \mathrm{Im}E_i; C, A)$ *and* $\mathcal{V}(A, B; \bigcap_{i=0}^{r} \mathrm{Ker}\, D_i)$ *have the minimal element* $\mathcal{V}_{1*}$ *and the maximal element* $\mathcal{V}_2^*$*, respectively. If* $\sum_{i=1}^{q} \mathrm{Im}B_i \subset \mathcal{V}_{1*} \subset \mathcal{V}_2^* \subset \bigcap_{i=1}^{r} \mathrm{Ker}\, C_i$, $A_i\mathcal{V}_2^* \subset \mathcal{V}_{1*}$ $(i = 1, \ldots, p)$*, and* $\mathcal{V}_2^* \subset D(A_0)$*, then the PIDRPDC is solvable.*

4. An Illustrative Example

Consider the following system with uncertain parameters $\alpha,\ \beta,\ \gamma \in \mathbf{R}$:

$$\begin{aligned}
\frac{\partial x(t,\eta)}{\partial t} &= \frac{\partial^2 x(t,\eta)}{\partial \eta^2} + \left\{\alpha \int_0^1 x(t,\mu)(\phi_1(\mu) + \phi_2(\mu) + \phi_3(\mu))d\mu\right\}(\phi_1(\eta) + \phi_2(\eta) + \phi_3(\eta)) \\
&\quad + (\phi_1(\eta) - 2\phi_2(\eta))u(t) + \beta\{\phi_1(\eta) + \phi_2(\eta) + \phi_3(\eta)\}u(t) \\
&\quad + \{\phi_1(\eta) + \phi_2(\eta) + \phi_3(\eta)\}\xi(t), \\
y(t) &= \int_0^1 x(t,\mu)(2\phi_1(\mu) - \phi_2(\mu))d\mu + \gamma\int_0^1 x(t,\mu)(\phi_1(\mu) + \phi_2(\mu) - 2\phi_3(\mu))d\mu, \\
z(t) &= \int_0^1 x(t,\mu)(\phi_1(\mu) + \phi_2(\mu) - 2\phi_3(\mu))d\mu,
\end{aligned} \tag{4.1}$$

where $x(t,\eta)$ is the temperature distribution of a bar of unit length at position $\eta \in (0,1)$ and time $t \geq 0$. Moreover, $u(t) \in \mathbf{R}$, $\xi(t) \in \mathbf{R}$, and $z(t) \in \mathbf{R}$ are the input, the disturbance, and the controlled output, respectively, and $\phi_k(\eta) = \sqrt{2}\sin(k\pi\eta)$, for $\eta \in (0,1),\ k \geq 1$.

Now, let a Hilbert space $\mathcal{X} := L^2(0,1)$ which is a set of all square integrable functions on $(0,1)$. Then, we remark that $\{\phi_k;\ \ k \geq 1\}$ is an orthonormal basis of $\mathcal{X} = L^2(0,1)$. Further, define the following operators as

$$\begin{aligned}
&A(\alpha) := A_0 + A_1, \quad \alpha \in \mathbf{R}, \\
&\quad (A_0 x)(\eta) := \frac{d^2 x(\eta)}{d\eta^2}, \quad x \in \mathrm{D}(A_0) := \left\{x = \sum_{k=1}^{\infty} x_k\phi_k \in \mathcal{X} \mid \sum_{k=1}^{\infty} |\lambda_k x_k|^2 < \infty\right\}, \\
&\quad (A_1 x) := \langle x, \phi_1 + \phi_2 + \phi_3\rangle(\phi_1 + \phi_2 + \phi_3), \quad x \in \mathrm{D}(A_1) := \mathcal{X},
\end{aligned}$$

$$
\begin{aligned}
&B(\beta) := B_0 + \beta B_1, \quad \beta \in \mathbf{R}, \\
&\quad B_0 u := (\phi_1 - 2\phi_2)u, \quad u \in \mathrm{D}(B_0) := \mathbf{R}, \\
&\quad B_1 u := (\phi_1 + \phi_2 + \phi_3)u, \quad u \in \mathrm{D}(B_1) := \mathbf{R}, \\
&C(\gamma) := C_0 + \gamma C_1, \quad \gamma \in \mathbf{R}, \\
&\quad C_0 := \langle x, 2\phi_1 - \phi_2 \rangle, \quad x \in \mathrm{D}(C_0) := \mathcal{X}, \\
&\quad C_1 := \langle x, \phi_1 + \phi_2 - 2\phi_3 \rangle, \quad x \in \mathrm{D}(C_1) := \mathcal{X}, \\
&Dx := \langle x, \phi_1 + \phi_2 - 2\phi_3 \rangle, \quad x \in \mathrm{D}(D) := \mathcal{X}, \\
&E\xi := (\phi_1 + \phi_2 + \phi_3)\xi, \quad \xi \in \mathrm{D}(E) := \mathbf{R},
\end{aligned}
\tag{4.2}
$$

where $\langle \cdot, \cdot \rangle$ means the inner product in $\mathcal{X}$.

Then, it is easily seen that A_0 is the infinitesimal generator of a C_0-semigroup $\{S_{A_0}(t); t \geq 0\}$ on $\mathcal{X}$ and that for each $k \geq 1$ ϕ_k is an eigenvector of A_0 belonging to an eigenvalue $\lambda_k = -k^2\pi^2$. Moreover, the operators A_1, B_0, B_1, C_0, C_1, D, and E are all bounded. Then, by using these operators we can rewrite the above system as

$$
\begin{aligned}
\frac{d}{dt}x(t) &= A(\alpha)x(t) + B(\beta)u(t) + E\xi(t), \\
y(t) &= C(\gamma)x(t), \\
z(t) &= Dx(t).
\end{aligned}
\tag{4.3}
$$

Since $S_{A(\alpha)}(t)\mathrm{Im}E \not\subset \mathrm{Ker}\, D$ for all $t > 0$, one can see that the controlled output $z(\cdot)$ of the original system (4.3) is influenced by disturbunces $\xi(\cdot)$.

Let us define two closed subspaces $\mathcal{V}_1$ and $\mathcal{V}_2$ of $\mathcal{X}$ as

$$
\mathcal{V}_1 := \{a(\phi_1 + \phi_2 + \phi_3) \mid a \in \mathbf{R}\}, \qquad \mathcal{V}_2 := \{a(\phi_1 + \phi_2 + \phi_3) + b(\phi_1 - \phi_2) \mid a, b \in \mathbf{R}\}. \tag{4.4}
$$

If we introduce an operator $G \in \mathbf{B}(\mathbf{R}; \mathcal{X})$ as

$$
Gs := \{(\lambda_2 + \lambda_3)\phi_1 + (\lambda_1 + \lambda_3)\phi_2 + (\lambda_1 + \lambda_2)\phi_3\}s, \quad s \in \mathbf{R}, \tag{4.5}
$$

then the condition (ii) of Theorem 2.4 for the system (4.3) is satisfied, and hence it follows from Theorem 2.4 that $\mathcal{V}_1$ is a generalized $S(C, A)$-invariant. Moreover, if we introduce an operator $F \in \mathbf{B}(\mathcal{X}; \mathbf{R})$ as

$$
Fx := \langle x, \lambda_1\phi_1 + \lambda_2\phi_2 - 2\lambda_3\phi_3 \rangle, \quad x \in \mathcal{X}, \tag{4.6}
$$

then it follows from Theorem 2.3 that $\mathcal{V}_2$ is a generalized $S(A,B)$-invariant. Thus, we can see that the $(\mathcal{V}_1,\mathcal{V}_2)$ is a generalized $S(C,A,B)$-pair and it is easily shown that the pair $(\mathcal{V}_1,\mathcal{V}_2)$ satisfies the conditions of Theorem 3.1 for the system (4.3), that is,

$$\{\mathrm{Im}B_1+\mathrm{Im}E\}\subset\mathcal{V}_1\subset\mathcal{V}_2\subset\{\mathrm{Ker}\,C_1\cap\mathrm{Ker}\,D\},\qquad A_1\mathcal{V}_2\subset\mathcal{V}_1,\qquad \mathcal{V}_2\subset\mathrm{D}(A_0). \tag{4.7}$$

Therefore, it follows from Theorem 3.1 that the PIDRPDC is solvable by using a dynamic compensator which is constructed in terms of the proof of Lemma 2.10. In fact, we can obtain the dynamic compensator Σ in a Hilbert space $\mathcal{W}$ described by

$$\Sigma:\begin{cases}\dfrac{d}{dt}w(t)=Nw(t)+My(t),\\ u(t)=Lw(t)+Ky(t),\end{cases} \tag{4.8}$$

where $\mathcal{W}:=\mathcal{V}_2\cap\mathcal{V}_1^{\perp}=\{a\psi;\psi=\phi_1-\phi_2,\ a\in\mathbf{R}\}$ and

$$\begin{gathered}Nw:=-(4\lambda_1+4\lambda_2-9\lambda_3)w,\quad Lw:=\langle w,-(\lambda_1+2\lambda_2-3\lambda_3)\psi\rangle,\quad \text{for } w\in\mathcal{W},\\ My:=(2\lambda_1+\lambda_2-3\lambda_3)y\psi,\quad Ky:=(\lambda_1+\lambda_2-2\lambda_3)y,\quad \text{for } y\in\mathbf{R},\end{gathered} \tag{4.9}$$

which solves the PIDRPDC.

5. Concluding Remarks

In this paper we studied the concept of generalized $S(C,A,B)$-pairs and its properties for infinite-dimensional systems. This concept is an extension of generalized (C,A,B)-pairs investigated by Otsuka [13] to infinite-dimensional systems. After that a parameter insensitive disturbance-rejection problem with dynamic compensator was formulated and its solvability conditions were given. Further, an illustrative example was also examined.

In the present investigation, it should be pointed out that the sufficient conditions of Theorem 3.1 is not easy to check. As future studies it is useful to investigate the existence conditions and computational algorithms of the minimal element $\mathcal{V}_{1*}$ and the maximal element $\mathcal{V}_2^*$ of Corollary 3.2 in order to check easily the solvability conditions of the PIDRPDC. Further, we need to study stabilizability problems for the parameter insensitive disturbance-rejected systems.

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