

Research Article

On the Fuzzy Convergence

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The aim of this paper is to introduce and study the fuzzy neighborhood, the limit fuzzy number, the convergent fuzzy sequence, the bounded fuzzy sequence, and the Cauchy fuzzy sequence on the base which is adopted by Abdul Hameed (every real number r is replaced by a fuzzy number \bar{r} (either triangular fuzzy number or singleton fuzzy set (fuzzy point))). And then, we will consider that some results respect effect of the upper sequence on the convergent fuzzy sequence, the bounded fuzzy sequence, and the Cauchy fuzzy sequence.

1. Introduction

Zadeh [1] introduced the concept of fuzzy set in 1965. Kramosil and Michálek [2] introduced the concept of fuzzy metric space using continuous t -norms in 1975.

Matloka [3] introduced bounded and convergent sequences of fuzzy numbers and studied some of their properties in 1986. Sequences of fuzzy numbers also were discussed by Nanda [4], Kwon [5], Esi [6], and many others.

In 2010, Al-Tai defined the fuzzy metric space, the fuzzy sequence, and many other concepts, on other base, that the family of fuzzy real numbers is $\bar{\bar{R}} = \bar{\bar{Z}} \cup \bar{\bar{Q}} \cup \bar{\bar{Q}}'$. $\bar{\bar{Z}}$ is the family of fuzzy integer numbers, where every $\bar{r} \in \bar{\bar{Z}}$ ($r \in Z$) is a singleton fuzzy set (fuzzy point) (see [7–12]). $\bar{\bar{Q}}$ and $\bar{\bar{Q}}'$ are the family of fuzzy rational numbers and the family of fuzzy irrational numbers, respectively, where every $\bar{r} \in \bar{\bar{Q}}$ ($r \in Q$) or $\bar{r}' \in \bar{\bar{Q}}'$ ($r' \in Q'$) is a triangular fuzzy number [13] and by using the representation Theorem (the resolution principle) [14].

In this research, we will consider the fuzzy neighborhood, the limit fuzzy number, the convergent fuzzy sequence, the bounded fuzzy sequence, and the Cauchy fuzzy sequence, and then we will introduce some results about properties of the above concepts on the base which depended by Al-Tai to define the fuzzy metric space.

Definition 1.1. Let $(\overline{\overline{X}}, \overline{\overline{d}})$ be a fuzzy metric space. If $\overline{\overline{x}} \in \overline{\overline{X}}$, then a neighborhood of $\overline{\overline{x}}$ is a family of fuzzy numbers $\overline{\overline{N}}_{\overline{\overline{\varepsilon}}}(\overline{\overline{x}})$ consisting of all fuzzy numbers $\overline{\overline{y}} \in \overline{\overline{X}}$, such that $\overline{\overline{d}}(\overline{\overline{x}}, \overline{\overline{y}}) < \overline{\overline{\varepsilon}}$ ($\overline{\overline{\varepsilon}} > \overline{\overline{0}}$). The fuzzy number $\overline{\overline{\varepsilon}}$ will be called the fuzzy radius of $\overline{\overline{N}}_{\overline{\overline{\varepsilon}}}(\overline{\overline{x}})$.

Definition 1.2. A fuzzy number $\overline{\overline{x}}$ is a limit fuzzy number of the family of the fuzzy numbers $\overline{\overline{E}}$, if every fuzzy neighborhood of $\overline{\overline{x}}$ contains a fuzzy number $\overline{\overline{y}} \neq \overline{\overline{x}}$, such that $\overline{\overline{y}} \in \overline{\overline{E}}$.

Definition 1.3 (Al-Tai [13]). If we have the family of the fuzzy numbers $\overline{\overline{X}}$ and the family of the fuzzy natural numbers $\overline{\overline{N}}$, the fuzzy sequence $\langle \overline{\overline{x}}_n \rangle$ in $\overline{\overline{X}}$ is a fuzzy function from $\overline{\overline{N}}$ to $\overline{\overline{X}}$.

A fuzzy sequence $\langle \overline{\overline{x}}_n \rangle$ consists of all ordered tuples (sequences) at degree α , for all $\alpha \in]0, 1]$; $\langle x_{j,n,\alpha} \rangle = \langle x_{j,1,\alpha}, x_{j,2,\alpha}, \dots \rangle \in \langle \overline{\overline{x}}_1[\alpha], \overline{\overline{x}}_2[\alpha], \dots \rangle = \langle \overline{\overline{x}}_1, \overline{\overline{x}}_2, \dots \rangle[\alpha]$ is its α -cut, $j \in N$. The fuzzy value of the fuzzy function at $\overline{\overline{n}} \in \overline{\overline{N}}$ is denoted by $\overline{\overline{x}}_n$.

Definition 1.4. A fuzzy sequence $\langle \overline{\overline{x}}_n \rangle$ in a fuzzy metric space $(\overline{\overline{X}}, \overline{\overline{d}})$ is said to be convergent, if there is a fuzzy number $\overline{\overline{x}} \in \overline{\overline{X}}$, such that for every $\overline{\overline{\varepsilon}} > \overline{\overline{0}}$, there is a fuzzy natural number $\overline{\overline{N}}$, with $\overline{\overline{n}} \geq \overline{\overline{N}}$, which implies that

$$\overline{\overline{d}}(\overline{\overline{x}}_n, \overline{\overline{x}}) < \overline{\overline{\varepsilon}}. \quad (1.1)$$

That is, for every $\varepsilon_{j,\alpha} \in \overline{\overline{\varepsilon}}[\alpha]$ α -cut of $\overline{\overline{\varepsilon}}$, there is a natural number N , with $n \geq N$, such that

$$d(x_{j,n,\alpha}, x_{j,\alpha}) < \varepsilon_{j,\alpha}, \quad (1.2)$$

where $x_{j,n,\alpha} \in \overline{\overline{x}}_n[\alpha]$ α -cut of $\overline{\overline{x}}_n$, $x_{j,\alpha} \in \overline{\overline{x}}[\alpha]$ α -cut of $\overline{\overline{x}}$, $N \in \overline{\overline{N}}$, and $n \in \overline{\overline{n}}$, for all $\alpha \in]0, 1]$. $\overline{\overline{x}}$ will be called the fuzzy limit of $\langle \overline{\overline{x}}_n \rangle$, and we write

$$\lim_{n \rightarrow \infty} \overline{\overline{x}}_n = \overline{\overline{x}}. \quad (1.3)$$

If $\langle x_{j,n,\alpha} \rangle$ is not convergent at any $\alpha \in]0, 1]$, then $\langle \overline{\overline{x}}_n \rangle$ is said to be divergent.

Definition 1.5. The family of the fuzzy numbers $\overline{\overline{E}}$ in the fuzzy metric space $(\overline{\overline{X}}, \overline{\overline{d}})$ is bounded, if there is a fuzzy real number $\overline{\overline{r}} > \overline{\overline{0}}$ and a fuzzy number $\overline{\overline{x}} \in \overline{\overline{X}}$, such that

$$\overline{\overline{d}}(\overline{\overline{y}}, \overline{\overline{x}}) < \overline{\overline{r}} \quad (1.4)$$

for all $\overline{\overline{y}} \in \overline{\overline{E}}$. That is, if $r_\alpha \in \overline{\overline{r}}[\alpha]$ α -cut of $\overline{\overline{r}}$ and $x_\alpha \in \overline{\overline{x}}[\alpha]$ α -cut of $\overline{\overline{x}} \in \overline{\overline{X}}$, then

$$d(x_\alpha, y_\alpha) < r_\alpha, \quad (1.5)$$

where $y_\alpha \in \overline{\overline{y}}[\alpha]$ α -cut of all $\overline{\overline{y}} \in \overline{\overline{E}}$, for all $\alpha \in]0, 1]$.

Definition 1.6. The fuzzy sequence $\langle \bar{x}_n \rangle$ in a fuzzy metric space (\bar{X}, \bar{d}) is said to be bounded if its fuzzy range is bounded.

Definition 1.7. A fuzzy sequence $\langle \bar{x}_n \rangle$ in a fuzzy metric space (\bar{X}, \bar{d}) is said to be a Cauchy fuzzy sequence if for every $\bar{\varepsilon} > \bar{0}$, there is a fuzzy natural number \bar{N} , such that

$$\bar{d}(\bar{x}_n, \bar{x}_m) < \bar{\varepsilon}, \quad (1.6)$$

whenever $\bar{n} \geq \bar{N}$ and $\bar{m} \geq \bar{N}$. In other words, every $x_{j,n,\alpha} \in \bar{x}_n[\alpha]$ α -cut of \bar{x}_n is a Cauchy sequence; that is, for every $\varepsilon_{j,\alpha} \in \bar{\varepsilon}[\alpha]$ α -cut of $\bar{\varepsilon}$, there is a natural number N , such that

$$d(x_{j,n,\alpha}, x_{j,m,\alpha}) < \varepsilon_{j,\alpha} \quad (1.7)$$

for every $n \geq N$, $m \geq N$, where $x_{j,n,\alpha} \in \bar{x}_n[\alpha]$ α -cut of \bar{x}_n , $x_{j,m,\alpha} \in \bar{x}_m[\alpha]$ α -cut of \bar{x}_m , $n \in \bar{n}$, $m \in \bar{m}$, and $N \in \bar{N}$, for all $\alpha \in]0, 1]$.

Examples 1.8. (1) $\langle 1/(n+1), 1/n, 1/(n-1) \rangle$ converges to $\bar{0}$, since for all $\langle x_{n,\alpha} \rangle \in [(n+\alpha)/n(n+1), (n-\alpha)/n(n-1)]$ α -cut of $\langle 1/(n+1), 1/n, 1/(n-1) \rangle$, for all $\varepsilon_\alpha > 0$, $\exists N \in \bar{N} \in \bar{N} : d_\alpha(x_{n,\alpha}, 0) < \varepsilon_\alpha$, for all $n > N$, for all $\alpha \in]0, 1]$.

Therefore,

$$\lim_{n \rightarrow \infty} \left\langle \frac{1}{n+1}, \frac{1}{n}, \frac{1}{n-1} \right\rangle = \bar{0}. \quad (1.8)$$

(2) $\langle 1/e^{n+1}, 1/e^n, 1/e^{n-1} \rangle$ converges to $\bar{0}$, since for all $\langle x_{n,\alpha} \rangle \in [((e-1)\alpha+1)/e^{n+1}, ((1-e)\alpha+e)/e^n]$ α -cut of $\langle 1/e^{n+1}, 1/e^n, 1/e^{n-1} \rangle$, for all $\varepsilon_\alpha > 0$, $\exists N \in \bar{N} \in \bar{N} : d_\alpha(x_{n,\alpha}, 0) < \varepsilon_\alpha$, for all $n > N$, for all $\alpha \in]0, 1]$.

Therefore,

$$\lim_{n \rightarrow \infty} \left\langle \frac{1}{e^{n+1}}, \frac{1}{e^n}, \frac{1}{e^{n-1}} \right\rangle = \bar{0}. \quad (1.9)$$

(3) $\overline{\sin(\bar{n} \cdot (x_1, x, x_2))} / \sqrt{\bar{n}} = (\sin(nx_1), \sin(nx), \sin(nx_2)) / ((\sqrt{n})_1, \sqrt{n}, (\sqrt{n})_2)$ converges to $\bar{0}$, where α -cut of $(\sin(nx_1), \sin(nx), \sin(nx_2))$ is $[(\sin(nx) - \sin(nx_1))\alpha + \sin(nx_1), (\sin(nx) - \sin(nx_2))\alpha + \sin(nx_2)] = [a_1(\alpha), a_2(\alpha)]$.

And α -cut of $((\sqrt{n})_1, \sqrt{n}, (\sqrt{n})_2)$ is

$$[(\sqrt{n} - (\sqrt{n})_1)\alpha + (\sqrt{n})_1, (\sqrt{n} - (\sqrt{n})_2)\alpha + (\sqrt{n})_2] = [b_1(\alpha), b_2(\alpha)]. \quad (1.10)$$

Since

$$\forall \langle x_{n,\alpha} \rangle \in \left[\min \left(\frac{a_1(\alpha)}{b_2(\alpha)}, \frac{a_1(\alpha)}{b_1(\alpha)}, \frac{a_2(\alpha)}{b_2(\alpha)}, \frac{a_2(\alpha)}{b_1(\alpha)} \right), \max \left(\frac{a_1(\alpha)}{b_2(\alpha)}, \frac{a_1(\alpha)}{b_1(\alpha)}, \frac{a_2(\alpha)}{b_2(\alpha)}, \frac{a_2(\alpha)}{b_1(\alpha)} \right) \right], \quad (1.11)$$

$$0 \notin [b_1(\alpha), b_2(\alpha)],$$

α -cut of $(\sin(nx_1), \sin(nx), \sin(nx_2)) / ((\sqrt{n})_1, \sqrt{n}, (\sqrt{n})_2)$, for all $\varepsilon_\alpha > 0$, $\exists N \in \bar{N} \in \bar{N}$: $d_\alpha(x_{n,\alpha}, 0) < \varepsilon_\alpha$, for all $n > N$, for all $\alpha \in]0, 1]$.

Therefore,

$$\lim_{n \rightarrow \infty} \frac{(\sin(nx_1), \sin(nx), \sin(nx_2))}{((\sqrt{n})_1, \sqrt{n}, (\sqrt{n})_2)} = \bar{0}. \quad (1.12)$$

(4) $\langle (1/2, 1/3, 1/4) \rangle$ converges to $(1/2, 1/3, 1/4)$.

(5) $\langle \bar{n} \rangle$ is divergent and unbounded.

(6) $\overline{\cos(\bar{n} \cdot (x_1, x, x_2))} = (\cos(nx_1), \cos(nx), \cos(nx_2))$ is divergent, since for all $\langle x_{n,\alpha} \rangle \in [(\cos(nx) - \cos(nx_1))\alpha + \cos(nx_1), (\cos(nx) - \cos(nx_2))\alpha + \cos(nx_2)]$ α -cut of $\overline{\cos(\bar{n} \cdot (x_1, x, x_2))}$ is divergent. And it is bounded.

Theorem 1.9. Let $\langle \bar{x}_n \rangle$ be a fuzzy sequence in the fuzzy metric space (\bar{R}, \bar{d}) . If the upper sequence of the α -cut of $\langle \bar{x}_n \rangle$ is bounded for all $\alpha \in]0, 1]$, then $\langle \bar{x}_n \rangle$ is bounded.

Proof. Let $x_{j,\alpha}$, be a supremum of a sequence $\langle x_{j,n,\alpha} \rangle$, $j = 1, 2, \dots$, for all $\alpha \in]0, 1]$. We have from the hypothesis, the upper sequence $\langle x_{1,n,\alpha} \rangle$ is bounded for all $\alpha \in]0, 1]$; that is, there is a real number $r_{1,\alpha} \in \bar{r}[\alpha]$ α -cut of $\bar{r} \in \bar{R}$, such that

$$d(x_{1,i,\alpha}, 0) < r_{1,\alpha} \quad (1.13)$$

for all $x_{1,i,\alpha} \in \langle x_{1,n,\alpha} \rangle$, $i \in N$, for all $\alpha \in]0, 1]$.

When $j = 2$,

$$\begin{aligned} d(x_{2,i,\alpha}, 0) &\leq d(x_{2,i,\alpha}, x_{1,\alpha}) + d(x_{1,\alpha}, 0) \\ &< r'_{2,\alpha} + r_{1,\alpha} = r_{2,\alpha}, \end{aligned} \quad (1.14)$$

where $r'_{2,\alpha} = \max\{d(x_{2,i,\alpha}, x_{1,\alpha}), x_{2,i,\alpha} \in \langle x_{2,n,\alpha} \rangle\}$. That is, there is a real number $r_{2,\alpha} \in \bar{r}[\alpha]$ α -cut of $\bar{r} \in \bar{R}$, such that

$$d(x_{2,i,\alpha}, 0) < r_{2,\alpha} \quad (1.15)$$

for all $x_{2,i,\alpha} \in \langle x_{2,n,\alpha} \rangle$, $i \in N$, for all $\alpha \in]0, 1]$.

When $j = 3$,

$$\begin{aligned} d(x_{3,i,\alpha}, 0) &\leq d(x_{3,i,\alpha}, x_{2,\alpha}) + d(x_{2,\alpha}, 0) \\ &< r'_{3,\alpha} + r_{2,\alpha} = r_{3,\alpha}, \end{aligned} \quad (1.16)$$

where $r'_{3,\alpha} = \max\{d(x_{3,i,\alpha}, x_{2,\alpha}), x_{3,i,\alpha} \in \langle x_{3,n,\alpha} \rangle\}$. That is, there is a real number $r_{3,\alpha} \in \bar{r}[\alpha]$ α -cut of $\bar{r} \in \bar{R}$, such that

$$d(x_{3,i,\alpha}, 0) < r_{3,\alpha} \quad (1.17)$$

for all $x_{3,i,\alpha} \in \langle x_{3,n,\alpha} \rangle$, $i \in N$, for all $\alpha \in]0, 1]$.

And so on, when $j = m + 1$,

$$\begin{aligned} d(x_{m+1,i,\alpha}, 0) &\leq d(x_{m+1,i,\alpha}, x_{m,\alpha}) + d(x_{m,\alpha}, 0) \\ &< r'_{m+1,\alpha} + r_{m,\alpha} = r_{m+1,\alpha}, \end{aligned} \quad (1.18)$$

where $r'_{m+1,\alpha} = \max\{d(x_{m+1,i,\alpha}, x_{m,\alpha}), x_{m+1,i,\alpha} \in \langle x_{m+1,n,\alpha} \rangle\}$. That is, there is a real number $r_{m+1,\alpha} \in \bar{r}[\alpha]$ α -cut of $\bar{r} \in \bar{R}$, such that

$$d(x_{m+1,i,\alpha}, 0) < r_{m+1,\alpha} \quad (1.19)$$

for all $x_{m+1,i,\alpha} \in \langle x_{m+1,n,\alpha} \rangle$, $i \in N$, $m \in N$, for all $\alpha \in]0, 1]$. By Definitions 1.5 and 1.6, $\langle \bar{x}_n \rangle$ is bounded. \square

Theorem 1.10. Let $\langle \bar{x}_n \rangle$ be a fuzzy sequence in the fuzzy metric space (\bar{R}, \bar{d}) . If the upper sequence of the α -cut of $\langle \bar{x}_n \rangle$ is bounded for all $\alpha \in]0, 1]$, then $\langle \bar{x}_n \rangle$ is convergent.

Proof. Suppose that the upper sequence $\langle x_{1,n,\alpha} \rangle$ of the α -cut of $\langle \bar{x}_n \rangle$ is bounded for all $\alpha \in]0, 1]$. By Theorem 1.9, $\langle \bar{x}_n \rangle$ is bounded; that is, every sequence $\langle x_{j,n,\alpha} \rangle$, $j = 2, 3, \dots$, of the α -cut of $\langle \bar{x}_n \rangle$ is bounded for all $\alpha \in]0, 1]$, but every sequence $\langle x_{j,n,\alpha} \rangle$, $j = 1, 2, \dots$, of the α -cut of $\langle \bar{x}_n \rangle$ is monotonic, for all $\alpha \in]0, 1]$; then, every sequence $\langle x_{j,n,\alpha} \rangle$, $j = 1, 2, \dots$, of the α -cut of $\langle \bar{x}_n \rangle$ is convergent, for all $\alpha \in]0, 1]$ (see [15, Theorem 3.14]). By Definition 1.4, $\langle \bar{x}_n \rangle$ is convergent. \square

Theorem 1.11. Let $\langle \bar{x}_n \rangle$ be a fuzzy sequence in the fuzzy metric space (\bar{R}, \bar{d}) . If the upper sequence of the α -cut of $\langle \bar{x}_n \rangle$ is convergent for all $\alpha \in]0, 1]$, then $\langle \bar{x}_n \rangle$ is convergent.

Proof. Suppose that the upper sequence $\langle x_{1,n,\alpha} \rangle$ of the α -cut of $\langle \bar{x}_n \rangle$ is convergent for all $\alpha \in]0, 1]$, so it is bounded (see [15, Theorem 3.2-c]). By Theorem 1.10, $\langle \bar{x}_n \rangle$ is convergent. \square

Theorem 1.12. Every convergent fuzzy sequence $\langle \bar{x}_n \rangle$ in a fuzzy metric space (\bar{X}, \bar{d}) is bounded.

Proof. Let $\langle \bar{x}_n \rangle$ be a convergent fuzzy sequence in a fuzzy metric space (\bar{X}, \bar{d}) . By Definition 1.4, every sequence $\langle x_{j,n,\alpha} \rangle$, $j = 1, 2, \dots$, of the α -cut of $\langle \bar{x}_n \rangle$ is convergent for all $\alpha \in]0, 1]$; that is, the upper sequence $\langle x_{1,n,\alpha} \rangle$ of $\langle \bar{x}_n \rangle$ is bounded for all $\alpha \in]0, 1]$ (see [15, Theorem 3.2-c]). By Theorem 1.9, $\langle \bar{x}_n \rangle$ is bounded. \square

Theorem 1.13. Every bounded fuzzy sequence $\langle \bar{x}_n \rangle$ in the fuzzy metric space (\bar{R}, \bar{d}) is convergent.

Proof. Let $\langle \bar{x}_n \rangle$ be a bounded fuzzy sequence in a (\bar{R}, \bar{d}) . By Definition 1.5, we get that the upper sequence $\langle x_{1,n,\alpha} \rangle$ of $\langle \bar{x}_n \rangle$ is bounded for all $\alpha \in]0, 1]$. By Theorem 1.10, $\langle \bar{x}_n \rangle$ is convergent. \square

Theorem 1.14. Let $\langle \bar{x}_n \rangle$ be a fuzzy sequence in a fuzzy metric space (\bar{X}, \bar{d}) ; then $\langle \bar{x}_n \rangle$ converges to $\bar{x} \in \bar{X}$ if and only if every fuzzy neighborhood of \bar{x} contains all but finitely many of the terms of $\langle \bar{x}_n \rangle$.

Proof. Suppose that $\overline{\overline{N_\varepsilon(\bar{x})}}$ is a fuzzy neighborhood of \bar{x} . For $\bar{\varepsilon} > \bar{0}$, $\bar{d}(\bar{y}, \bar{x}) < \bar{\varepsilon}$, $\bar{y} \in \overline{\overline{X}}$, imply $\bar{y} \in \overline{\overline{N_\varepsilon(\bar{x})}}$. We have from the hypothesis $\langle \bar{x}_n \rangle$ converges to \bar{x} , so for same $\bar{\varepsilon} > \bar{0}$, there exists $\bar{N} \in \overline{\overline{N}}$, such that $\bar{n} \geq \bar{N}$, implying $\bar{d}(\bar{x}_n, \bar{x}) < \bar{\varepsilon}$. That is, $\bar{x}_n \in \overline{\overline{N_\varepsilon(\bar{x})}}$.

Conversely, suppose that every fuzzy neighborhood of \bar{x} contains all but finitely many of $\langle \bar{x}_n \rangle$. Fix $\bar{\varepsilon} > \bar{0}$, and let $\overline{\overline{N_\varepsilon(\bar{x})}}$ be the set of all $\bar{y} \in \overline{\overline{X}}$, such that $\bar{d}(\bar{y}, \bar{x}) < \bar{\varepsilon}$.

By assumption, there exists $\bar{N} \in \overline{\overline{N}}$ corresponding to $\overline{\overline{N_\varepsilon(\bar{x})}}$, such that $\bar{x}_n \in \overline{\overline{N_\varepsilon(\bar{x})}}$ if $\bar{n} \geq \bar{N}$. Thus $\bar{d}(\bar{x}_n, \bar{x}) < \bar{\varepsilon}$ if $\bar{n} \geq \bar{N}$. Hence $\langle \bar{x}_n \rangle$ converges to \bar{x} . \square

Theorem 1.15. *In any fuzzy metric space $(\overline{\overline{X}}, \overline{\overline{d}})$, every convergent fuzzy sequence is a Cauchy fuzzy sequence.*

Proof. Let $\langle \bar{x}_n \rangle$ be a convergent fuzzy sequence in a fuzzy metric space $(\overline{\overline{X}}, \overline{\overline{d}})$. By Definition 1.4, every sequence $\langle x_{j,n,\alpha} \rangle$, $j = 1, 2, \dots$, of the α -cut of $\langle \bar{x}_n \rangle$ is convergent for all $\alpha \in]0, 1]$; then every sequence $\langle x_{j,n,\alpha} \rangle$, $j = 1, 2, \dots$, of the α -cut of $\langle \bar{x}_n \rangle$ is a Cauchy sequence for all $\alpha \in]0, 1]$ (see [15, Theorem 3.11-a]); by Definition 1.7, $\langle \bar{x}_n \rangle$ is a Cauchy fuzzy sequence. \square

Theorem 1.16. *Let $\langle \bar{x}_n \rangle$ be a fuzzy sequence in the fuzzy metric space $(\overline{\overline{R}}, \overline{\overline{d}})$. If the upper sequence of the α -cut of $\langle \bar{x}_n \rangle$ is bounded for all $\alpha \in]0, 1]$, then $\langle \bar{x}_n \rangle$ is a Cauchy fuzzy sequence.*

Proof. Suppose that the upper sequence $\langle x_{1,n,\alpha} \rangle$ of the α -cut of $\langle \bar{x}_n \rangle$ is bounded for all $\alpha \in]0, 1]$. By Theorem 1.10, $\langle \bar{x}_n \rangle$ is convergent. By Theorem 1.15, $\langle \bar{x}_n \rangle$ is a Cauchy fuzzy sequence. \square

Theorem 1.17. *In the fuzzy metric space $(\overline{\overline{R}}, \overline{\overline{d}})$, every bounded fuzzy sequence is a Cauchy fuzzy sequence.*

Proof. Let $\langle \bar{x}_n \rangle$ be a bounded fuzzy sequence in $(\overline{\overline{R}}, \overline{\overline{d}})$. By Theorem 1.13, $\langle \bar{x}_n \rangle$ is a convergent fuzzy sequence. By Theorem 1.15, $\langle \bar{x}_n \rangle$ is a Cauchy fuzzy sequence. \square

Theorem 1.18. *Let $\langle \bar{x}_n \rangle$ be a fuzzy sequence in the fuzzy metric space $(\overline{\overline{R}}, \overline{\overline{d}})$. If the upper sequence of the α -cut of $\langle \bar{x}_n \rangle$ is convergent for all $\alpha \in]0, 1]$, then $\langle \bar{x}_n \rangle$ is a bounded fuzzy sequence.*

Proof. Suppose that the upper sequence $\langle x_{1,n,\alpha} \rangle$ of the α -cut of $\langle \bar{x}_n \rangle$ is convergent for all $\alpha \in]0, 1]$. By Theorem 1.11, $\langle \bar{x}_n \rangle$ is a convergent fuzzy sequence. By Theorem 1.12, $\langle \bar{x}_n \rangle$ is a bounded fuzzy sequence. \square

Theorem 1.19. *Let $\langle \bar{x}_n \rangle$ be a fuzzy sequence in the fuzzy metric space $(\overline{\overline{R}}, \overline{\overline{d}})$. If the upper sequence of the α -cut of $\langle \bar{x}_n \rangle$ is convergent for all $\alpha \in]0, 1]$, then $\langle \bar{x}_n \rangle$ is a Cauchy fuzzy sequence.*

Proof. Suppose that the upper sequence $\langle x_{1,n,\alpha} \rangle$ of the α -cut of $\langle \bar{x}_n \rangle$ is convergent for all $\alpha \in]0, 1]$. By Theorem 1.11, $\langle \bar{x}_n \rangle$ is a convergent fuzzy sequence. By Theorem 1.15, $\langle \bar{x}_n \rangle$ is a Cauchy fuzzy sequence. \square

Theorem 1.20. *Every Cauchy fuzzy sequence in $\overline{\overline{R}}^n$, the n -dimensional fuzzy Euclidean space, is convergent.*

Proof. Let $\langle \bar{x}_m \rangle$ be a Cauchy fuzzy sequence in $\overline{\overline{R}}$. By Definitions 1.7 and 1.3, \bar{x}_m consists of all Cauchy sequences at degree α , $\langle x_{m,\alpha} \rangle = \langle x_{1,\alpha}, x_{2,\alpha}, \dots \rangle \in \langle \bar{x}_1, \bar{x}_2, \dots \rangle[\alpha]$ α -cut of $\langle \bar{x}_m \rangle$ for all $\alpha \in]0, 1]$, where every element $x_{i,\alpha} \in \langle x_{m,\alpha} \rangle$ is $(x_{i,1,\alpha}, x_{i,2,\alpha}, \dots, x_{i,n,\alpha}) \in (\bar{x}_{i,1}, \bar{x}_{i,2}, \dots, \bar{x}_{i,n})[\alpha]$ α -cut of $\bar{x}_i = (\bar{x}_{i,1}, \bar{x}_{i,2}, \dots, \bar{x}_{i,n})$ for all $\alpha \in]0, 1]$ (Al-Tai [13]). By Theorem 3.11-c (see [15]), $\langle x_{m,\alpha} \rangle$ is a convergent sequence for all $\alpha \in]0, 1]$. By Definition 1.4, $\langle \bar{x}_m \rangle$ is a convergent fuzzy sequence. \square

Theorem 1.21. *Let $\langle \bar{x}_n \rangle$ be a fuzzy sequence in the fuzzy metric space $(\overline{\overline{R}}, \overline{\overline{d}})$. If the upper sequence of the α -cut of $\langle \bar{x}_n \rangle$ is a Cauchy sequence for all $\alpha \in]0, 1]$; then $\langle \bar{x}_n \rangle$ is convergent.*

Proof. Suppose that the upper sequence $\langle x_{1,n,\alpha} \rangle$ of the α -cut of $\langle \bar{x}_n \rangle$ is a Cauchy sequence for all $\alpha \in]0, 1]$, so it is convergent (see [15, Theorem 3.11-c]). By Theorem 1.11, $\langle \bar{x}_n \rangle$ is convergent. \square

Theorem 1.22. *Every Cauchy fuzzy sequence in the fuzzy metric space $(\overline{\overline{R}}, \overline{\overline{d}})$ is bounded.*

Proof. Let $\langle \bar{x}_n \rangle$ be a Cauchy fuzzy sequence in $(\overline{\overline{R}}, \overline{\overline{d}})$. By Theorem 1.20, $\langle \bar{x}_n \rangle$ is a convergent fuzzy sequence. By Theorem 1.12, $\langle \bar{x}_n \rangle$ is bounded. \square

Theorem 1.23. *Let $\langle \bar{x}_n \rangle$ be a fuzzy sequence in the fuzzy metric space $(\overline{\overline{R}}, \overline{\overline{d}})$. If the upper sequence of the α -cut of $\langle \bar{x}_n \rangle$ is a Cauchy sequence for all $\alpha \in]0, 1]$, then $\langle \bar{x}_n \rangle$ is bounded.*

Proof. Suppose that the upper sequence $\langle x_{1,n,\alpha} \rangle$ of the α -cut of $\langle \bar{x}_n \rangle$ is a Cauchy sequence for all $\alpha \in]0, 1]$. By Theorem 1.21, $\langle \bar{x}_n \rangle$ is convergent. By Theorem 1.12, $\langle \bar{x}_n \rangle$ is bounded. \square

Theorem 1.24. *Let $\langle \bar{x}_n \rangle$ be a fuzzy sequence in the fuzzy metric space $(\overline{\overline{R}}, \overline{\overline{d}})$. If the upper sequence of the α -cut of $\langle \bar{x}_n \rangle$ is a Cauchy sequence for all $\alpha \in]0, 1]$, then $\langle \bar{x}_n \rangle$ is Cauchy.*

Proof. Suppose that the upper sequence $\langle x_{1,n,\alpha} \rangle$ of the α -cut of $\langle \bar{x}_n \rangle$ is a Cauchy sequence for all $\alpha \in]0, 1]$; then it is convergent (see [15, Theorem 3.11-c]). By Theorem 1.19, $\langle \bar{x}_n \rangle$ is Cauchy. \square

Theorem 1.25. *The limit fuzzy number of a convergent fuzzy sequence $\langle \bar{x}_n \rangle$ in a fuzzy metric space $(\overline{\overline{X}}, \overline{\overline{d}})$ is unique.*

Proof. Let $\langle \bar{x}_n \rangle$ be a fuzzy sequence in a fuzzy metric space $(\overline{\overline{X}}, \overline{\overline{d}})$ which converges to $\bar{x} \in (\overline{\overline{X}}, \overline{\overline{d}})$. By Definition 4 and Theorem 3.2-b [15], we have that every sequence $\langle x_{n,\alpha} \rangle$ of the α -cut of $\langle \bar{x}_n \rangle$ has a unique limit point $x_\alpha \in \bar{x}[\alpha]$ α -cut of \bar{x} for all $\alpha \in]0, 1]$, and hence, $\langle \bar{x}_n \rangle$ converges to a unique limit fuzzy number \bar{x} . \square

Theorem 1.26. *If $\langle \bar{y}_n \rangle$ is a convergent fuzzy sequence in the fuzzy metric space $(\overline{\overline{R}}, \overline{\overline{d}})$, then a fuzzy sequence $\langle \bar{x}_n \rangle$ is convergent for all $\bar{n} \geq \bar{N}$, as $\bar{0} \leq \langle \bar{x}_n \rangle \leq \langle \bar{y}_n \rangle$.*

Proof. Let $\langle y_{1,n,\alpha} \rangle$ be the upper sequence of the α -cut of $\langle \bar{y}_n \rangle$ for all $\alpha \in]0, 1]$. Since $\bar{0} \leq \langle \bar{x}_n \rangle \leq \langle \bar{y}_n \rangle$, then $0 \leq \langle x_{1,n,\alpha} \rangle \leq \langle y_{1,n,\alpha} \rangle$ [16], where $\langle x_{1,n,\alpha} \rangle$ is an upper sequence of the α -cut of $\langle \bar{x}_n \rangle$ for all $\alpha \in]0, 1]$. By Theorem 7.4.4 (Comparison Test) [17], $\langle x_{1,n,\alpha} \rangle$ is convergent for all $\alpha \in]0, 1]$. By Theorem 1.11, $\langle \bar{x}_n \rangle$ is convergent. \square

2. Conclusion

Fuzzy neighborhood, limit fuzzy point, convergent fuzzy sequence, bounded fuzzy sequence, and Cauchy fuzzy sequence can be defined on the dependent base by Al-Tai, these concepts help us to show that the fuzzy sequence in $(\overline{\overline{R}}, \overline{\overline{d}})$ is convergent, if an upper sequence of the α -cut of it is convergent, bounded, or Cauchy for all $\alpha \in]0, 1]$. The fuzzy sequence in $(\overline{\overline{R}}, \overline{\overline{d}})$ is bounded, if an upper sequence of the α -cut of it is convergent, bounded, or Cauchy for all $\alpha \in]0, 1]$. The fuzzy sequence in $(\overline{\overline{R}}, \overline{\overline{d}})$ is Cauchy, if an upper sequence of the α -cut of it is convergent, bounded, or, Cauchy for all $\alpha \in]0, 1]$. The limit fuzzy number of a convergent fuzzy sequence is unique. And other results about above concepts.

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