Hindawi Publishing Corporation
Journal of Applied Mathematics
Volume 2009, Article ID 818269, 12 pages
doi:10.1155/2009/818269

Research Article

A Numerical Algorithm for a Kirchhoff-Type Nonlinear Static Beam

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Received 5 March 2009; Accepted 22 June 2009

Recommended by James Buchanan

A boundary value problem is posed for an integro-differential beam equation. An approximate solution is found using the Galerkin method and the Jacobi nonlinear iteration process. A theorem on the algorithm error is proved.

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1. Introduction

1.1. Statement of the Problem

We consider the equation

$$u^{\mathrm{IV}}(x) - \left(\alpha + \beta \int_0^L \left(u'(\xi)\right)^2 d\xi\right) u''(x) = f(x), \quad 0 < x < L, \tag{1.1}$$

with the conditions

$$u(0) = u(L) = 0, \qquad u''(0) = u''(L) = 0. \tag{1.2}$$

Here α, β, and L are some positive constants, $f(x)$ is a given function, and $u(x)$ is the function we want to define.

1.2. Background of the Problem

Equation (1.1) is the stationary problem associated with

$$u_{tt} + \frac{EI}{\rho A} u_{xxxx} - \left(\frac{H}{\rho} + \frac{E}{2\rho L} \int_0^L u_x^2 dx \right) u_{xx} = 0 \tag{1.3}$$

which was proposed by Woinowsky-Krieger [1] as a model for the deflection of an extensible beam with hinged ends. Here H, E, ρ, I, A, and L denote, respectively, the tension at rest, Young's elasticity modulus, density, cross-sectional moment of inertia, cross-section area and length of the beam. The nonlinear term in brackets is the correction to the classical Euler-Bernoulli equation

$$u_{tt} + \frac{EI}{\rho A} u_{xxxx} = 0, \tag{1.4}$$

where tension changes induced by the vibration of the beam during deflection are not taken into account. This nonlinear term was for the first time proposed by Kirchhoff [2] who generalized D'Alembert's classical model. Therefore (1.3) is often called a Kirchhoff-type equation for a dynamic beam. Note that Arosio [3] calls the function of the integral $\int_0^L u_x^2 dx$ the Kirchhoff correction (briefly, the K-correction) and makes a reasonable statement that the K-correction is inherent in a lot of physical phenomena.

The works dealing with the mathematical aspects of (1.3) and its generalization

$$\begin{gathered} u_{tt} + u_{xxxx} - M\left(\int_0^L u_x^2 dx \right) u_{xx} = f(x, t, u), \\ M(\lambda) \geq \text{const} > 0, \end{gathered} \tag{1.5}$$

as well as some modifications of (1.3) and (1.5) belong to Ball [4, 5], Biler [6], Henriques de Brito [7], Dickey [8], B.-Z. Guo and W. Guo [9], Kouémou-Patcheu [10], Medeiros [11], Menezes et al. [12], Panizzi [13], Pereira [14], and to others. The subject of investigation concerned the questions of the existence and uniqueness of a solution [4, 5, 9–14], its asymptotic behavior [6–8, 10], stabilization and control problems [9], and so on.

As to the static Kirchhoff-type equation for a beam, its more general form than (1.1), namely,

$$\begin{gathered} u^{\text{IV}} - m\left(\int_0^L u'^2 dx \right) u'' = f(x, u), \\ m(\lambda) \geq \text{const} > 0, \end{gathered} \tag{1.6}$$

was considered in Ma [15, 16], where the solvability under nonlinear boundary conditions is studied.

The topic of approximate solution of Kirchhoff equations, which the present paper is concerned with, was treated by Choo and Chung [17], Choo et al. [18], Clark et al. [19], and Geveci and Christie [20] for a dynamic beam, while Ma [16] and Tsai [21] studied the problem for the static case. Speaking more exactly, the finite difference and finite element Galerkin approximate solutions are investigated and the corresponding error estimates are derived in [17, 18]. Numerical analysis of solutions for a beam with moving boundary is carried out in [19]. The question of the stability and convergence of a semidiscrete and fully discrete Galerkin approximation is dealt with in [20]. To solve the problem with nonlinear boundary conditions, Ma [16] applies the difference method and the Gauss-Seidel iteration process. Finally, in [21] for the discretization of the problem, in particular the finite difference, finite element and spectral methods are used, while nonlinear systems of equations are solved by the Newton iteration and other methods.

In the present paper, a numerical algorithm is constructed and its total error estimated for (1.1). Formulas are given allowing us to calculate the upper bound of the error by using the initial data of the problem. The algorithm includes the Galerkin approximation reducing the problem to a system of cubic algebraic equations which are solved by means of the nonlinear Jacobi iteration process. We also use the Cardano formula due to which the current iteration approximation is expressed through the already found approximation in explicit form.

1.3. Assumptions

Let for each $i = 1, 2, \ldots$ there exists an integral

$$f_i = \frac{2}{L}\int_0^L f(x)\sin\frac{i\pi x}{L}\,dx, \tag{1.7}$$

and let the inequality

$$|f_i| \le \frac{\omega}{i^m}, \quad i = 1, 2, \ldots \tag{1.8}$$

be fulfilled with ω and m being some known positive constants.

Assume that there exists a solution of problem (1.1)-(1.2) representable as a series

$$u(x) = \sum_{i=1}^{\infty} u_i \sin\frac{i\pi x}{L}, \tag{1.9}$$

whose coefficients satisfy the system of equations

$$\left(\frac{i\pi}{L}\right)^4 u_i + \left(\frac{i\pi}{L}\right)^2\left(\alpha + \frac{\beta L}{2}\sum_{j=1}^{\infty}\left(\frac{j\pi}{L}\right)^2 u_j^2\right)u_i = f_i, \quad i = 1, 2, \ldots. \tag{1.10}$$

2. The Algorithm

2.1. Galerkin Method

An approximate solution of problem (1.1)-(1.2) will be sought for in the form of a finite series

$$u_n(x) = \sum_{i=1}^{n} u_{ni} \sin \frac{i\pi x}{L}, \tag{2.1}$$

where the coefficient u_{ni} is defined by the Galerkin method from the system

$$\left(\frac{i\pi}{L}\right)^4 u_{ni} + \left(\frac{i\pi}{L}\right)^2 \left(\alpha + \frac{\beta L}{2} \sum_{j=1}^{n} \left(\frac{j\pi}{L}\right)^2 u_{nj}^2\right) u_{ni} = f_i, \quad i = 1, 2, \ldots, n. \tag{2.2}$$

Here, incidentally, note that vast literature is available (e.g., see [22–25]) on the application of the Galerkin method to differential equations of second and fourth order.

2.2. Jacobi Iteration Process

To solve the nonlinear system (2.2) we use the Jacobi iteration process [26]

$$\begin{aligned} &\left(\frac{i\pi}{L}\right)^4 u_{ni,k+1} + \left(\frac{i\pi}{L}\right)^2 \left[\alpha + \frac{\beta L}{2} \left(\left(\frac{i\pi}{L}\right)^2 u_{ni,k+1}^2 + \sum_{\substack{j=1 \\ j \neq i}}^{n} \left(\frac{j\pi}{L}\right)^2 u_{nj,k}^2\right)\right] u_{ni,k+1} \\ &\quad = f_i, \quad k = 0, 1, \ldots, \; i = 1, 2, \ldots, n, \end{aligned} \tag{2.3}$$

where $u_{ni,k+l}$ denotes the $(k+l)$th iteration approximation of u_{ni}, $l = 0, 1$.

For fixed i, (2.3) is a cubic equation with respect to $(i\pi/L)u_{ni,k+1}$ (here $u_{ni,k+1}$ is taken with weight $i\pi/L$ just for convenience). Using the Cardano formula [27], we express $(i\pi/L)u_{ni,k+1}$ through the kth iteration approximation

$$\frac{i\pi}{L} u_{ni,k+1} = \sigma_{i1,k} - \sigma_{i2,k}, \tag{2.4}$$

where

$$\begin{aligned} \sigma_{ip,k} &= \left[(-1)^p s_i + \left(s_i^2 + r_{i,k}^3\right)^{1/2}\right]^{1/3}, \\ r_{i,k} &= \frac{1}{3}\left[\frac{2}{\beta L}\left(\alpha + \left(\frac{i\pi}{L}\right)^2\right) + \sum_{\substack{j=1 \\ j \neq i}}^{n} \left(\frac{j\pi}{L}\right)^2 u_{nj,k}^2\right], \quad s_i = -\frac{1}{\beta i\pi} f_i, \\ &\qquad k = 0, 1, \ldots, \; i = 1, 2, \ldots, n. \end{aligned} \tag{2.5}$$

The algorithm we have considered should be understood as the counting carried out by formula (2.4). Having $u_{ni,k}$, $i = 1, 2, \ldots, n$, we construct the approximate solution of the problem

$$u_{n,k}(x) = \sum_{i=1}^{n} u_{ni,k} \sin \frac{i\pi x}{L}. \tag{2.6}$$

2.3. Algorithm Error Definition

Let us compare the approximate solution (2.6) with the nth truncation of the exact solution (1.9)

$$p_n u(x) = \sum_{i=1}^{n} u_i \sin \frac{i\pi x}{L}. \tag{2.7}$$

This means that the algorithm error is defined as a difference

$$p_n u(x) - u_{n,k}(x) \tag{2.8}$$

which we write as a sum

$$p_n u(x) - u_{n,k}(x) = \Delta u_n(x) + \Delta u_{n,k}(x), \tag{2.9}$$

where $\Delta u_n(x)$ is the Galerkin method error and $\Delta u_{n,k}(x)$ the Jacobi process error which are equal, respectively, to

$$\Delta u_n(x) = p_n u(x) - u_n(x), \qquad \Delta u_{n,k}(x) = u_n(x) - u_{n,k}(x). \tag{2.10}$$

3. The Algorithm Error

We set ourselves the task of estimating the $L_2(0, L)$-norm of the algorithm error. For this we have to estimate the errors of the Galerkin method and the Jacobi process.

3.1. Galerkin Method Error

Let us expand $\Delta u_n(x)$ into a series. Taking (2.10), (2.7), and (2.1) into account we write

$$\Delta u_n(x) = \sum_{i=1}^{n} \Delta u_{ni} \sin \frac{i\pi x}{L}, \tag{3.1}$$

where

$$\Delta u_{ni} = u_i - u_{ni}, \quad i = 1, 2, \ldots, n. \tag{3.2}$$

By virtue of (3.1) we have

$$\|\Delta u_n(x)\|_{L_2(0,L)} = \left(\frac{L}{2}\sum_{i=1}^{n}(\Delta u_{ni})^2\right)^{1/2}. \tag{3.3}$$

We will come back to (3.3) later, while now we denote

$$\gamma_{ln} = (2-l)\left(\frac{i\pi}{L}\right)^4 + \frac{1}{2}\left(\frac{i\pi}{L}\right)^2\left[\alpha + \frac{\beta L}{2}\sum_{j=1}^{n}\left(\frac{j\pi}{L}\right)^2 u_j^2 + (-1)^{l+1}\left(\alpha + \frac{\beta L}{2}\sum_{j=1}^{n}\left(\frac{j\pi}{L}\right)^2 u_{nj}^2\right)\right], \tag{3.4}$$

$$\varepsilon_n = \frac{1}{2}\beta L\left(\frac{i\pi}{L}\right)^2\sum_{j=n+1}^{\infty}\left(\frac{j\pi}{L}\right)^2 u_j^2, \tag{3.5}$$

$$\nabla_n = \frac{1}{4}\beta L\left(\frac{i\pi}{L}\right)^2\sum_{j=1}^{n}\left(\frac{j\pi}{L}\right)^2(u_j + u_{nj})\Delta u_{nj}, \tag{3.6}$$

and rewrite (1.10) and (2.2) in the form $(\gamma_{1n} + \gamma_{2n} + \varepsilon_n)u_i = f_i$ and $(\gamma_{1n} - \gamma_{2n})u_{ni} = f_i$. Since by virtue of (3.4), (3.2), and (3.6) we have $\gamma_{2n} = \nabla_n$ and therefore $(\gamma_{1n} + \nabla_n + \varepsilon_n)u_i = f_i$ and $(\gamma_{1n} - \nabla_n)u_{ni} = f_i$. Subtracting the last two equalities from each other and taking (3.2) into account, we obtain $\gamma_{1n}\Delta u_{ni} + \nabla_n(u_i + u_{ni}) + \varepsilon_n u_i = 0$ which we multiply by Δu_{ni} and sum over $i = 1, 2, \ldots, n$. Using (3.4), (3.5), and the inequality $\sum_{i=1}^{n}\nabla_n(u_i + u_{ni})\Delta u_{ni} \geq 0$ following from (3.6), we see that

$$\sum_{i=1}^{n}\left(\alpha + \left(\frac{i\pi}{L}\right)^2\right)\left(\frac{i\pi}{L}\right)^2(\Delta u_{ni})^2 \leq \frac{1}{2}\beta L\sum_{i=1}^{n}\left(\frac{i\pi}{L}\right)^2|u_i\Delta u_{ni}|\sum_{i=n+1}^{\infty}\left(\frac{i\pi}{L}\right)^2 u_i^2. \tag{3.7}$$

By the Cauchy-Bunyakowsky-Schwarz inequality, we therefore have

$$\left(\sum_{i=1}^{n}\left(\alpha + \left(\frac{i\pi}{L}\right)^2\right)\left(\frac{i\pi}{L}\right)^2(\Delta u_{ni})^2\right)^{1/2} \leq \frac{1}{2}\,\beta L\left(\sum_{i=1}^{n}u_i^2\right)^{1/2}\sum_{i=n+1}^{\infty}\left(\frac{i\pi}{L}\right)^2 u_i^2. \tag{3.8}$$

Let us estimate the right-hand side of inequality (3.8). After multiplying (1.10) by $(i\pi/L)^t u_i$ and summing the resulting relation over $i = 1, 2, \ldots, n$ in one case and over $i = n+1, n+2, \ldots$ in the other, we come to the formula common for both cases

$$\sum_{i=v}^{w}\left(\frac{i\pi}{L}\right)^{4+t}u_i^2 + \left(\alpha + \frac{\beta L}{2}\sum_{j=1}^{\infty}\left(\frac{j\pi}{L}\right)^2 u_j^2\right)\sum_{i=v}^{w}\left(\frac{i\pi}{L}\right)^{2+t}u_i^2 = \sum_{i=v}^{w}\left(\frac{i\pi}{L}\right)^t f_i u_i, \tag{3.9}$$

where $v = 1$, $w = n$ or $v = n+1$, $w = \infty$. Thus

$$\left(\alpha + \frac{\beta L}{2}\sum_{j=1}^{\infty}\left(\frac{j\pi}{L}\right)^2 u_j^2\right)\sum_{i=v}^{w}\left(\frac{i\pi}{L}\right)^{2+t} u_i^2 \leq \frac{1}{4}\sum_{i=v}^{w}\left(\frac{i\pi}{L}\right)^{t-4} f_i^2. \tag{3.10}$$

Let us put $v = 1$, $w = n$, $t = -2$ in (3.10) and use the fact that $\sum_{j=1}^{\infty}(j\pi/L)^2 u_j^2 \geq (\pi^2/L^2)\sum_{j=1}^{n} u_j^2$. We obtain

$$\sum_{i=1}^{n} u_i^2 \leq a_n, \tag{3.11}$$

where

$$a_n = \frac{L}{\pi^2\beta}\left[\left(\alpha^2 + \frac{1}{2L}\pi^2\beta\sum_{i=1}^{n}\left(\frac{L}{i\pi}\right)^6 f_i^2\right)^{1/2} - \alpha\right]. \tag{3.12}$$

Now assuming $v = n+1$, $w = \infty$, $t = 0$ in (3.10) and using in addition to this the inequality $\sum_{j=1}^{\infty}(j\pi/L)^2 u_j^2 \geq \sum_{j=n+1}^{\infty}(j\pi/L)u_j^2$, we get

$$\sum_{i=n+1}^{\infty}\left(\frac{i\pi}{L}\right)u_i^2 \leq b_n, \tag{3.13}$$

where

$$b_n = \frac{1}{\beta L}\left[\left(\alpha^2 + \frac{1}{2}\beta L\sum_{i=n+1}^{\infty}\left(\frac{L}{i\pi}\right)^4 f_i^2\right)^{1/2} - \alpha\right]. \tag{3.14}$$

The use of (3.11) and (3.13) in (3.8) brings us to the inequality

$$\left(\sum_{i=1}^{n}\left(\alpha + \left(\frac{i\pi}{L}\right)^2\right)\left(\frac{i\pi}{L}\right)^2(\Delta u_{ni})^2\right)^{1/2} \leq \frac{1}{2}\,\beta L(a_n)^{1/2} b_n \tag{3.15}$$

which together with (3.3) gives

$$\|\Delta u_n(x)\|_{L_2(0,L)} \leq \frac{1}{2\pi}\beta L^2\left(\frac{L}{2}\frac{a_n}{\alpha + (\pi/L)^2}\right)^{1/2} b_n. \tag{3.16}$$

Let us substitute (3.12) and (3.14) into (3.16) and apply condition (1.8) and also the integral test for series convergence. As a result, if $n > 1$, for the Galerkin method error we obtain the estimate

$$\|\Delta u_n(x)\|_{L_2(0,L)} \le c_0\left[\left(1+c_1\frac{1}{n^{2m+3}}\right)^{1/2}-1\right]\left\{\left[1+c_2\left(1+c_3\left(1-\frac{1}{n^{2m+5}}\right)\right)\right]^{1/2}-1\right\}^{1/2}, \tag{3.17}$$

where the coefficients c_0, c_1, c_2, and c_3 do not depend on n and are defined by

$$c_0 = \frac{\alpha}{2}\left(\frac{L}{\pi}\right)^2\left(\frac{1}{2\beta\left(1+(1/\alpha)(\pi/L)^2\right)}\right)^{1/2},$$
$$c_l = \frac{\beta L\omega^2}{2\alpha^2(1-2(l-2)(m+1))}\left(\frac{L}{\pi}\right)^4, \quad l=1,2, \qquad c_3 = \frac{1}{2m+5}. \tag{3.18}$$

3.2. *Jacobi Process Error*

Taking (2.10), (2.1), and (2.6) into account, we represent $\Delta u_{n,k}(x)$ as a series

$$\Delta u_{n,k}(x) = \sum_{i=1}^{n}\Delta u_{ni,k}\sin\frac{i\pi x}{L}, \tag{3.19}$$

where

$$\Delta u_{ni,k} = u_{ni} - u_{ni,k}, \quad i = 1,2,\ldots,n. \tag{3.20}$$

Series (3.19) implies the formula

$$\|\Delta u_{n,k}(x)\|_{L_2(0,L)} = \left(\frac{L}{2}\sum_{i=1}^{n}(\Delta u_{ni,k})^2\right)^{1/2} \tag{3.21}$$

to be used later.

Let us rewrite (2.4) in the form

$$\frac{i\pi}{L}u_{ni,k+1} = \varphi_i\left(\frac{\pi}{L}u_{n1,k}, \frac{2\pi}{L}u_{n2,k},\ldots,\frac{n\pi}{L}u_{nn,k}\right), \tag{3.22}$$

and introduce into consideration the Jacobian

$$J = \left(\frac{\partial \varphi_i}{\partial((j\pi/L)u_{nj,k})} \right)_{i,j=1}^{n} \tag{3.23}$$

(in this paper this is the second notion associated with the name of C. Jacobi, 1804–1851).

To establish the convergence condition for process (3.22) we have to estimate the norm of the matrix J. By virtue of (2.4), (2.9), and (3.22) there are zeros on the principal diagonal of this matrix,

$$\frac{\partial \varphi_i}{\partial((i\pi/L)u_{ni,k})} = 0. \tag{3.24}$$

As to the nondiagonal elements, $i \neq j$, they are defined by the formula

$$\frac{\partial \varphi_i}{\partial((j\pi/L)u_{nj,k})} = \frac{1}{3} r_{i,k}^2 \left(s_i^2 + r_{i,k}^3 \right)^{-1/2} \left(\sigma_{i1,k}^{-2} - \sigma_{i2,k}^{-2} \right) \frac{j\pi}{L} u_{nj,k}. \tag{3.25}$$

Using the relations

$$\sigma_{i1,k}\sigma_{i2,k} = r_{i,k}, \quad \left(s_i^2 + r_{i,k}^3 \right)^{1/2} = \frac{1}{2}\left(\sigma_{i1,k}^3 + \sigma_{i2,k}^3 \right), \quad \sigma_{i2,k}^3 - \sigma_{i1,k}^3 = 2s_i, \tag{3.26}$$

which follow from (2.5), we rewrite (3.25) as the equality

$$\frac{\partial \varphi_i}{\partial((j\pi/L)u_{nj,k})} = -\frac{4}{3} s_i \left(\sigma_{i1,k}^4 + r_{i,k}^2 + \sigma_{i2,k}^4 \right)^{-1} \frac{j\pi}{L} u_{nj,k}. \tag{3.27}$$

Apply to the latter equality the estimate $\sigma_{i1,k}^4 + \sigma_{i2,k}^4 \geq 2r_{i,k}^2$, which is obtained from the first relation in (3.26) and (2.5). Also use the fact that the maximal value of the function $z(x) = x(a^2 + x^2)^{-2}$, $x \geq 0$, is equal to $(1/16)(3/a^2)^{3/2}$. Thus we obtain the inequalities

$$\begin{aligned} \left| \frac{\partial \varphi_i}{\partial((j\pi/L)u_{nj,k})} \right| &\leq \frac{4}{9} \frac{|s_i|}{r_{i,k}^2} \frac{j\pi}{L} |u_{nj,k}| \\ &\leq \frac{4|f_i|}{\beta i \pi} \left(\frac{2}{\beta L} \left(\alpha + \left(\frac{i\pi}{L} \right)^2 \right) + \left(\frac{j\pi}{L} u_{nj,k} \right)^2 \right)^{-2} \frac{j\pi}{L} |u_{nj,k}| \\ &\leq \frac{3}{8} \left(\frac{3}{2} \beta L \right)^{1/2} \frac{|f_i|}{(i\pi/L)\left(\alpha + (i\pi/L)^2 \right)^{3/2}}, \quad i \neq j, \end{aligned} \tag{3.28}$$

which are fulfilled for the nondiagonal elements of the matrix J.

Let us use the vector and matrix norms equal, respectively, to $\sum_{i=1}^{n} |v_i|$ and $\max_{1 \leq j \leq n} \sum_{i=1}^{n} |m_{ij}|$ for the vector $v = (v_i)_{i=1}^{n}$ and the matrix $M = (m_{ij})_{i,j=1}^{n}$. Assume that for an arbitrary set of values $u_{nj,k}$, $j = 1, 2, \ldots, n$, $k = 0, 1, \ldots$, the elements of the matrix J satisfy the condition $\max_{1\le j\le n} \sum_{i=1}^{n} |\partial\varphi_i / \partial((j\pi/L)u_{nj,k})| \le q < 1$. For this, by virtue of (3.28), (3.24), and (1.8) it is sufficient that

$$\frac{3}{8\pi} L\omega \left(\frac{3}{2}\beta L\right)^{1/2} \sum_{\substack{i=1 \\ i\neq j}}^{n} \frac{1}{i^{m+1}\left(\alpha + (i\pi/L)^2\right)^{3/2}} \le q < 1, \quad j = 1, 2, \ldots, n. \tag{3.29}$$

Then, according to the map compression principle, the system of (2.2) has a unique solution u_{ni}, $i = 1, 2, \ldots, n$, the iteration process (2.4) converges, $\lim_{k\to\infty} u_{ni,k} = u_{ni}$, $i = 1, 2, \ldots, n$, with the rate which in view of notation (3.20) is defined by the inequality $\sum_{i=1}^{n} i|\Delta u_{ni,k}| \le (q^k/(1-q)) \sum_{i=1}^{n} i|u_{ni,1} - u_{ni,0}|$. From this and (3.21) we obtain the estimate for the Jacobi process error

$$\|\Delta u_{n,k}(x)\|_{L_2(0,L)} \le \frac{q^k}{1-q}\left(\frac{L}{2}\right)^{1/2} \sum_{i=1}^{n} i|u_{ni,1} - u_{ni,0}|, \quad k = 0, 1, \ldots. \tag{3.30}$$

To conclude this section, we would like to touch upon one auxiliary question. Let us see how condition (3.29) will change if we apply to it the integral test for the convergence of series and ignore $i \neq j$ under the summation sign. Besides, we restrict ourselves to the case where m is an integer number and apply the inequality $\alpha^{1/2} + i\pi/L \le [2(\alpha + (i\pi/L)^2)]^{1/2}$. Then using the formula for the integral $\int dx/x^{m+1}(a+bx)^3$, $a, b > 0$, [28] instead of (3.29), we obtain

$$\begin{aligned} \frac{3}{4}\omega(3\beta L)^{1/2} \Bigg\{ &\frac{L}{2\pi\sqrt{2}(\alpha + (\pi/L)^2)^{3/2}} + \frac{1}{\alpha^2}\left(\frac{\pi}{L\sqrt{\alpha}}\right)^{m-1} (m+2)! \\ &\times \sum_{l=0}^{m+2} (-1)^l \frac{1}{l!(m-l+2)!(m-l)} \left[\left(1 + \frac{L\sqrt{\alpha}}{\pi}\right)^{m-l} - \left(1 + \frac{L\sqrt{\alpha}}{\pi n}\right)^{m-l}\right] \Bigg\} \le q < 1. \end{aligned} \tag{3.31}$$

3.3. Algorithm Error

Let us estimate error (2.8). By (2.9) we have

$$\left\|p_n u(x) - u_{n,k}(x)\right\|_{L_2(0,L)} \le \|\Delta u_n(x)\|_{L_2(0,L)} + \|\Delta u_{n,k}(x)\|_{L_2(0,L)}, \tag{3.32}$$

and therefore the application of (3.17) and (3.30) gives the inequality

$$\begin{aligned} &\left\| p_n u(x) - u_{n,k}(x) \right\|_{L_2(0,L)} \\ &\quad \le c_0 \left[\left(1 + c_1 \frac{1}{n^{2m+3}} \right)^{1/2} - 1 \right] \left\{ \left[1 + c_2 \left(1 + c_3 \left(1 - \frac{1}{n^{2m+5}} \right) \right) \right]^{1/2} - 1 \right\}^{1/2} \\ &\qquad + \frac{q^k}{1-q} \left(\frac{L}{2} \right)^{1/2} \sum_{i=1}^{n} i \left| u_{ni,1} - u_{ni,0} \right|. \end{aligned} \tag{3.33}$$

The obtained result can be summarized as follows.

Theorem 3.1. *Let $n > 1$ and q be some number from the interval $(0, 1)$. Assume that the conditions of Section 1.3 and restriction* (3.29) (*or* (3.31) *in the case of integer m*) *are fulfilled. Then the algorithm error is estimated by inequality* (3.33)*, where the coefficients c_0, c_1, c_2, and c_3 are calculated by formulas* (3.18).

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