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Research Article

Strong Convergence Algorithms for Hierarchical Fixed Points Problems and Variational Inequalities

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We introduce a new iterative scheme that converges strongly to a common fixed point of a countable family of nonexpansive mappings in a Hilbert space such that the common fixed point is a solution of a hierarchical fixed point problem. Our results extend the ones of Moudafi, Xu, Cianciaruso et al., and Yao et al.

1. Introduction

Let H be a real Hilbert space and C a nonempty closed convex subset of H. A mapping $T : C \to C$ is called nonexpansive if one has

$$\|Tx - Ty\| \le \|x - y\|, \quad \forall x, y \in C. \tag{1.1}$$

If there exists a point $x \in C$ such that $x = Tx$, then x is said to be a fixed point of T. We denote the set of all fixed points of T by $F(T)$. It is well known that $F(T)$ is closed and convex if T is nonexpansive.

Let $S : C \to H$ be a mapping. The following problem is called a hierarchical fixed point problem: find $x^* \in F(T)$ such that

$$\langle x^* - Sx^*, x^* - x \rangle \le 0, \quad \forall x \in F(T). \tag{1.2}$$

It is known that the hierarchical fixed point problem (1.2) links with some monotone variational inequalities and convex programming problems (see [1]).

In order to solve the hierarchical fixed point problem (1.2), Moudafi [2] introduced the following Krasnoselski-Mann algorithm:

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n(\sigma_n S x_n + (1-\sigma_n)Tx_n), \quad \forall n \geq 0, \tag{1.3}$$

where $\{\alpha_n\}$ and $\{\sigma_n\}$ are two sequences in $(0,1)$, and he proved that $\{x_n\}$ converges weakly to a fixed point of T which is a solution of the problem (1.2).

Let $f : C \to C$ be a mapping. The mapping f is called a contraction if there exists a constant $\lambda \in [0,1)$ such that $\|fx - fy\| \leq \lambda\|x-y\|$ for all $x, y \in C$. For obtaining a strong convergence result, Mainge and Moudafi in [3] and Marino and Xu in [4] introduced the following algorithm:

$$x_{n+1} = (1-\alpha_n)f(x_n) + \alpha_n(\sigma_n S x_n + (1-\sigma_n)Tx_n), \quad \forall n \geq 0, \tag{1.4}$$

where $S : C \to C$ is a nonexpansive mapping and $\{\alpha_n\}$ and $\{\sigma_n\}$ are two sequences in $(0,1)$, and they proved that $\{x_n\}$ converges strongly to a fixed point of T which is a solution of the problem (1.2). Recently, for solving the hierarchical fixed point problem (1.2), Cianciaruso et al. [5] also studied the following iterative scheme:

$$\begin{aligned} y_n &= \beta_n S x_n + (1-\beta_n)x_n, \\ x_{n+1} &= \alpha_n f(x_n) + (1-\alpha_n)Ty_n, \quad \forall n \geq 0, \end{aligned} \tag{1.5}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences in $[0,1]$. The authors proved some strong convergence results. Very recently, Yao et al. [1] introduced the following strong convergence iterative algorithm to solve the problem (1.2):

$$\begin{aligned} y_n &= \beta_n S x_n + (1-\beta_n)x_n, \\ x_{n+1} &= P_C\left[\alpha_n f(x_n) + (1-\alpha_n)Ty_n\right], \quad \forall n \geq 0, \end{aligned} \tag{1.6}$$

where $f : C \to H$ is a contraction and $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences in $(0,1)$. Under some certain restrictions on parameters, the authors proved that the sequence $\{x_n\}$ generated by (1.6) converges strongly to $z \in F(T)$, which is the unique solution of the following variational inequality:

$$\langle (I-f)z, x-z\rangle \geq 0, \quad \forall x \in F(T). \tag{1.7}$$

By changing the restrictions on parameters, the authors obtained another result on the iterative scheme (1.6), that is, the sequence $\{x_n\}$ generated by (1.6)converges strongly to a point $x^* \in F(T)$, which is the unique solution of the following variational inequality:

$$\left\langle \frac{1}{\tau}(I-f)x^* + (I-S)x^*, y - x^* \right\rangle \geq 0, \quad \forall y \in F(T), \tag{1.8}$$

where $\tau \in (0,\infty)$ is a constant.

Let $S : C \to H$ be a nonexpansive mapping and $\{T_i\}_{i=1}^{\infty} : C \to C$ a countable family of nonexpansive mappings. In this paper, motivated and inspired by the results of Yao et al. [1] and Marino and Xu [4], we introduce and study the following iterative scheme:

$$\begin{aligned} y_n &= P_C\left[\beta_n S x_n + (1-\beta_n)x_n\right], \\ x_{n+1} &= P_C\left[\alpha_n f(x_n) + \sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)T_i y_n\right], \quad \forall n \geq 1, \end{aligned} \tag{1.9}$$

where $\alpha_0 = 1$, $\{\alpha_n\}$ is a strictly decreasing sequence in $(0,1)$ and $\{\beta_n\}$ is a sequence in $(0,1)$. Under some certain conditions on parameters, we first prove that the sequence $\{x_n\}$ generated by (1.9) converges strongly to $x^* \in \bigcap_{i=1}^{\infty} F(T_i)$, which is the unique solution of the following variational inequality:

$$\langle (I-f)x^*, x - x^* \rangle \geq 0, \quad \forall x \in \bigcap_{i=1}^{\infty} F(T_i). \tag{1.10}$$

By changing the restrictions on parameters, we also prove that the sequence $\{x_n\}$ converges strongly to $x^* \in \cap_{i=1}^{\infty} F(T_i)$, which is the unique solution of the following variational inequality:

$$\left\langle \frac{1}{\tau}(I-f)x^* + (I-S)x^*, y - x^* \right\rangle \geq 0, \quad \forall y \in \bigcap_{i=1}^{\infty} F(T_i), \tag{1.11}$$

where $\tau \in (0,\infty)$ is a constant. It is easy to see that, if $T_i = T$ for each $i \geq 1$ and S is a self-mapping of C into itself, then our algorithm (1.9) is reduced to (1.6) of Yao et al. [1] Also, our results extend the corresponding ones of Moudafi [6], Xu [7], and Cianciaruso et al. [5].

2. Preliminaries

Let H be a Hilbert space and C a nonempty closed convex subset of H. Let T be a nonexpansive mapping of C into itself such that $F(T) \neq \emptyset$. For all $\hat{x} \in F(T)$ and all $x \in C$, we have

$$\begin{aligned} \|x-\hat{x}\|^2 &\geq \|Tx - T\hat{x}\|^2 = \|Tx-\hat{x}\|^2 = \|Tx - x + (x-\hat{x})\|^2 \\ &= \|Tx-x\|^2 + \|x-\hat{x}\|^2 + 2\langle Tx - x, x - \hat{x}\rangle \end{aligned} \tag{2.1}$$

and hence

$$\|Tx - x\|^2 \leq 2\langle x - Tx, x - \hat{x}\rangle, \quad \forall \hat{x} \in F(T),\ x \in C. \tag{2.2}$$

Let $x \in H$ be an arbitrary point. There exists a unique nearest point in C, denoted by $P_C x$, such that

$$\|P_C x - x\| \leq \|y - x\|, \quad \forall y \in C. \tag{2.3}$$

Moreover, we have the following:

$$z = P_C x \Longleftrightarrow \langle x - z, z - y\rangle \geq 0, \quad \forall y \in C. \tag{2.4}$$

Let I denote the identity operator of H, and let $\{x_n\}$ be a sequence in a Hilbert space H and $x \in H$. Throughout this paper, $x_n \to x$ denotes that $\{x_n\}$ strongly converges to x and $x_n \rightharpoonup x$ denotes that $\{x_n\}$ weakly converges to x.

The following lemmas will be used in the next section.

Lemma 2.1 (see [8]). *Let $T : C \to C$ be a nonexpansive mapping with $F(T) \neq \emptyset$. If $\{x_n\}$ is a sequence in C weakly converging to a point $x \in C$ and $\{(I - T)x_n\}$ converges strongly to a point $y \in C$, then $(I - T)x = y$. In particular, if $y = 0$, then $x \in F(T)$.*

Lemma 2.2 (see [9]). *Let $f : C \to H$ be a contraction with coefficient $\lambda \in [0, 1)$ and $T : C \to C$ a nonexpansive mapping. Then one has the following.*

(1) *The mapping $(I - f)$ is strongly monotone with coefficient $(1 - \lambda)$, that is,*

$$\langle x - y, (I - f)x - (I - f)y\rangle \geq (1 - \lambda)\|x - y\|^2, \quad \forall x, y \in C. \tag{2.5}$$

(2) *The mapping $I - T$ is monotone, that is,*

$$\langle x - y, (I - T)x - (I - T)y\rangle \geq 0, \quad \forall x, y \in C. \tag{2.6}$$

Lemma 2.3 (see [10]). *Let $\{s_n\}$, $\{c_n\}$ be the sequences of nonnegative real numbers, and let $\{a_n\} \subset (0, 1)$. Suppose that $\{b_n\}$ is a real number sequence such that*

$$s_{n+1} \leq (1 - a_n)s_n + b_n + c_n, \quad \forall n \geq 0. \tag{2.7}$$

Assume that $\sum_{n=0}^{\infty} c_n < \infty$. Then the following results hold.

(1) *If $b_n \leq \beta a_n$, where $\beta \geq 0$, then $\{s_n\}$ is a bounded sequence.*

(2) *If one has*

$$\sum_{n=0}^{\infty} a_n = \infty, \qquad \limsup_{n \to \infty} \frac{b_n}{a_n} \leq 0, \tag{2.8}$$

then $\lim_{n \to \infty} s_n = 0$.

3. Main Results

Now, we give the main results in this paper.

Theorem 3.1. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $f : C \to H$ be a λ-contraction with $\lambda \in [0,1)$. Let $S : C \to H$ be a nonexpansive mapping and $\{T_i\}_{i=1}^{\infty}$ a countable family of nonexpansive mappings of C into itself such that $F = \bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$. Set $\alpha_0 = 1$, and let $\{\alpha_n\} \subset (0,1)$ be a strictly decreasing sequence and $\{\beta_n\} \subset (0,1)$ a sequence satisfying the following conditions:*

(a) $\lim_{n\to\infty}\alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(b) $\lim_{n\to\infty}(\beta_n/\alpha_n) = 0$,

(c) $\sum_{n=1}^{\infty}(\alpha_{n-1} - \alpha_n) < \infty$ *and* $\sum_{n=1}^{\infty} |\beta_{n-1} - \beta_n| < \infty$.

Then the sequence $\{x_n\}$ generated by (1.9) *converges strongly to a point $x^* \in F$, which is the unique solution of the variational inequality*

$$\langle (I - f)x^*, x - x^* \rangle \geq 0, \quad \forall x \in F. \tag{3.1}$$

Proof. First, $P_F f$ is a contraction from C into itself with a constant λ and C is complete, and there exists a unique $x^* \in C$ such that $x^* = P_F f(x^*)$. From (2.4), it follows that x^* is the unique solution of the problem (3.1).

Now, we prove that $\{x_n\}$ converges strongly to x^*. To this end, we first prove that $\{x_n\}$ is bounded. Take $p \in F$. Then it follows from (1.9) that

$$\begin{aligned} \|y_n - p\| &= \|P_C[\beta_n S x_n + (1-\beta_n)x_n] - P_C p\| \\ &\leq \|\beta_n S x_n + (1-\beta_n)x_n - p\| \\ &\leq \beta_n\|Sx_n - Sp\| + \beta_n\|Sp - p\| + (1-\beta_n)\|x_n - p\| \\ &\leq \beta_n\|x_n - p\| + \beta_n\|Sp - p\| + (1-\beta_n)\|x_n - p\| \\ &= \|x_n - p\| + \beta_n\|Sp - p\| \end{aligned} \tag{3.2}$$

and hence (note that $\{\alpha_n\}$ is strictly decreasing)

$$\begin{aligned} \|x_{n+1} - p\| &= \left\| P_C\left[\alpha_n f(x_n) + \sum_{i=1}^{n}(\alpha_{i-1} - \alpha_i)T_i y_n\right] - P_C p \right\| \\ &\leq \left\| \alpha_n f(x_n) + \sum_{i=1}^{n}(\alpha_{i-1} - \alpha_i)T_i y_n - p \right\| \\ &= \left\| \alpha_n(f(x_n) - p) + \sum_{i=1}^{n}(\alpha_{i-1} - \alpha_i)(T_i y_n - p) \right\| \end{aligned}$$

$$\begin{aligned}
&\leq \alpha_n\|f(x_n)-f(p)\|+\alpha_n\|f(p)-p\|+\sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)\|T_iy_n-p\| \\
&\leq \alpha_n\lambda\|x_n-p\|+\alpha_n\|f(p)-p\|+\sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)\|y_n-p\| \\
&\leq \alpha_n\lambda\|x_n-p\|+\alpha_n\|f(p)-p\|+\sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)(\|x_n-p\|+\beta_n\|Sp-p\|) \\
&= \alpha_n\lambda\|x_n-p\|+\alpha_n\|f(p)-p\|+(1-\alpha_n)(\|x_n-p\|+\beta_n\|Sp-p\|).
\end{aligned} \tag{3.3}$$

By condition (b), we can assume that $\beta_n \leq \alpha_n$ for all $n \geq 1$. Hence, from above inequality, we get

$$\begin{aligned}
\|x_{n+1}-p\| &\leq \alpha_n\lambda\|x_n-p\|+\alpha_n\|f(p)-p\|+(1-\alpha_n)(\|x_n-p\|+\alpha_n\|Sp-p\|) \\
&\leq \alpha_n\lambda\|x_n-p\|+\alpha_n\|f(p)-p\|+(1-\alpha_n)\|x_n-p\|+\alpha_n\|Sp-p\| \\
&= [1-\alpha_n(1-\lambda)]\|x_n-p\|+\alpha_n(\|f(p)-p\|+\|Sp-p\|).
\end{aligned} \tag{3.4}$$

For each $n \geq 1$, let $a_n = \alpha_n(1-\lambda)$, $b_n = \alpha_n(\|f(p)-p\|+\|Sp-p\|)$, and $c_n = 0$. Then $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ satisfy the condition of Lemma 2.3(1). Hence, it follows from Lemma 2.3(1) that $\{x_n\}$ is bounded and so are $\{f(x_n)\}$, $\{y_n\}$, $\{T_ix_n\}$, and $\{T_iy_n\}$ for all $i \geq 1$. Set $u_n = \alpha_n f(x_n)+\sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)T_iy_n$ for each $n \geq 1$. From (1.9), we have

$$\begin{aligned}
\|x_{n+1}-x_n\| &= \|P_Cu_n-P_Cu_{n-1}\| \leq \|u_n-u_{n-1}\| \\
&= \Bigg\|\alpha_n(f(x_n)-f(x_{n-1}))+(\alpha_n-\alpha_{n-1})f(x_{n-1}) \\
&\quad +\sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)(T_iy_n-T_iy_{n-1})+(\alpha_{n-1}-\alpha_n)T_ny_{n-1}\Bigg\| \\
&\leq \alpha_n\|f(x_n)-f(x_{n-1})\|+\sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)\|y_n-y_{n-1}\| \\
&\quad +(\alpha_{n-1}-\alpha_n)(\|f(x_{n-1})\|+\|T_ny_{n-1}\|) \\
&\leq \alpha_n\lambda\|x_n-x_{n-1}\|+\sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)\|y_n-y_{n-1}\|+(\alpha_{n-1}-\alpha_n)M \\
&= \alpha_n\lambda\|x_n-x_{n-1}\|+(1-\alpha_n)\|y_n-y_{n-1}\|+(\alpha_{n-1}-\alpha_n)M,
\end{aligned} \tag{3.5}$$

where M is a constant such that

$$\sup_{n\geq 1}\{\|f(x_{n-1})\|+\|T_ny_{n-1}\|+\|Sx_{n-1}\|+\|x_{n-1}\|\} \leq M. \tag{3.6}$$

From (1.9), we have

$$\begin{aligned}\|y_n - y_{n-1}\| &= \|P_C(\beta_n S x_n + (1-\beta_n)x_n) - P_C(\beta_{n-1} S x_{n-1} + (1-\beta_{n-1})x_{n-1})\| \\ &\le \|(\beta_n S x_n + (1-\beta_n)x_n) - (\beta_{n-1} S x_{n-1} + (1-\beta_{n-1})x_{n-1})\| \\ &= \|\beta_n(Sx_n - Sx_{n-1}) + (\beta_n - \beta_{n-1})Sx_{n-1} + (1-\beta_n)(x_n - x_{n-1}) + (\beta_{n-1} - \beta_n)x_{n-1}\| \\ &\le \|x_n - x_{n-1}\| + |\beta_n - \beta_{n-1}|M.\end{aligned} \tag{3.7}$$

Substituting (3.7) into (3.5), we get that

$$\begin{aligned}\|x_{n+1} - x_n\| &\le \|u_n - u_{n-1}\| \\ &\le \alpha_n\lambda\|x_n - x_{n-1}\| + (1-\alpha_n)\left[\|x_n - x_{n-1}\| + |\beta_n - \beta_{n-1}|M\right] + (\alpha_{n-1} - \alpha_n)M \\ &\le (1-(1-\lambda)\alpha_n)\|x_n - x_{n-1}\| + M\left[(\alpha_{n-1} - \alpha_n) + |\beta_n - \beta_{n-1}|\right].\end{aligned} \tag{3.8}$$

Let $a_n = (1-\lambda)\alpha_n$, $b_n = 0$, and $c_n = (\alpha_{n-1} - \alpha_n) + |\beta_n - \beta_{n-1}|$. Then conditions (a) and (c) imply that $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ satisfy the condition of Lemma 2.3(2). Thus, by Lemma 2.3(2), we can conclude that

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0. \tag{3.9}$$

Since $T_i x_n \in C$ for each $i \ge 1$ and $\sum_{i=1}^{n}(\alpha_{i-1} - \alpha_i) + \alpha_n = 1$, we have

$$\sum_{i=1}^{n}(\alpha_{i-1} - \alpha_i)T_i x_n + \alpha_n z \in C, \quad \forall z \in C. \tag{3.10}$$

Now, fixing a $z \in F$, from (1.9) we have

$$\begin{aligned}&\sum_{i=1}^{n}(\alpha_{i-1} - \alpha_i)(x_n - T_i x_n) \\ &\quad= P_C u_n + (1-\alpha_n)x_n - \left(\sum_{i=1}^{n}(\alpha_{i-1} - \alpha_i)T_i x_n + \alpha_n z\right) + \alpha_n z - x_{n+1} \\ &\quad= P_C u_n - P_C\left[\sum_{i=1}^{n}(\alpha_{i-1} - \alpha_i)T_i x_n + \alpha_n z\right] + (1-\alpha_n)(x_n - x_{n+1}) + \alpha_n(z - x_{n+1}).\end{aligned} \tag{3.11}$$

Hence it follows that

$$\begin{aligned}
&\sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)\langle x_n - T_i x_n, x_n - p\rangle \\
&\quad = \left\langle P_C u_n - P_C\left[\sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)T_i x_n + \alpha_n z\right], x_n - p\right\rangle \\
&\qquad + \langle (1-\alpha_n)(x_n - x_{n+1}), x_n - p\rangle + \alpha_n\langle z - x_{n+1}, x_n - p\rangle \\
&\quad \le \left\| u_n - \sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)T_i x_n - \alpha_n z\right\| \|x_n - p\| + (1-\alpha_n)\|x_n - x_{n+1}\|\|x_n - p\| \\
&\qquad + \alpha_n\|z - x_{n+1}\|\|x_n - p\| \\
&\quad = \left\| \alpha_n(f(x_n) - z) + \sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)(T_i y_n - T_i x_n)\right\| \|x_n - p\| \\
&\qquad + (1-\alpha_n)\|x_n - x_{n+1}\|\|x_n - p\| + \alpha_n\|z - x_{n+1}\|\|x_n - p\| \\
&\quad \le \alpha_n\|f(x_n) - z\|\|x_n - p\| + \sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)\|y_n - x_n\|\|x_n - p\| \\
&\qquad + (1-\alpha_n)\|x_n - x_{n+1}\|\|x_n - p\| + \alpha_n\|z - x_{n+1}\|\|x_n - p\| \\
&\quad \le \alpha_n\|f(x_n) - z\|\|x_n - p\| + \sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)\beta_n\|Sx_n - x_n\|\|x_n - p\| \\
&\qquad + (1-\alpha_n)\|x_n - x_{n+1}\|\|x_n - p\| + \alpha_n\|z - x_{n+1}\|\|x_n - p\| \\
&\quad = \alpha_n\|f(x_n) - z\|\|x_n - p\| + (1-\alpha_n)\beta_n\|Sx_n - x_n\|\|x_n - p\| \\
&\qquad + (1-\alpha_n)\|x_n - x_{n+1}\|\|x_n - p\| + \alpha_n\|z - x_{n+1}\|\|x_n - p\| \\
&\quad \le (2\alpha_n + \beta_n)M' + (1-\alpha_n)\|x_n - x_{n+1}\|\|x_n - p\|,
\end{aligned} \tag{3.12}$$

where

$$M' = \sup_{n\ge 1}\{\|f(x_n) - z\|\|x_n - p\|, \|Sx_n - x_n\|\|x_n - p\|, \|z - x_{n+1}\|\|x_n - p\|\}. \tag{3.13}$$

Now, from (2.2) and (3.12), we get

$$\begin{aligned}
\frac{1}{2}\sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)\|x_n - T_i x_n\|^2 &\le \sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)\langle x_n - T_i x_n, x_n - p\rangle \\
&\le (2\alpha_n + \beta_n)M' + (1-\alpha_n)\|x_n - x_{n+1}\|\|x_n - p\|.
\end{aligned} \tag{3.14}$$

From (a), (b), (3.9), and (3.14), we have

$$\lim_{n\to\infty}\sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)\|x_n-T_ix_n\|=0. \tag{3.15}$$

Since $(\alpha_{i-1}-\alpha_i)\|x_n-T_ix_n\| \le \sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)\|x_n-T_ix_n\|$ for each $i\ge 1$ and $\{\alpha_n\}$ is strictly decreasing, one has

$$\lim_{n\to\infty}\|x_n-T_ix_n\|=0, \quad \forall i\ge 1. \tag{3.16}$$

Next, we prove that

$$\limsup_{n\to\infty}\langle f(x^*)-x^*, x_n-x^*\rangle \le 0. \tag{3.17}$$

Since $\{x_n\}$ is bounded, we can take a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k}\rightharpoonup x'$ and

$$\limsup_{n\to\infty}\langle f(x^*)-x^*, x_n-x^*\rangle = \lim_{k\to\infty}\langle f(x^*)-x^*, x_{n_k}-x^*\rangle. \tag{3.18}$$

From (3.16) and Lemma 2.1, we conclude that $x'\in F(T_i)$ for each $i\ge 1$, that is, $x'\in\bigcap_{i=1}^{\infty}F(T_i)$. Then

$$\lim_{k\to\infty}\langle f(x^*)-x^*, x_{n_k}-x^*\rangle = \langle f(x^*)-x^*, x'-x^*\rangle \le 0. \tag{3.19}$$

By (2.4), we have

$$\langle P_Cu_n-u_n, P_Cu_n-x^*\rangle \le 0. \tag{3.20}$$

Also,

$$\begin{aligned}
&\sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)\langle T_iy_n-x^*, x_{n+1}-x^*\rangle \\
&\quad\le \sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)\|T_iy_n-x^*\|\|x_{n+1}-x^*\| \\
&\quad\le \sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)\|y_n-x^*\|\|x_{n+1}-x^*\| \\
&\quad= (1-\alpha_n)\|y_n-x^*\|\|x_{n+1}-x^*\| \\
&\quad= (1-\alpha_n)\|P_C(\beta_nSx_n+(1-\beta_n)x_n)-P_Cx^*\|\|x_{n+1}-x^*\| \\
&\quad\le (1-\alpha_n)\|\beta_nSx_n+(1-\beta_n)x_n-x^*\|\|x_{n+1}-x^*\|
\end{aligned}$$

$$
\begin{aligned}
&= (1-\alpha_n)\left\|\beta_n(Sx_n - Sx^*) + \beta_n(Sx^* - x^*) + \left(1-\beta_n\right)(x_n - x^*)\right\|\|x_{n+1} - x^*\| \\
&\le (1-\alpha_n)\left[\beta_n\|x_n - x^*\| + \beta_n\|Sx^* - x^*\| + \left(1-\beta_n\right)\|x_n - x^*\|\right]\|x_{n+1} - x^*\| \\
&= (1-\alpha_n)\|x_n - x^*\|\|x_{n+1} - x^*\| + (1-\alpha_n)\beta_n\|Sx^* - x^*\|\|x_{n+1} - x^*\|.
\end{aligned}
\tag{3.21}
$$

Thus,

$$
\begin{aligned}
\|x_{n+1} - x^*\|^2 &= \langle P_C u_n - u_n, P_C u_n - x^*\rangle + \langle u_n - x^*, x_{n+1} - x^*\rangle \\
&\le \langle u_n - x^*, x_{n+1} - x^*\rangle = \alpha_n\left\langle f(x_n) - f(x^*), x_{n+1} - x^*\right\rangle \\
&\quad + \sum_{i=1}^{n}(\alpha_{i-1} - \alpha_i)\left\langle T_i y_n - x^*, x_{n+1} - x^*\right\rangle + \alpha_n\left\langle f(x^*) - x^*, x_{n+1} - x^*\right\rangle \\
&\le \alpha_n\lambda\|x_n - x^*\|\|x_{n+1} - x^*\| + (1-\alpha_n)\|x_n - x^*\|\|x_{n+1} - x^*\| \\
&\quad + (1-\alpha_n)\beta_n\|Sx^* - x^*\|\|x_{n+1} - x^*\| + \alpha_n\langle f(x^*) - x^*, x_{n+1} - x^*\rangle \\
&= [1 - \alpha_n(1-\lambda)]\|x_n - x^*\|\|x_{n+1} - x^*\| + (1-\alpha_n)\beta_n\|Sx^* - x^*\|\|x_{n+1} - x^*\| \\
&\quad + \alpha_n\left\langle f(x^*) - x^*, x_{n+1} - x^*\right\rangle \\
&\le \frac{1 - \alpha_n(1-\lambda)}{2}\left[\|x_n - x^*\|^2 + \|x_{n+1} - x^*\|^2\right] + (1-\alpha_n)\beta_n\|Sx^* - x^*\|\|x_{n+1} - x^*\| \\
&\quad + \alpha_n\left\langle f(x^*) - x^*, x_{n+1} - x^*\right\rangle.
\end{aligned}
\tag{3.22}
$$

It follows that

$$
\begin{aligned}
\|x_{n+1} - x^*\|^2 &\le \left[1 - \frac{2(1-\lambda)\alpha_n}{1 + (1-\lambda)\alpha_n}\right]\|x_n - x^*\|^2 + \frac{2(1-\alpha_n)\beta_n}{1 + (1-\lambda)\alpha_n}\|Sx^* - x^*\|\|x_{n+1} - x^*\| \\
&\quad + \frac{2\alpha_n}{1 + (1-\lambda)\alpha_n}\left\langle f(x^*) - x^*, x_{n+1} - x^*\right\rangle \\
&= \left[1 - \frac{2(1-\lambda)\alpha_n}{1 + (1-\lambda)\alpha_n}\right]\|x_n - x^*\|^2 + \frac{2(1-\lambda)\alpha_n}{1 + (1-\lambda)\alpha_n} \\
&\quad \times \left\{\frac{(1-\alpha_n)\beta_n}{(1-\lambda)\alpha_n}\|Sx^* - x^*\|\|x_{n+1} - x^*\| + \frac{1}{1-\lambda}\left\langle f(x^*) - x^*, x_{n+1} - x^*\right\rangle\right\}.
\end{aligned}
\tag{3.23}
$$

Let $a_n = 2(1-\lambda)\alpha_n/(1+(1-\lambda))\alpha_n$, $b_n = (2(1-\lambda)\alpha_n/(1+(1-\lambda)\alpha_n))\{(1-\alpha_n)\beta_n/(1-\lambda)\alpha_n\|Sx^* - x^*\|\|x_{n+1} - x^*\| + (1/1-\lambda)\langle f(x^*) - x^*, x_{n+1} - x^*\rangle\}$, and $c_n = 0$ for all $n \ge 1$. Since

$$
\limsup_{n\to\infty}\left\{\frac{(1-\alpha_n)\beta_n}{(1-\lambda)\alpha_n}\|Sx^* - x^*\|\|x_{n+1} - x^*\| + \frac{1}{1-\lambda}\left\langle f(x^*) - x^*, x_{n+1} - x^*\right\rangle\right\} \le 0,
\tag{3.24}
$$

$\sum_{n=1}^{\infty} \alpha_n = \infty$, and $2(1-\lambda)\alpha_n/(1+(1-\lambda)\alpha_n) \geq (1-\lambda)\alpha_n$, we have

$$\sum_{n=1}^{\infty} a_n = \infty, \qquad \limsup_{n\to\infty} \frac{b_n}{a_n} = 0, \qquad \sum_{n=1}^{\infty} c_n = 0. \tag{3.25}$$

Therefore, it follows from Lemma 2.3(2) that

$$\lim_{n\to\infty} \|x_n - x^*\| = 0. \tag{3.26}$$

This completes the proof. □

Remark 3.2. In (1.9), if $f = 0$, then it follows that $x_n \to x^* = P_F 0$. In this case, from (3.1), it follows that

$$\langle x^*, x^* - x \rangle \leq 0, \quad \forall x^* \in F, \tag{3.27}$$

that is,

$$\|x^*\|^2 \leq \langle x^*, x \rangle \leq \|x^*\| \|x\|, \quad \forall x \in F. \tag{3.28}$$

Therefore, the point x^* is the unique solution to the quadratic minimization problem

$$x^* = \arg\min_{x\in F} \|x\|^2. \tag{3.29}$$

In Theorem 3.1, if $T_i = T$ for all $i \geq 1$, then we get the following result.

Corollary 3.3. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $f : C \to H$ be a λ-contraction with $\lambda \in [0,1)$. Let $S : C \to H$ be a nonexpansive mapping and $T : C \to C$ a nonexpansive mapping such that $F(T) \neq \emptyset$. Let $x_1 \in C$ and define a sequence $\{x_n\}$ by*

$$\begin{aligned} y_n &= P_C[\beta_n S x_n + (1-\beta_n)x_n], \\ x_{n+1} &= P_C[\alpha_n f(x_n) + (1-\alpha_n)T y_n], \quad \forall n \geq 1, \end{aligned} \tag{3.30}$$

where $\{\alpha_n\} \subset (0,1)$ and $\{\beta_n\} \subset (0,1)$ are two sequences satisfying the following conditions:

(a) $\lim_{n\to\infty} \alpha_n = 0$, *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(b) $\lim_{n\to\infty} (\beta_n/\alpha_n) = 0$,

(c) $\sum_{n=1}^{\infty} |\alpha_{n-1} - \alpha_n| < \infty$ *and* $\sum_{n=1}^{\infty} |\beta_{n-1} - \beta_n| < \infty$.

Then the sequence $\{x_n\}$ converges strongly to a point $x^* \in F(T)$, which is the unique solution of the variational inequality

$$\langle (I-f)x^*, x - x^* \rangle \geq 0, \quad \forall x \in F(T). \tag{3.31}$$

In Corollary 3.3, if S is a self-mapping of C into itself, then we get the following result.

Corollary 3.4. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $f : C \to H$ be a λ-contraction with $\lambda \in [0,1)$. Let $S : C \to C$ be a nonexpansive mapping and $T : C \to C$ a nonexpansive mapping such that $F(T) \neq \emptyset$. Let $x_1 \in C$ and define a sequence $\{x_n\}$ by*

$$y_n = \beta_n S x_n + (1-\beta_n) x_n,$$
$$x_{n+1} = P_C\left[\alpha_n f(x_n) + (1-\alpha_n) T x_n\right], \quad \forall n \geq 1, \tag{3.32}$$

where the sequences $\{\alpha_n\} \subset (0,1)$ and $\{\beta_n\} \subset (0,1)$ are two sequences satisfying the following conditions:

(a) $\lim_{n\to\infty}\alpha_n = 0$ *and* $\sum_{n=1}^{\infty}\alpha_n = \infty$,

(b) $\lim_{n\to\infty}(\beta_n/\alpha_n) = 0$,

(c) $\sum_{n=1}^{\infty}|\alpha_{n-1} - \alpha_n| < \infty$ *and* $\sum_{n=1}^{\infty}|\beta_{n-1} - \beta_n| < \infty$.

Then the sequence $\{x_n\}$ converges strongly to a point $x^* \in F(T)$, which is the unique solution of the variational inequality

$$\left\langle (I-f)x^*, x - x^* \right\rangle \geq 0, \quad \forall x \in F(T). \tag{3.33}$$

By changing the restrictions on parameters in Theorem 3.1, we obtain the following.

Theorem 3.5. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $f : C \to H$ be a λ-contraction with $\lambda \in [0,1)$. Let $S : C \to C$ be a nonexpansive mapping and $\{T_i\}_{i=1}^{\infty} : C \to C$ a countable family of nonexpansive mappings such that $F = \bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$. Set $\alpha_0 = 1$. Let $x_1 \in C$ and define a sequence $\{x_n\}$ by*

$$y_n = \beta_n S x_n + (1-\beta_n) x_n,$$
$$x_{n+1} = P_C\left[\alpha_n f(x_n) + \sum_{i=1}^{n}(\alpha_{i-1} - \alpha_i) T_i y_n\right], \quad \forall n \geq 1, \tag{3.34}$$

where $\{\alpha_n\} \subset (0,1)$ is a strictly decreasing sequence and $\{\beta_n\} \subset (0,1)$ is a sequence satisfying the following conditions:

(a) $\lim_{n\to\infty}\alpha_n = 0$ *and* $\sum_{n=1}^{\infty}\alpha_n = \infty$,

(b) $\lim_{n\to\infty}(\beta_n/\alpha_n) = \tau \in (0,\infty)$,

(c) $\sum_{n=1}^{\infty}(\alpha_{n-1} - \alpha_n) < \infty$ *and* $\sum_{n=1}^{\infty}|\beta_{n-1} - \beta_n| < \infty$,

(d) $\lim_{n\to\infty}((\alpha_{n-1} - \alpha_n) + |\beta_n - \beta_{n-1}|)/\alpha_n\beta_n = 0$,

(e) *there exists a constant $K > 0$ such that* $(1/\alpha_n)|1/\beta_n - 1/\beta_{n-1}| \leq K$.

Then the sequence $\{x_n\}$ *generated by* (3.34) *converges strongly to a point* $x^* \in F$, *which is the unique solution of the variational inequality*

$$\left\langle \frac{1}{\tau}(I-f)x^* + (I-S)x^*, x - x^* \right\rangle \geq 0, \quad \forall x \in F. \tag{3.35}$$

Proof. First, the proof of Theorem 3.2 of [1] shows that (3.35) has the unique solution. By a similar argument as in that of Theorem 3.1, we can conclude that $\{x_n\}$ is bounded, $\|x_{n+1} - x_n\| \to 0$ and $\|x_n - T_i x_n\| \to 0$ as $n \to \infty$. Note that conditions (a) and (b) imply that $\beta_n \to 0$ as $n \to \infty$. Hence we have

$$\left\|y_n - x_n\right\| = \beta_n \|Sx_n - x_n\| \longrightarrow 0 \quad (n \longrightarrow \infty). \tag{3.36}$$

It follows that, for all $i \geq 1$,

$$\left\|y_n - T_i x_n\right\| \leq \left\|y_n - x_n\right\| + \|x_n - T_i x_n\| \longrightarrow 0 \quad (n \longrightarrow \infty). \tag{3.37}$$

Now, it follows from (3.36) and (3.37) that, for all $i \geq 1$,

$$\left\|y_n - T_i y_n\right\| \leq \left\|y_n - T_i x_n\right\| + \left\|T_i x_n - T_i y_n\right\| \leq \left\|y_n - T_i x_n\right\| + \left\|x_n - y_n\right\| \longrightarrow 0 \quad (n \longrightarrow \infty). \tag{3.38}$$

From (3.8), we get

$$\begin{aligned}
\frac{\|x_{n+1} - x_n\|}{\beta_n} &\leq \frac{\|u_n - u_{n-1}\|}{\beta_n} \\
&\leq (1-(1-\lambda)\alpha_n)\frac{\|x_n - x_{n-1}\|}{\beta_n} + M\left[\frac{|\beta_n - \beta_{n-1}|}{\beta_n} + \frac{\alpha_{n-1} - \alpha_n}{\beta_n}\right] \\
&= (1-(1-\lambda)\alpha_n)\frac{\|x_n - x_{n-1}\|}{\beta_{n-1}} + (1-(1-\lambda)\alpha_n)\|x_n - x_{n-1}\|\left(\frac{1}{\beta_n} - \frac{1}{\beta_{n-1}}\right) \\
&\quad + M\left[\frac{|\beta_n - \beta_{n-1}|}{\beta_n} + \frac{\alpha_{n-1} - \alpha_n}{\beta_n}\right].
\end{aligned} \tag{3.39}$$

Note that

$$(1-(1-\lambda)\alpha_n)\left(\frac{1}{\beta_n} - \frac{1}{\beta_{n-1}}\right) \leq \alpha_n \frac{1}{\alpha_n}\left|\frac{1}{\beta_n} - \frac{1}{\beta_{n-1}}\right| \leq \alpha_n K. \tag{3.40}$$

Hence, from (3.39), we have

$$\begin{aligned}\frac{\|x_{n+1}-x_n\|}{\beta_n} \le \frac{\|u_n-u_{n-1}\|}{\beta_n} &\le (1-(1-\lambda)\alpha_n)\frac{\|x_n-x_{n-1}\|}{\beta_{n-1}} + \alpha_n K\|x_n-x_{n-1}\| \\ &\quad + M\left[\frac{|\beta_n-\beta_{n-1}|}{\beta_n} + \frac{\alpha_{n-1}-\alpha_n}{\beta_n}\right] \\ &\le (1-(1-\lambda)\alpha_n)\frac{\|u_{n-1}-u_{n-2}\|}{\beta_{n-1}} + \alpha_n K\|x_n-x_{n-1}\| \\ &\quad + M\left[\frac{|\beta_n-\beta_{n-1}|}{\beta_n} + \frac{\alpha_{n-1}-\alpha_n}{\beta_n}\right].\end{aligned} \tag{3.41}$$

Let $a_n = (1-\lambda)\alpha_n$ and $b_n = \alpha_n K\|x_n-x_{n-1}\| + M[|\beta_n-\beta_{n-1}|/\beta_n + (\alpha_{n-1}-\alpha_n)/\beta_n]$. From conditions (a) and (d), we have

$$\sum_{n=1}^{\infty} a_n = \infty, \qquad \lim_{n\to\infty}\frac{b_n}{a_n} = 0. \tag{3.42}$$

By Lemma 2.3(2), we get

$$\lim_{n\to\infty}\frac{\|x_{n+1}-x_n\|}{\beta_n} = 0, \qquad \lim_{n\to\infty}\frac{\|u_n-u_{n-1}\|}{\beta_n} = \lim_{n\to\infty}\frac{\|u_n-u_{n-1}\|}{\alpha_n} = 0. \tag{3.43}$$

From (3.34), we have

$$x_{n+1} = P_C u_n - u_n + \alpha_n f(x_n) + \sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)(T_i y_n - y_n) + (1-\alpha_n)y_n. \tag{3.44}$$

Hence it follows that

$$\begin{aligned}x_n - x_{n+1} &= (1-\alpha_n)x_n + \alpha_n x_n \\ &\quad - \left[P_C u_n - u_n + \alpha_n f(x_n) + \sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)(T_i y_n - y_n) + (1-\alpha_n)y_n\right] \\ &= (1-\alpha_n)\beta_n(x_n - Sx_n) + (u_n - P_C u_n) \\ &\quad + \sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)(y_n - T_i y_n) + \alpha_n(x_n - f(x_n))\end{aligned} \tag{3.45}$$

and hence

$$\begin{aligned}\frac{x_n - x_{n+1}}{(1-\alpha_n)\beta_n} &= x_n - Sx_n + \frac{1}{(1-\alpha_n)\beta_n}(u_n - P_C u_n) \\ &\quad + \frac{1}{(1-\alpha_n)\beta_n}\sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)(y_n - T_i y_n) + \frac{\alpha_n}{(1-\alpha_n)\beta_n}(x_n - f(x_n)).\end{aligned} \tag{3.46}$$

Let $v_n = (x_n - x_{n+1})/(1-\alpha_n)\beta_n$. For any $z \in F$, we have

$$\begin{aligned}\langle v_n, x_n - z\rangle &= \frac{1}{(1-\alpha_n)\beta_n}\langle u_n - P_C u_n, P_C u_{n-1} - z\rangle + \frac{\alpha_n}{(1-\alpha_n)\beta_n}\langle (I-f)x_n, x_n - z\rangle \\ &\quad + \langle x_n - Sx_n, x_n - z\rangle + \frac{1}{(1-\alpha_n)\beta_n}\sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)\langle y_n - T_i y_n, x_n - z\rangle.\end{aligned} \tag{3.47}$$

By Lemma 2.2, we have

$$\begin{aligned}\langle x_n - Sx_n, x_n - z\rangle &= \langle (I-S)x_n - (I-S)z, x_n - z\rangle + \langle (I-S)z, x_n - z\rangle \\ &\geq \langle (I-S)z, x_n - z\rangle,\end{aligned} \tag{3.48}$$

$$\begin{aligned}\langle (I-f)x_n, x_n - z\rangle &= \langle (I-f)x_n - (I-f)z, x_n - z\rangle + \langle (I-f)z, x_n - z\rangle \\ &\geq (1-\lambda)\|x_n - z\|^2 + \langle (I-f)z, x_n - z\rangle,\end{aligned} \tag{3.49}$$

$$\begin{aligned}\langle y_n - T_i y_n, x_n - z\rangle &= \langle (I-T_i)y_n - (I-T_i)z, x_n - y_n\rangle + \langle (I-T_i)y_n - (I-T_i)z, y_n - z\rangle \\ &\geq \langle (I-T_i)y_n - (I-T_i)z, x_n - y_n\rangle \\ &= \beta_n\langle (I-T_i)y_n, x_n - Sx_n\rangle, \quad i \geq 1.\end{aligned} \tag{3.50}$$

By (2.4), we have

$$\begin{aligned}\langle u_n - P_C u_n, P_C u_{n-1} - z\rangle &= \langle u_n - P_C u_n, P_C u_{n-1} - P_C u_n\rangle \\ &\quad + \langle u_n - P_C u_n, P_C u_n - z\rangle \\ &\geq \langle u_n - P_C u_n, P_C u_{n-1} - P_C u_n\rangle.\end{aligned} \tag{3.51}$$

Now, from (3.47)–(3.51) it follows that

$$\begin{aligned}\langle v_n, x_n - z\rangle &\geq \frac{1}{(1-\alpha_n)\beta_n}\langle u_n - P_C u_n, P_C u_{n-1} - P_C u_n\rangle + \frac{\alpha_n}{(1-\alpha_n)\beta_n}\langle (I-f)z, x_n - z\rangle \\ &\quad + \langle (I-S)z, x_n - z\rangle + \frac{1}{(1-\alpha_n)}\sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)\langle (I-T_i)y_n, x_n - Sx_n\rangle \\ &\quad + \frac{(1-\lambda)\alpha_n}{(1-\alpha_n)\beta_n}\|x_n - z\|^2.\end{aligned} \tag{3.52}$$

Observe that (3.52) implies that

$$\|x_n - z\|^2 \le \frac{(1-\alpha_n)\beta_n}{(1-\lambda)\alpha_n}\left[\langle v_n, x_n - z\rangle - \langle (I-S)z, x_n - z\rangle\right] - \frac{1}{1-\lambda}\left\langle (I-f)z, x_n - z\right\rangle$$
$$- \frac{\beta_n}{(1-\lambda)\alpha_n}\sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)\left\langle (I-T_i)y_n, x_n - Sx_n\right\rangle + \frac{\|u_{n-1}-u_n\|}{(1-\lambda)\alpha_n}\|u_n - P_C u_n\|. \tag{3.53}$$

Since $v_n \to 0$, $y_n - T_i y_n \to 0$ for all $i \ge 1$, and $\|u_{n-1} - u_n\|/\alpha_n \to 0$ as $n \to \infty$, every weak cluster point of $\{x_n\}$ is also a strong cluster point. Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ converging to a point $x^* \in C$. Note that $x_n - T_i x_n \to 0$ as $n \to \infty$ for all $i \ge 1$. By the demiclosed principle for a nonexpansive mapping, we have $x^* \in F(T_i)$ for all $i \ge 1$ and so $x^* \in F = \bigcap_{i=1}^{\infty} F(T_i)$. From (3.47), (3.48), (3.50), and (3.51), it follows that, for all $z \in F$,

$$\begin{aligned}
&\left\langle (I-f)x_{n_k}, x_{n_k} - z\right\rangle \\
&\quad= \frac{(1-\alpha_{n_k})\beta_{n_k}}{\alpha_{n_k}}\langle v_{n_k}, x_{n_k} - z\rangle - \frac{1}{\alpha_{n_k}}\langle u_{n_k} - P_C u_{n_k}, P_C u_{n_k-1} - z\rangle \\
&\qquad- \frac{(1-\alpha_{n_k})\beta_{n_k}}{\alpha_{n_k}}\langle x_{n_k} - Sx_{n_k}, x_{n_k} - z\rangle - \frac{1}{\alpha_{n_k}}\sum_{i=1}^{n_k}(\alpha_{i-1}-\alpha_i)\langle y_{n_k} - T_i y_{n_k}, x_{n_k} - z\rangle \\
&\quad\le \frac{(1-\alpha_{n_k})\beta_{n_k}}{\alpha_{n_k}}\langle v_{n_k}, x_{n_k} - z\rangle - \frac{1}{\alpha_{n_k}}\|u_{n_k} - P_C u_{n_k}\|\|u_{n_k-1} - u_{n_k}\| \\
&\qquad- \frac{(1-\alpha_{n_k})\beta_{n_k}}{\alpha_{n_k}}\langle (I-S)z, x_{n_k} - z\rangle - \frac{\beta_{n_k}}{\alpha_{n_k}}\sum_{i=1}^{n_k}(\alpha_{i-1}-\alpha_i)\left\langle (I-T_i)y_{n_k}, x_{n_k} - Sx_{n_k}\right\rangle.
\end{aligned} \tag{3.54}$$

Since $v_n \to 0$, $(I - T_i)y_n \to 0$ for all $i \ge 1$, and $\|u_n - u_{n-1}\|/\alpha_n = 0$, letting $k \to \infty$ in (3.54), we obtain

$$\left\langle (I-f)x^*, x^* - z\right\rangle \le -\tau\langle (I-S)z, x^* - z\rangle, \quad \forall z \in F. \tag{3.55}$$

Since (3.35) has the unique solution, it follows that $\omega_w(x_n) = \{\overline{x}\}$. Since every weak cluster point of $\{x_n\}$ is also a strong cluster point, we conclude that $x_n \to \overline{x}$ as $n \to \infty$. This completes the proof. □

In Theorem 3.5, if $T_i = T$ for each $i \ge 1$, then we have the following result, which is Theorem 3.2 of Yao et al. [1].

Corollary 3.6. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $f : C \to H$ be a λ-contraction with $\lambda \in [0,1)$. Let $S : C \to C$ be a nonexpansive mapping and $T : C \to C$ a nonexpansive mapping such that $F(T) \ne \emptyset$. Let $x_1 \in C$ and define a sequence $\{x_n\}$ by*

$$\begin{aligned}
y_n &= \beta_n S x_n + (1-\beta_n)x_n, \\
x_{n+1} &= P_C\left[\alpha_n f(x_n) + (1-\alpha_n)T y_n\right], \quad \forall n \ge 1,
\end{aligned} \tag{3.56}$$

where $\{\alpha_n\} \subset (0,1)$ *and* $\{\beta_n\} \subset (0,1)$ *are the sequences satisfying the following conditions:*

(a) $\lim_{n\to\infty}\alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(b) $\lim_{n\to\infty}(\beta_n/\alpha_n) = \tau \in (0,\infty)$,

(c) $\sum_{n=1}^{\infty}(\alpha_{n-1} - \alpha_n) < \infty$ *and* $\sum_{n=1}^{\infty} |\beta_{n-1} - \beta_n| < \infty$,

(d) $\lim_{n\to\infty}((\alpha_{n-1} - \alpha_n) + |\beta_n - \beta_{n-1}|)/\alpha_n\beta_n = 0$,

(e) *there exists a constant* $K > 0$ *such that* $(1/\alpha_n)|(1/\beta_n) - 1/\beta_{n-1}| \leq K$.

Then the sequence $\{x_n\}$ *generated by* (3.56) *converges strongly to a point* $x^* \in F(T)$, *which is the unique solution of the variational inequality*

$$\left\langle \frac{1}{\tau}(I - f)x^* + (I - S)x^*, x - x^* \right\rangle \geq 0, \quad \forall x \in F(T). \tag{3.57}$$

Remark 3.7. In (1.9), if $S = I$ and f is a self-contraction of C, then we get

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)Ty_n, \quad \forall n \geq 0, \tag{3.58}$$

which is well known as the viscosity method studied by Moudafi [6] and Xu [7]. If S and f are both self-mappings of C in (1.9), then we get the algorithm of Cianciaruso et al. [5].

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