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Research Article

Some Common Fixed Point Theorems in Partial Metric Spaces

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Many problems in pure and applied mathematics reduce to a problem of common fixed point of some self-mapping operators which are defined on metric spaces. One of the generalizations of metric spaces is the partial metric space in which self-distance of points need not to be zero but the property of symmetric and modified version of triangle inequality is satisfied. In this paper, some well-known results on common fixed point are investigated and generalized to the class of partial metric spaces.

1. Introduction and Preliminaries

Partial metric spaces, introduced by Matthews [1, 2], are a generalization of the notion of the metric space in which in definition of metric the condition $d(x,x) = 0$ is replaced by the condition $d(x,x) \leq d(x,y)$. Different approaches in this area have been reported including applications of mathematical techniques to computer science [3–7].

In [2], Matthews discussed some properties of convergence of sequences and proved the fixed point theorems for contractive mapping on partial metric spaces: any mapping T of a complete partial metric space X into itself that satisfies, where $0 \leq k < 1$, the inequality $d(Tx,Ty) \leq kd(x,y)$, for all $x,y \in X$, has a unique fixed point. Recently, many authors (see e.g., [8–16]) have focused on this subject and generalized some fixed point theorems from the class of metric spaces to the class of partial metric spaces.

The definition of partial metric space is given by Matthews (see e.g., [1]) as follows.

Definition 1.1. Let X be a nonempty set and let $p : X \times X \to \mathbb{R}_0^+$ satisfy

$$\text{(PM1)} \quad x = y \Longleftrightarrow p(x,x) = p(y,y) = p(x,y),$$
$$\text{(PM2)} \quad p(x,x) \leq p(x,y),$$

$$\begin{aligned} &\text{(PM3)} \quad p(x,y) = p(y,x), \\ &\text{(PM4)} \quad p(x,y) \le p(x,z) + p(z,y) - p(z,z) \end{aligned} \tag{1.1}$$

for all x, y, and $z \in X$, where $\mathbb{R}_0^+ = [0,\infty)$. Then the pair (X,p) is called a partial metric space (in short PMS) and p is called a partial metric on X.

Let (X,p) be a PMS. Then, the functions $d_p, d_m : X \times X \to \mathbb{R}_0^+$ given by

$$\begin{gathered} d_p(x,y) = 2p(x,y) - p(x,x) - p(y,y), \\ d_m(x,y) = \max\{p(x,y) - p(x,x), p(x,y) - p(y,y)\} \end{gathered} \tag{1.2}$$

are (usual) metrics on X. It is clear that d_p and d_m are equivalent. Each partial metric p on X generates a T_0 topology τ_p on X with a base of the family of open p-balls $\{B_p(x,\varepsilon) : x \in X,\ \varepsilon > 0\}$, where $B_p(x,\varepsilon) = \{y \in X : p(x,y) < p(x,x) + \varepsilon\}$ for all $x \in X$ and $\varepsilon > 0$. A basic example of partial metric is $(\mathbb{R}_0^+, p)$, where $p(x,y) = \max\{x,y\}$. One can easily deduce that $d_p(x,y) = |x-y| = d_m(x,y)$.

Example 1.2 (See [1, 2]). Let $X = \{[a,b] : a,b \in \mathbb{R},\ a \le b\}$ and define $p([a,b],[c,d]) = \max\{b,d\} - \min\{a,c\}$. Then (X,p) is a partial metric spaces.

We give same topological definitions on partial metric spaces.

Definition 1.3 (see e.g., [1, 2, 13]). (i) A sequence $\{x_n\}$ in a PMS (X,p) converges to $x \in X$ if and only if $p(x,x) = \lim_{n\to\infty} p(x,x_n)$.

(ii) A sequence $\{x_n\}$ in a PMS (X,p) is called a Cauchy sequence if and only if $\lim_{n,m\to\infty} p(x_n,x_m)$ exists (and finite).

(iii) A PMS (X,p) is said to be complete if every Cauchy sequence $\{x_n\}$ in X converges, with respect to τ_p, to a point $x \in X$ such that $p(x,x) = \lim_{n,m\to\infty} p(x_n,x_m)$.

(iv) A mapping $f : X \to X$ is said to be continuous at $x_0 \in X$ if for every $\varepsilon > 0$, there exists $\delta > 0$ such that $f(B(x_0,\delta)) \subset B(f(x_0),\varepsilon)$.

The following lemmas will be frequently used in the proofs of the main results.

Lemma 1.4 (see e.g., [1, 2, 13]). *(A) A sequence $\{x_n\}$ is Cauchy in a PMS (X,p) if and only if $\{x_n\}$ is Cauchy in a metric space (X,d_p).*

(B) A PMS (X,p) is complete if and only if the metric space (X,d_p) is complete. Moreover,

$$\lim_{n\to\infty} d_p(x,x_n) = 0 \Longleftrightarrow p(x,x) = \lim_{n\to\infty} p(x,x_n) = \lim_{n,m\to\infty} p(x_n,x_m), \tag{1.3}$$

where x is a limit of $\{x_n\}$ in (X,d_p).

Remark 1.5. Let (X,p) be a PMS. Therefore,

(A) if $p(x,y) = 0$, then $x = y$;

(B) if $x \ne y$, then $p(x,y) > 0$.

This follows immediately from the definition and can easily be verified by the reader.

Lemma 1.6 (See e.g., [15]). *Assume $x_n \to z$ as $n \to \infty$ in a PMS (X, p) such that $p(z, z) = 0$. Then $\lim_{n\to\infty} p(x_n, y) = p(z, y)$ for every $y \in X$.*

In this paper, we extend some common fixed point theorems for two self-mappings without commuting property from the class of usual metric spaces (see e.g., [17]) to the class of partial metric spaces (see Theorems 2.6 and 2.2). We also consider some common fixed point theorems with commuting property on partial metric spaces (see Theorem 2.7, Corollary 2.8).

2. Main Results

We first recall the definition of a common fixed point of two self-mappings.

Definition 2.1. Let (X, p) be a PMS and S, T two self-mappings on (X, p). A point $z \in X$ is said to be a common fixed point of S and T if $Sz = Tz = z$.

In the sequel, we give the first results about a common fixed point theorem. We prove the existence and uniqueness of a common fixed point of two self-mappings under certain conditions. Notice that here the operators need not commute with each other.

Theorem 2.2. *Suppose that (X, p) is a complete PMS and T, S are self-mappings on X. If there exists an $r \in [0, 1)$ such that*

$$p(Tx, Sy) \leq rM(x, y) \tag{2.1}$$

for any $x, y \in X$, where

$$M(x, y) = \max\left\{p(Tx, x), p(Sy, y), p(x, y), \frac{1}{2}[p(Tx, y) + p(Sy, x)]\right\}, \tag{2.2}$$

then there exists $z \in X$ such that $Tz = Sz = z$.

Proof. Let $x_0 \in X$. Define the sequence $\{x_n\}_{n=1}^{\infty}$ in a way that $x_2 = Tx_1$ and $x_1 = Sx_0$ and inductively

$$x_{2k+2} = Tx_{2k+1}, \qquad x_{2k+1} = Sx_{2k} \quad \text{for } k = 0, 1, 2, \ldots. \tag{2.3}$$

If there exists a positive integer N such that $x_{2N} = x_{2N+1}$, then x_{2N} is a fixed point of S and hence a fixed point of T. Indeed, since $x_{2N} = x_{2N+1} = Sx_{2N}$, then

$$Sx_{2N} = Sx_{2N+1} = S^2x_{2N} \Longrightarrow x_{2N} = Sx_{2N} = Sx_{2N+1} = x_{2N+1}. \tag{2.4}$$

Also, due to (2.1) we have

$$p(x_{2N+2}, x_{2N+1}) = p(Tx_{2N+1}, Sx_{2N}) \leq rM(x_{2N+1}, x_{2N}), \tag{2.5}$$

where

$$\begin{aligned} M(x_{2N+1}, x_{2N}) &= \max\Big\{ p(Tx_{2N+1}, x_{2N+1}), p(Sx_{2N}, x_{2N}), p(x_{2N+1}, x_{2N}), \\ &\qquad \frac{1}{2}\left[p(Tx_{2N+1}, x_{2N}) + p(Sx_{2N}, x_{2N+1})\right] \Big\} \\ &= \max\Big\{ p(Tx_{2N+1}, x_{2N+1}), p(x_{2N}, x_{2N}), p(x_{2N+1}, x_{2N}), \\ &\qquad \frac{1}{2}\left[p(Tx_{2N+1}, x_{2N}) + p(x_{2N+1}, x_{2N+1})\right] \Big\} \\ &= p(Tx_{2N+1}, x_{2N+1}) = p(Tx_{2N+1}, x_{2N}) = p(x_{2N+2}, x_{2N+1}). \end{aligned} \tag{2.6}$$

Thus we have (2.5) implies $(1-r)p(x_{2N+2}, x_{2N+1}) \leq 0$. Since $r < 1$, then $p(x_{2N+2}, x_{2N+1}) = 0$, which yields that $Tx_{2N+1} = x_{2N+2} = x_{2N+1}$. Notice that $x_{2N+1} = x_{2N}$ is the fixed point of S. As a result, $x_{2N+1} = x_{2N}$ is the common fixed point of S and T. A similar conclusion holds if $x_{2N+1} = x_{2N+2}$ for some positive integer N. Therefore, we may assume that $x_k \neq x_{k+1}$ for all k. If k is odd, due to (2.1), we have

$$p(x_{k+1}, x_{k+2}) = p(Tx_k, Sx_{k+1}) \leq rM(x_k, x_{k+1}), \tag{2.7}$$

where

$$M(x_k, x_{k+1}) = \max\left\{ p(x_{k+1}, x_k), p(x_{k+2}, x_{k+1}), p(x_k, x_{k+1}), \frac{1}{2}\left[p(x_{k+1}, x_{k+1}) + p(x_{k+2}, x_k)\right] \right\}. \tag{2.8}$$

In view of (PM4), we have

$$p(x_{k+1}, x_{k+1}) + p(x_{k+2}, x_k) \leq p(x_{k+2}, x_{k+1}) + p(x_{k+1}, x_k). \tag{2.9}$$

Thus, (2.8) turns into

$$\begin{aligned} M(x_k, x_{k+1}) &= \max\left\{ p(x_{k+1}, x_k), p(x_{k+2}, x_{k+1}), p(x_k, x_{k+1}), \frac{1}{2}\left[p(x_{k+2}, x_{k+1}) + p(x_{k+1}, x_k)\right] \right\} \\ &= \max\{p(x_{k+1}, x_k), p(x_{k+2}, x_{k+1})\}. \end{aligned} \tag{2.10}$$

If $M(x_k, x_{k+1}) = p(x_{k+2}, x_{k+1})$, then since $r < 1$, the inequality (2.7) yields a contradiction. Hence, $M(x_k, x_{k+1}) = p(x_{k+1}, x_k)$ and by (2.7) we have

$$p(x_{k+2}, x_{k+1}) \leq rp(x_{k+1}, x_k). \tag{2.11}$$

If k is even, the same inequality (2.11) can be obtained analogously.

We get that $\{p(x_k, x_{k+1})\}$ is a nonnegative, nonincreasing sequence of real numbers. Regarding (2.11), one can observe that

$$p(x_k, x_{k+1}) \leq r^k p(x_0, x_1), \quad \forall k = 0, 1, 2, \ldots. \tag{2.12}$$

Consider now

$$\begin{aligned} d_p(x_{k+1}, x_{k+2}) &= 2p(x_{k+1}, x_{k+2}) - p(x_{k+1}, x_{k+1}) - p(x_{k+2}, x_{k+2}) \\ &\leq 2p(x_{k+1}, x_{k+2}) \\ &\leq 2r^{k+1} p(x_0, x_1). \end{aligned} \tag{2.13}$$

Hence, regarding (2.12), we have $\lim_{k\to\infty} d_p(x_{k+1}, x_{k+2}) = 0$. Moreover,

$$\begin{aligned} d_p(x_{k+1}, x_{k+s}) &\leq d_p(x_{k+s-1}, x_{k+s}) + \cdots + d_p(x_{k+1}, x_{k+2}) \\ &\leq 2r^{k+s} p(x_0, x_1) + \cdots + 2r^{n+1} p(x_0, x_1). \end{aligned} \tag{2.14}$$

After standard calculation, we obtain that $\{x_k\}$ is a Cauchy sequence in (X, d_p) that is, $d_p(x_k, x_m) \to 0$ as $k, m \to \infty$. Since (X, p) is complete, by Lemma 1.4, (X, d_p) is complete and the sequence $\{x_k\}$ is convergent in (X, d_p) to, say, $z \in X$.

Again by Lemma 1.4,

$$p(z, z) = \lim_{k\to\infty} p(x_k, z) = \lim_{k,m\to\infty} p(x_k, x_m). \tag{2.15}$$

Since $\{x_k\}$ is a Cauchy sequence in (X, d_p), we have $\lim_{k,m\to\infty} d_p(x_k, x_m) = 0$. We assert that $\lim_{k,m\to\infty} p(x_k, x_m) = 0$. Without loss of generality, we assume that $n > m$. Now observe that

$$\begin{aligned} p(x_{n+2}, x_n) &\leq p(x_{n+2}, x_{n+1}) + p(x_{n+1}, x_n) - p(x_{n+1}, x_{n+1}) \\ &\leq p(x_{n+2}, x_{n+1}) + p(x_{n+1}, x_n). \end{aligned} \tag{2.16}$$

Analogously,

$$\begin{aligned} p(x_{n+3}, x_n) &\leq p(x_{n+3}, x_{n+2}) + p(x_{n+2}, x_n) - p(x_{n+2}, x_{n+2}) \\ &\leq p(x_{n+3}, x_{n+2}) + p(x_{n+2}, x_n). \end{aligned} \tag{2.17}$$

Taking into account (2.16), the expression (2.17) yields

$$p(x_{n+3}, x_n) \leq p(x_{n+3}, x_{n+2}) + p(x_{n+2}, x_{n+1}) + p(x_{n+1}, x_n). \tag{2.18}$$

Inductively, we obtain

$$p(x_m, x_n) \leq p(x_m, x_{m-1}) + \cdots + p(x_{n+2}, x_{n+1}) + p(x_{n+1}, x_n). \tag{2.19}$$

Due to (2.12), the expression (2.19) turns into

$$\begin{aligned} p(x_m, x_n) &\leq r^{m-1} p(x_1, x_0) + \cdots + r^{n+1} p(x_1, x_0) + r^n p(x_1, x_0) \\ &\leq r^n \left(1 + r + r^{m-n-1}\right) p(x_1, x_0). \end{aligned} \tag{2.20}$$

Regarding $r < 1$, by simple calculations, one can observe that

$$\lim_{k,m \to \infty} p(x_k, x_m) = 0. \tag{2.21}$$

Therefore, from (2.15), we have

$$p(z, z) = \lim_{k \to \infty} p(x_k, z) = \lim_{k,m \to \infty} p(x_k, x_m) = 0. \tag{2.22}$$

We assert that $Tz = z$. On the contrary, assume $Tz \neq z$. Then $p(z, Tz) > 0$. Let $\{x_{2k(i)}\}$ be a subsequence of $\{x_{2k}\}$ and hence of $\{x_k\}$. Due to (2.1), we have

$$p(Sx_{2k(i)}, Tz) \leq r M(x_{2k(i)}, z), \tag{2.23}$$

where

$$M(x_{2k(i)}, z) = \max\left\{ p(x_{2k(i)}, x_{2k(i)+1}), p(Tz, z), p(x_{2k(i)}, z), \frac{1}{2}\left[p(Tz, x_{2k(i)}) + p(z, x_{2k(i)+1})\right] \right\}. \tag{2.24}$$

Letting $k \to \infty$ and taking into account (2.22), the expression (2.24) implies that

$$M(x_{2k(i)}, z) = \max\left\{0, p(Tz, z), 0, \frac{1}{2} p(Tz, z)\right\} = p(Tz, z). \tag{2.25}$$

Thus,

$$p(z, Tz) \leq r p(Tz, z). \tag{2.26}$$

Since $r < 1$, we have $p(Tz, z) = 0$. By Remark 1.5, we get $Tz = z$. Analogously, if we choose a subsequence $\{x_{2k(i)+1}\}$ of $\{x_{2k+1}\}$, we obtain $Sz = z$. Hence $Tz = Sz = z$. □

Remark 2.3. We notice that Theorem 2.2 can be obtained from Theorem 2.1 in [18] or Theorem 5 in [19] by simple manipulations.

However, explicit proof of Theorem 2.2 has a crucial role in the proofs of Proposition 2.5, Theorems 2.6 and 2.7.

These results cannot be obtained from thementioned papers [18, 19].

Example 2.4. Let $X = [0,1]$ and $p(x,y) = \max\{x,y\}$. Then (X,p) is a complete PMS. Clearly, p is not a metric. Suppose $S,T : X \to X$ such that $Sx = Tx = x/3$ and $r = 1/2$. Without loss of generality, assume $x \geq y$. Then

$$\begin{aligned} p(Tx,Sy) &= \max\left\{\frac{x}{3},\frac{y}{3}\right\} = \frac{x}{3} \\ &\leq \frac{1}{2}M(x,y), \end{aligned} \tag{2.27}$$

where

$$M(x,y) = \max\left\{x,y,x,\frac{1}{2}[x+p(y,Sx)]\right\} = x. \tag{2.28}$$

Thus, all conditions of Theorem 2.2 are satisfied, and 0 is the common fixed point of S and T.

Proposition 2.5. *Suppose (X,p) is a complete PMS and T, S are self-mappings on X. If there exists an $r \in [0,1)$ such that*

$$p(T^m x, S^n y) \leq rM(x,y) \tag{2.29}$$

for any $x,y \in X$ and some positive integers m,n, where

$$M(x,y) = \max\left\{p(T^m x,x),p(S^n y,y),p(x,y),\frac{1}{2}[p(T^m x,y)+p(S^n y,x)]\right\}, \tag{2.30}$$

then there exists $z \in X$ such that $T^m z = S^n z = z$.

Proof. Let $x_0 \in X$. As in the proof of the previous theorem, we define a sequence $\{x_n\}_{n=1}^{\infty}$ in a way that $x_2 = T^m x_1$ and $x_1 = S^n x_0$; we get inductively

$$x_{2k+2} = T^m x_{2k+1}, \qquad x_{2k+1} = S^n x_{2k} \quad \text{for } k = 0,1,2,\ldots. \tag{2.31}$$

If there exists a positive integer N such that $x_{2N} = x_{2N+1}$, then x_{2N} is a fixed point of T^m and hence a fixed point of S^n. A similar conclusion holds if $x_{2N+1} = x_{2N+2}$ for some positive integer N. Therefore, we may assume that $x_k \neq x_{k+1}$ for all k.

If k is odd, due to (2.29), we have

$$p(x_{k+1},x_{k+2}) = p(T^m x_k, S^n x_{k+1}) \leq rM(x_k,x_{k+1}), \tag{2.32}$$

where

$$M(x_k,x_{k+1}) = \max\left\{p(x_{k+1},x_k),p(x_{k+2},x_{k+1}),p(x_k,x_{k+1}),\frac{1}{2}[p(x_{k+1},x_{k+1})+p(x_{k+2},x_k)]\right\}. \tag{2.33}$$

By means of (PM4), we have

$$p(x_{k+1}, x_{k+1}) + p(x_{k+2}, x_k) \leq p(x_{k+2}, x_{k+1}) + p(x_{k+1}, x_k). \tag{2.34}$$

Thus, (2.33) becomes

$$\begin{aligned} M(x_k, x_{k+1}) &= \max\left\{ p(x_{k+1}, x_k), p(x_{k+2}, x_{k+1}), p(x_k, x_{k+1}), \frac{1}{2}\left[p(x_{k+2}, x_{k+1}) + p(x_{k+1}, x_k)\right] \right\} \\ &= \max\{p(x_{k+1}, x_k), p(x_{k+2}, x_{k+1})\}. \end{aligned} \tag{2.35}$$

If $M(x_k, x_{k+1}) = p(x_{k+2}, x_{k+1})$, then since $r < 1$, the inequality (2.32) yields a contradiction. Hence, $M(x_k, x_{k+1}) = p(x_{k+1}, x_k)$ and by (2.32), we have

$$p(x_{k+2}, x_{k+1}) \leq r p(x_{k+1}, x_k). \tag{2.36}$$

If k is even, analogously we obtain the same inequality (2.36).

We obtain that $\{p(x_k, x_{k+1})\}$ is a nonnegative, nonincreasing sequence of real numbers. Regarding (2.36), one has

$$p(x_k, x_{k+1}) \leq r^k p(x_0, x_1), \quad \forall k = 0, 1, 2, \ldots. \tag{2.37}$$

Consider now

$$\begin{aligned} d_p(x_{k+1}, x_{k+2}) &= 2p(x_{k+1}, x_{k+2}) - p(x_{k+1}, x_{k+1}) - p(x_{k+2}, x_{k+2}) \\ &\leq 2p(x_{k+1}, x_{k+2}) \\ &\leq 2r^{k+1} p(x_0, x_1). \end{aligned} \tag{2.38}$$

Hence, regarding (2.37), we have $\lim_{k\to\infty} d_p(x_{k+1}, x_{k+2}) = 0$. Moreover,

$$\begin{aligned} d_p(x_{k+1}, x_{k+s}) &\leq d_p(x_{k+s-1}, x_{k+s}) + \cdots + d_p(x_{k+1}, x_{k+2}) \\ &\leq 2r^{k+s} p(x_0, x_1) + \cdots + 2r^{k+1} p(x_0, x_1) \end{aligned} \tag{2.39}$$

which implies that $\{x_k\}$ is a Cauchy sequence in (X, d_p), that is, $d_p(x_k, x_m) \to 0$ as $k, m \to \infty$. Since (X, p) is complete, by Lemma 1.4, (X, d_p) is complete and the sequence $\{x_k\}$ is convergent in (X, d_p) to, say, $z \in X$.

By Lemma 1.4,

$$p(z, z) = \lim_{k\to\infty} p(x_k, z) = \lim_{k,m\to\infty} p(x_k, x_m). \tag{2.40}$$

Since $\{x_k\}$ is a Cauchy sequence in (X, d_p), we have

$$\lim_{k,m\to\infty} d_p(x_k, x_m) = 0. \tag{2.41}$$

We claim that $\lim_{k,m\to\infty} p(x_k, x_m) = 0$. Following the steps (2.16)–(2.22) in the proof of Theorem 2.2, we conclude the result. Thus,

$$\begin{aligned}2p(x_m, x_n) - p(x_m, x_{m+1}) - p(x_n, x_{n+1}) &\le 2p(x_m, x_n) - p(x_m, x_m) - p(x_n, x_n) \\ &= d_p(x_m, x_n).\end{aligned} \tag{2.42}$$

Thus, letting $n, m \to \infty$ in view of (2.37), (2.41), the expression (2.42) yields that $\lim_{k,m\to\infty} p(x_k, x_m) = 0$. Therefore, from (2.40) we have

$$p(z, z) = \lim_{k\to\infty} p(x_k, z) = \lim_{k,m\to\infty} p(x_k, x_m) = 0. \tag{2.43}$$

We assert that $T^m z = z$. Assume the contrary, that is, $T^m z \neq z$, then $p(z, T^m z) > 0$. Let $\{x_{2k(i)}\}$ be a subsequence of $\{x_{2k}\}$ and hence of $\{x_k\}$. Due to (2.29), we have

$$\begin{aligned}p\left(S^n x_{2k(i)}, T^m z\right) &\le rM\left(x_{2k(i)}, z\right) \\ &\le r\max\Big\{p\left(x_{2k(i)}, x_{2k(i)+1}\right), p(T^m z, z), \\ &\qquad p\left(x_{2k(i)}, z\right), \frac{1}{2}\left[p\left(T^m z, x_{2k(i)}\right) + p\left(z, x_{2k(i)+1}\right)\right]\Big\}.\end{aligned} \tag{2.44}$$

Letting $k \to \infty$ and taking into account (2.43), the expression (2.44) implies that

$$p(z, T^m z) \le r\max\left\{0, p(T^m z, z), 0, \frac{1}{2}p(T^m z, z)\right\} \le rp(T^m z, z). \tag{2.45}$$

Since $r < 1$, we have $p(T^m z, z) = 0$. By Remark 1.5, we get $T^m z = z$. Analogously, if we choose a subsequence $\{x_{2k(i)+1}\}$ of $\{x_{2k+1}\}$, we obtain $S^n z = z$. Hence $T^m z = S^n z = z$. □

The following theorem is a generalization of a common fixed point theorem that requires no commuting criteria (see e.g., [17]).

Theorem 2.6. *Suppose (X, p) is a complete PMS and T, S are self-mappings on X. If there exists an $r \in [0, 1)$ such that*

$$p(T^m x, S^n y) \le rM(x, y) \tag{2.46}$$

for any $x, y \in X$ and some positive integers m, n, where

$$M(x, y) = \max\left\{p(T^m x, x), p(S^n y, y), p(x, y), \frac{1}{2}\left[p(T^m x, y) + p(S^n y, x)\right]\right\}, \tag{2.47}$$

then T and S have a unique common fixed point $z \in X$.

Proof. Due to Proposition 2.5, we have

$$T^m z = S^n z = z. \tag{2.48}$$

We claim that z is a common fixed point of S and T. From (2.46) and (2.48), it follows that

$$p(Tz, z) = p(TT^m z, S^n z) = p(T^m Tz, S^n z) \le rM(Tz, z), \tag{2.49}$$

where

$$\begin{aligned} M(Tz, z) &= \max\left\{p(T^m Tz, Tz), p(S^n z, z), p(Tz, z), \frac{1}{2}\left[p(T^m Tz, z) + p(S^n z, Tz)\right]\right\} \\ &= \max\left\{p(TT^m z, Tz), p(z, z), p(Tz, z), \frac{1}{2}\left[p(TT^m z, z) + p(z, Tz)\right]\right\} \\ &= \max\left\{p(Tz, Tz), p(Tz, z), \frac{1}{2}\left[p(Tz, z) + p(z, Tz)\right]\right\} \\ &= \max\{p(Tz, Tz), p(Tz, z)\}. \end{aligned} \tag{2.50}$$

Due to (PM3), we have $p(Tz, Tz) \le p(Tz, z)$. Hence

$$M(Tz, z) = \max\{p(Tz, Tz), p(Tz, z)\} = p(Tz, z). \tag{2.51}$$

Regarding the assumption $r < 1$ and the expression (2.49), we get $p(Tz, z) \le rp(Tz, z)$ which implies that $p(Tz, z) = 0$, and by Remark 1.5, we obtain $Tz = z$. Analogously, one can show that $Sz = z$. Hence, $Tz = Sz = z$.

For the uniqueness of the common fixed point z, assume the contrary. Suppose w is another common fixed point of S and T. Then,

$$p(z, w) = p(T^m z, S^n w) \le rM(z, w), \tag{2.52}$$

where

$$\begin{aligned} M(z, w) &= \max\left\{p(T^m z, z), p(S^n w, w), p(z, w), \frac{1}{2}\left[p(T^m z, w) + p(S^n w, z)\right]\right\} \\ &= \max\left\{p(z, z), p(w, w), p(z, w), \frac{1}{2}\left[p(z, w) + p(w, z)\right]\right\} \\ &= p(z, w). \end{aligned} \tag{2.53}$$

Therefore, $p(z, w) \le rp(z, w)$. Since $0 \le r < 1$, one has $p(z, w) = 0$ which yields $z = w$ by Remark 1.5. Hence, z is a unique common fixed point of S and T. □

Theorem 2.7. *Let (X, p) be a complete PMS. Suppose that T, S, F, and G are self-mappings on X, and F and G are continuous. Suppose also that T, F and S, G are commuting pairs and that*

$$T(X) \subset F(X), \qquad S(X) \subset G(X). \tag{2.54}$$

If there exists an $r \in [0,1)$, and $m, n \in \mathbb{N}$ such that

$$p(Tx, Sy) \leq rM(x, y) \tag{2.55}$$

for any x, y in X, where

$$M(x, y) = \max\left\{p(Tx, Gx), p(Sy, Fy), p(Gx, Fy), \frac{1}{2}[p(Tx, Fy) + p(Sy, Gx)]\right\}, \tag{2.56}$$

then T, S, F, and G have a unique common fixed point z in X.

Proof. Fix $x_0 \in X$. Since $T(X) \subset F(X)$ and $S(X) \subset G(X)$, we can choose x_1, x_2 in X such that $y_1 = Fx_1 = Tx_0$ and $y_2 = Gx_2 = Sx_1$. In general, we can choose x_{2n-1}, x_{2n} in X such that

$$y_{2n-1} = Fx_{2n-1} = Tx_{2n-2}, \qquad y_{2n} = Gx_{2n} = Sx_{2n-1}, \quad n = 1, 2, \ldots. \tag{2.57}$$

We claim that the constructive sequence $\{y_n\}$ is a Cauchy sequence. If there exists a positive integer N such that $y_N = y_{N+1}$, then $y_N = y_{N+1} = y_{N+2} = \cdots = y_{N+k}$ for all $k \in \mathbb{N}$. Therefore, $\{y_n\}$ is a Cauchy sequence and we proved claim. Thus, we may assume that $y_n \neq y_{n+1}$ for all n.

By (2.55) and (2.57),

$$\begin{aligned} p(y_{2n+1}, y_{2n+2}) &= p(Fx_{2n+1}, Gx_{2n+2}) \\ &= p(Tx_{2n}, Sx_{2n+1}) \\ &\leq rM(x_{2n}, x_{2n+1}), \end{aligned} \tag{2.58}$$

where

$$\begin{aligned} M(x_{2n}, x_{2n+1}) = \max\Big\{&p(Tx_{2n}, Gx_{2n}), p(Sx_{2n+1}, Fx_{2n+1}), p(Gx_{2n}, Fx_{2n+1}), \\ &\frac{1}{2}[p(Tx_{2n}, Fx_{2n+1}) + p(Sx_{2n+1}, Gx_{2n})]\Big\} \\ = \max\Big\{&p(Tx_{2n}, Sx_{2n-1}), p(Sx_{2n+1}, Tx_{2n}), p(Sx_{2n-1}, Tx_{2n}), \\ &\frac{1}{2}[p(Tx_{2n}, Tx_{2n}) + p(Sx_{2n+1}, Sx_{2n-1})]\Big\}. \end{aligned} \tag{2.59}$$

Due to (PM4), we have

$$p(Sx_{2n-1}, Sx_{2n+1}) + p(Tx_{2n}, Tx_{2n}) \leq p(Sx_{2n-1}, Tx_{2n}) + p(Sx_{2n+1}, Tx_{2n}). \tag{2.60}$$

Hence

$$M(x_{2n}, x_{2n+1}) = \max\{p(Tx_{2n}, Sx_{2n-1}), p(Sx_{2n+1}, Tx_{2n})\}. \tag{2.61}$$

But if $M(x_{2n}, x_{2n+1}) = p(Sx_{2n+1}, Tx_{2n})$, then by (2.58)

$$p(Sx_{2n+1}, Tx_{2n}) \leq rp(Sx_{2n+1}, Tx_{2n}), \quad r \in [0,1), \tag{2.62}$$

which implies $p(Sx_{2n+1}, Tx_{2n}) = 0$. Thus, $M(x_{2n}, x_{2n+1}) = p(Sx_{2n-1}, Tx_{2n})$, and consequently,

$$p(Sx_{2n+1}, Tx_{2n}) \leq rp(Sx_{2n-1}, Tx_{2n}), \tag{2.63}$$

or, equivalently,

$$p(y_{2n+2}, y_{2n+1}) \leq rp(y_{2n+1}, y_{2n}). \tag{2.64}$$

Analogously, one can show that

$$p(y_{2n+3}, y_{2n+2}) \leq rp(y_{2n+2}, y_{2n+1}). \tag{2.65}$$

Indeed, from (2.55) and (2.57),

$$p(y_{2n+3}, y_{2n+2}) = p(Sx_{2n+1}, Tx_{2n+2}) \leq rM(x_{2n+1}, x_{2n+2}), \tag{2.66}$$

where

$$\begin{aligned}
M(x_{2n+2}, x_{2n+1}) &= \max\Big\{p(Tx_{2n+1}, Gx_{2n+1}), p(Sx_{2n+2}, Fx_{2n+2}), p(Gx_{2n+1}, Fx_{2n+2}), \\
&\qquad \frac{1}{2}\left[p(Tx_{2n+1}, Fx_{2n+2}) + p(Sx_{2n+2}, Gx_{2n+1})\right]\Big\} \\
&= \max\Big\{p(Tx_{2n+2}, Sx_{2n+1}), p(Sx_{2n+1}, Tx_{2n}), p(Sx_{2n+1}, Tx_{2n}), \\
&\qquad \frac{1}{2}\left[p(Tx_{2n+2}, Tx_{2n}) + p(Sx_{2n+1}, Sx_{2n+1})\right]\Big\} \\
&= \max\{p(Tx_{2n+2}, Sx_{2n+1}), p(Sx_{2n+1}, Tx_{2n})\}.
\end{aligned} \tag{2.67}$$

If $M(x_{2n+2}, x_{2n+1}) = p(Tx_{2n+2}, Sx_{2n+1}) = p(y_{2n+3}, y_{2n+2})$, then by (2.66), we have a contradiction. Thus $M(x_{2n+2}, x_{2n+1}) = p(Sx_{2n+1}, Tx_{2n}) = p(y_{2n+2}, y_{2n+1})$ which proves (2.65).

Thus, we conclude that $p(y_{n+1}, y_n) \leq rp(y_n, y_{n-1})$, for all $n \in \mathbb{N}$.

By elementary calculation, regarding $0 < r < 1$, we conclude that $\{y_n\}$ is a Cauchy sequence. Since $\{X, d\}$ is complete, $\{y_n\}$ converges to a point $z \in X$. Consequently, the subsequences $\{T^m x_{2n}\}$, $\{S^n x_{2n-1}\}$, $\{Gx_{2n}\}$, and $\{Fx_{2n-1}\}$ converge to z.

Regarding that T, F and S, G are commuting pairs and the continuity of G and F, the sequences $\{FFx_{2n-1}\}, \{SFx_{2n-1}\}$ tend to Fz, and the sequences $\{GGx_{2n}\}, \{TGx_{2n}\}$ tend to Gz, as $n \to \infty$.

Thus,

$$p(Gz, Fz) = \lim_{n\to\infty} p(TGx_{2n}, SFx_{2n-1}) \leq r \lim_{n\to\infty} M(Gx_{2n}, Fx_{2n-1}), \tag{2.68}$$

where

$$\begin{aligned} M(Gx_{2n}, Fx_{2n-1}) = \max\Big\{ & p(TGx_{2n}, GGx_{2n}), p(SFx_{2n-1}, FFx_{2n-1}), p(GGx_{2n}, FFx_{2n-1}), \\ & \frac{1}{2}\left[p(TGx_{2n}, FFx_{2n-1}) + p(GGx_{2n}, S^n Fx_{2n-1})\right]\Big\}. \end{aligned} \tag{2.69}$$

Since $\lim_{n\to\infty} M(Gx_{2n}, Fx_{2n-1}) = p(Gz, Fz)$, one has $p(Gz, Fz) \leq rp(Gz, Fz)$, that is, $Gz = Fz$. Analogously, one obtains

$$Tz = Sz = Fz = Gz. \tag{2.70}$$

To conclude the proof, consider

$$p(z, Fz) = \lim_{n\to\infty} p(Fx_{2n-1}, SFx_{2n-1}) = \lim_{n\to\infty} p(Tx_{2n-2}, SFx_{2n-1}) \leq r \lim_{n\to\infty} M(x_{2n-2}, Fx_{2n-1}), \tag{2.71}$$

where

$$\begin{aligned} M(x_{2n-2}, Fx_{2n-1}) = \max\Big\{ & p(Tx_{2n-2}, Gx_{2n-2}), p(SFx_{2n-1}, FFx_{2n-1}), p(Gx_{2n-2}, FFx_{2n-1}), \\ & \frac{1}{2}\left[p(Tx_{2n-2}, FFx_{2n-1}) + p(SFx_{2n-1}, Gx_{2n-2})\right]\Big\}, \end{aligned} \tag{2.72}$$

Letting $n \to \infty$ in (2.72) and having in mind (2.70), we get that $\lim_{n\to\infty} M(x_{2n-2}, Fx_{2n-1}) = p(z, Fz)$. Due to (2.71), we have $Fz = z$. Thus, we have

$$Tz = Sz = Fz = Gz = z. \tag{2.73}$$

We assert that z is unique. Suppose on the contrary that there is another common fixed point w of S, T, F, and G. Then $p(z,w) = p(Tz,Sw) \le rM(z,w)$, where

$$\begin{aligned} M(z,w) &= \max\left\{p(Tz,Gz),p(Sw,Fw),p(Gz,Fw),\frac{1}{2}[p(Tz,Fw)+p(Sw,Gz)]\right\} \\ &= \max\left\{p(z,z),p(w,w),p(z,w),\frac{1}{2}[p(z,w)+p(w,z)]\right\} \\ &= p(z,w). \end{aligned} \tag{2.74}$$

Since $M(z,w) = p(z,w)$,

$$p(z,w) \le rp(z,w). \tag{2.75}$$

Therefore, $p(z,w) = 0$, and by Remark 1.5, we have $z = w$. Hence z is the unique common fixed point of S, T, F, and G. □

Regarding the relation between Theorems 2.2 and 2.6, one concludes the following corollary in view of Theorem 2.7.

Corollary 2.8. *Let (X,p) be a complete PMS. Suppose that A, B, F, and G are self-mappings on X, and F and G are continuous. Suppose also that A, B, and F, G are commuting pairs and also A and G commutes each other and*

$$A(X) \subset F(X), \qquad B(X) \subset G(X). \tag{2.76}$$

If there exists $r \in [0,1)$, and $m,n \in \mathbb{N}$ such that

$$p(A^m x, B^n y) \le rM(x,y) \tag{2.77}$$

for any x, y in X, where

$$M(x,y) = \max\left\{p(A^m x,Gx),p(B^n y,Fy),p(Gx,Fy),\frac{1}{2}[p(A^m x,Fy)+p(B^n y,Gx)]\right\}, \tag{2.78}$$

then A, B, F, and G have a unique common fixed point z in X.

Proof. Due to Theorem 2.7,

$$A^m z = B^n z = Fz = Gz = z. \tag{2.79}$$

Following the steps of the proof of Theorem 2.7 with $T = A^m$, $S = B^n$, we get (2.70) which is equivalent to (2.79). Thus, A^m, B^n, F, and G have a unique common fixed point z in X.

We claim that

$$Az = Bz = z. \tag{2.80}$$

By (2.79),

$$p(Az, z) = p(AA^m z, B^n z) = p(A^m Az, B^n z) \le r \lim_{n\to\infty} M(Az, z), \tag{2.81}$$

where

$$\begin{aligned} M(Az, z) &= \max\left\{p(A^m Az, GAz), p(B^n z, Fz), p(GAz, Fz), \frac{1}{2}\left[p(A^m Az, Fz) + p(B^n z, GAz)\right]\right\} \\ &= \max\left\{p(AA^m z, AGz), p(z, z), p(AGz, z), \frac{1}{2}\left[p(AA^m z, z) + p(z, AGz)\right]\right\} \\ &= \max\left\{p(Az, Az), 0, p(Az, z), \frac{1}{2}\left[p(Az, z) + p(z, Az)\right]\right\} \\ &= p(Az, z). \end{aligned} \tag{2.82}$$

Hence, (2.81) is equivalent to $p(Az, z) \le rp(Az, z)$ which yields $p(Az, z) = 0$, that is, $Az = z$. Analogously, one can get $Bz = z$. Indeed, By (2.79),

$$p(z, Bz) = p(A^m z, BB^n z) = p(A^m z, B^n Bz) \le r \lim_{n\to\infty} M(z, Bz), \tag{2.83}$$

where

$$\begin{aligned} M(z, Bz) &= \max\left\{p(A^m z, Gz), p(B^n Bz, FBz), p(Gz, FBz), \frac{1}{2}\left[p(A^m z, FBz) + p(B^n Bz, Gz)\right]\right\} \\ &= \max\left\{p(z, z), p(Bz, FBz), p(z, FBz), \frac{1}{2}\left[p(z, FBz) + p(Bz, z)\right]\right\} \\ &= \max\left\{0, p(Bz, Bz), p(z, Bz), \frac{1}{2}\left[p(Bz, z) + p(z, Bz)\right]\right\} \\ &= p(Bz, z). \end{aligned} \tag{2.84}$$

Hence, (2.83) is equivalent to $p(Bz, z) \le rp(Bz, z)$ which yields $p(z, Bz) = 0$, that is, $z = Bz$. Hence,

$$Az = Bz = z. \tag{2.85}$$

Combining (2.80) and (2.85), we obtain $Gz = Fz = Az = Bz = z$. □

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