Hindawi Publishing Corporation
Journal of Applied Mathematics
Volume 2011, Article ID 375838, 11 pages
doi:10.1155/2011/375838

Research Article

Canonical Quantization of Higher-Order Lagrangians

Khaled I. Nawafleh

Department of Physics, Mu'tah University, Karak 61710, Jordan

Correspondence should be addressed to Khaled I. Nawafleh, knawafleh@mutah.edu.jo

Received 4 July 2011; Accepted 22 August 2011

Academic Editor: Nicola Guglielmi

Copyright © 2011 Khaled I. Nawafleh. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

After reducing a system of higher-order regular Lagrangian into first-order singular Lagrangian using constrained auxiliary description, the Hamilton-Jacobi function is constructed. Besides, the quantization of the system is investigated using the canonical path integral approximation.

1. Introduction

The efforts to quantize systems with constraints started with the work of Dirac [1, 2], who first set up a formalism for treating singular systems and the constraints involved for the purpose of quantizing his field, with special emphasis on the gravitational field. In Dirac's canonical quantization method, the Poisson brackets of classical mechanics are replaced with quantum commutators.

A new formalism for investigating first-order singular systems-, the canonical-, was developed by Rabei and Guler [3]. These authors obtained a set of Hamilton-Jacobi partial differential equations (HJPDEs) for singular systems using Caratheodory's equivalent-Lagrangian method [4]. In this formalism, the equations of motion are obtained as total differential equations and the set of HJPDEs was determined. Recently, the formalism has been extended to second- and higher-order Lagrangians [5, 6]. Depending on this method, the path-integral quantization of first-and higher-order constrained Lagrangian systems has been applied [7–10].

Moreover, the quantization of constrained systems has been studied for first-order singular Lagrangians using the WKB approximation [11]. The HJPDEs for these systems have been constructed using the canonical method; the Hamilton-Jacobi functions have then been obtained by solving these equations.

The Hamiltonian formulation for systems with higher-order regular Lagrangians was first developed by Ostrogradski [12]. This led to Euler's and Hamilton's equations of motion.

However, in Ostrogradski's construction the structure of phase space and in particular of its local simplistic geometry is not immediately transparent which leads to confusion when considering canonical path integral quantization.

In Ostrogradski's construction, this problem can be resolved within the well-established context of constrained systems [13] described by Lagrangians depending on coordinates and velocities only. Therefore, higher-order systems can be set in the form of ordinary constrained systems [14]. These new systems will be functions only of first-order time derivative of the degrees of freedom and coordinates which can be treated using the theory of constrained systems [1–11].

The purpose of the present paper is to study the canonical path integral quantization for singular systems with arbitrary higher-order Lagrangian. In fact, this work is a continuation of the previous work [15], where the path integral for certain kinds of higher-order Lagrangian systems has been obtained.

The present work is organized as follows: in Section 2, a review of the canonical method is introduced. In Section 3, Ostrogradski's formalism of higher-order Lagrangians is discussed. In Section 4, the formulation of the canonical Hamiltonian is reviewed briefly. In Section 5, the canonical path integral quantization of the extended Lagrangian is applied. In Section 6, two illustrative examples are investigated in detail. The work closes with some concluding remarks in Section 7.

2. Review of the Canonical Method

The starting point is a singular Lagrangian $L = L(q_i, \dot{q}_i)$, $i = 1, 2, \ldots, N$, with the Hessian matrix $\partial^2 L/\partial\dot{q}_i\partial\dot{q}_j$ of rank N-R, $R < N$.

The canonical formulation [3] gives the set of the Hamilton-Jacobi partial differential equations as

$$\begin{gathered} H'_0 = p_0 + H_0 \equiv \frac{\partial S}{\partial t} + H_0\left(q_\beta, q_a, p_a = \frac{\partial S}{\partial q_a}\right) = 0, \\ H'_\mu = p_\mu + H_\mu \equiv \frac{\partial S}{\partial q_\mu} + H_\mu\left(q_\beta, q_a, p_a = \frac{\partial S}{\partial q_a}\right) = 0, \\ a = 1, \ldots, N - R,\ \mu = N - R + 1, \ldots, N, \end{gathered} \tag{2.1}$$

where p_0 and q_μ are the momenta conjugate to t and q_μ, respectively,

$$p_0 = \frac{\partial S(q_i, t)}{\partial t}, \qquad p_\mu = \frac{\partial S(q_i, t)}{\partial q_\mu}. \tag{2.2}$$

The canonical Hamiltonian H_0 is given by

$$H_0 = p_a\dot{q}_a + p_\mu\dot{q}_\mu - L. \tag{2.3}$$

The equations of motion are obtained as total differential equations in many variables as follows:

$$dq_a = \frac{\partial H_0'}{\partial p_a}dt + \frac{\partial H_\mu'}{\partial p_a}dq_\mu,$$
$$dp_i = -\frac{\partial H_0'}{\partial q_i}dt - \frac{\partial H_\mu'}{\partial q_i}dq_\mu, \tag{2.4}$$

$$dz = -H_0 dt - H_\mu dq_\mu + p_a\frac{\partial H_0'}{\partial p_a}dt + p_a\frac{\partial H_\mu'}{\partial p_a}dq_\mu, \tag{2.5}$$

where $z = S(t, q_a, q_\mu)$. The set of equations (2.4) and (2.5) is integrable if and only if

$$dH_0' = 0, \qquad \partial H_\mu' = 0 \tag{2.6}$$

are identically satisfied. If they are not, one could consider them as new constraints and again should test the consistency conditions. Thus, in repeating this procedure one may obtain a new set of conditions. Equations (2.4) then can be solved to obtain the coordinates q_a and momenta p_i as functions of q_μ and t.

3. Ostrogradski's Formalism of Higher-Order Lagrangians

Consider a higher-order Lagrangian system of N generalized coordinates $q_n(t)$:

$$L_0\left(q_n, \dot{q}_n, \ldots, q_n^{(m)}\right), \quad m \geq 1, \tag{3.1}$$

where $q_n^{(s)} = d^s q_n / dt^s$, $s = 0, 1, \ldots, m$ and $n = 1, \ldots, N$.

The Euler-Lagrange equations of motion are obtained as [12]

$$\sum_{s=0}^{m} (-1)^s \frac{d^s}{dt^s}\left(\frac{\partial L_0}{\partial q_n^{(s)}}\right) = 0. \tag{3.2}$$

Theories with higher derivatives, which have been first developed by Ostrogradski [12], treat the derivatives $q_n^{(s)}$ $(s = 0, \ldots, m-1)$ as independent coordinates. Therefore, we will indicate this by writing them as $q_n^{(s)} = q_{n,s}$. In Ostrogradski's formalism, the momenta conjugated, respectively, to $q_{n,m-1}$ and $q_{n,s-1}$, $(s = 1, \ldots, m-1)$ read as

$$p_{n,m-1} \equiv \frac{\partial L_0}{\partial q_n^{(m)}},$$
$$p_{n,s-1} \equiv \frac{\partial L_0}{\partial q_n^{(s)}} - \dot{p}_{n,s}, \quad s = 1, \ldots, m-1. \tag{3.3}$$

Therefore, the canonical Hamiltonian is given by

$$H_0\left(q_{n,0}, \ldots, q_{n,m-1}; p_{n,0}, \ldots, p_{n,m-1}\right) = \sum_{s=0}^{m-2} p_{n,s} q_{n,s+1} + p_{n,m-1}\dot{q}_{n,m-1} - L_0\left(q_{n,0}, \ldots, q_{n,m-1}, \dot{q}_{n,m-1}\right). \tag{3.4}$$

Hamilton's equations of motion are written using Poisson bracket as [5, 6]

$$\dot{q}_{n,s} = \frac{\partial H_0}{\partial p_{n,s}} = \{q_{n,s}, H_0\}, \tag{3.5}$$

$$\dot{p}_{n,s} = \frac{\partial H_0}{\partial q_{n,s}} = \{p_{n,s}, H_0\}, \tag{3.6}$$

where $\{\ ,\ \}$ is the Poisson bracket defined as

$$\{A, B\} = \sum_{s=0}^{m-1} \frac{\partial A}{\partial q_{n,s}} \frac{\partial B}{\partial p_{n,s}} - \frac{\partial B}{\partial q_{n,s}} \frac{\partial A}{\partial p_{n,s}}. \tag{3.7}$$

The fundamental Poisson brackets are

$$\{q_{n,s} p_{n',s'}\} = \delta_{nn'} \delta_{ss'}, \qquad \{q_{n,s}, q_{n',s'}\} = \{p_{n,s}, p_{n',s'}\} = 0, \tag{3.8}$$

where $n, n' = 1, \ldots, N,$ and $s, s' = 0, \ldots, m-1$.

With this procedure, the phase space, described in terms of the canonical variables $q_{n,s}$ and $p_{n,s}$, is obeying the equations of motion that are given by (3.5) and (3.6), which are first-order differential equations.

4. Formulation of the Canonical Hamiltonian

Recall the higher-order Lagrangian given in (3.1), and let us introduce new independent variables $(q_{n,m-1}, q_{n,i},\ i = 0, 1, \ldots, m-2)$ such that the following recursion relations would hold [13, 14]:

$$\dot{q}_{n,i} = q_{n,i+1}. \tag{4.1}$$

Clearly, the variables $(q_{n,m-1}, q_{n,i})$, would then correspond to the time derivatives $(q_n^{(m-1)}, q_n^{(i)})$ respectively, that is,

$$q_n^{(0)} = q_{n,0}, \qquad \dot{q}_n = q_{n,1}, \ \ldots, \qquad q_n^{(m-1)} = q_{n,m-1}, \qquad q_n^{(m)} = \dot{q}_{n,m-1}. \tag{4.2}$$

Equation (4.1) represents relations between the new variables. In order to enforce these relations for independent variables $(q_{n,m-1}, q_{n,i})$, additional Lagrange multipliers $\lambda_{n,i}(t)$ are introduced [14]. The variables $(q_{n,m-1}, q_{n,i}, \lambda_{n,i})$, thus, determine the set of independent degrees of freedom of the extended Lagrangian system. The extended Lagrangian of this auxiliary description of the system is given by

$$L_T(q_{n,i}, q_{n,m-1}, \dot{q}_{n,i}, \dot{q}_{n,m-1}, \lambda_{n,i}) = L_0(q_{n,i}, q_{n,m-1}, \dot{q}_{n,m-1}) + \sum_{i=0}^{m-2} \lambda_{n,i}(\dot{q}_{n,i} - q_{n,i+1}). \tag{4.3}$$

The new Lagrangian in (4.3) is singular, and one can use the standard methods of singular systems like Dirac's method or the canonical approach to investigate this Lagrangian.

Upon introducing the canonical momenta:

$$p_{n,m-1} = \frac{\partial L_T}{\partial \dot{q}_{n,m-1}}, \tag{4.4}$$

$$p_{n,i} = \frac{\partial L_T}{\partial \dot{q}_{n,i}} = \lambda_{n,i} = -H_{n,i}, \tag{4.5}$$

$$\pi_{n,i} = \frac{\partial L_T}{\partial \dot{\lambda}_{n,i}} = 0 = -\Phi_{n,i}, \tag{4.6}$$

the canonical Hamiltonian can be obtained as

$$\begin{aligned} &H_0\left(q_{n,i}, q_{n,m-1}, p_{n,m-1}, \lambda_{n,i}\right) \\ &\quad = p_{n,m-1}\dot{q}_{n,m-1} + \sum_{i=0}^{m-2} p_{n,i}\dot{q}_{n,i} + \sum_{i=0}^{m-2} \pi_{n,i}\dot{\lambda}_{n,i} - L_T\left(q_{n,i}, q_{n,m-1}, \dot{q}_{n,i}, \dot{q}_{n,m-1}, \lambda_{n,i}\right), \end{aligned} \tag{4.7}$$

Equations (4.5) and (4.6) represent primary constraints [1, 2]. Their Hamilton-Jacobi partial differential equations can be obtained as

$$H_0' = p_0 + H_0\left(q_{n,i}, q_{n,m-1}, p_{n,m-1}, \lambda_{n,i}\right) = 0, \tag{4.8}$$

$$\Phi_{n,i}' = \pi_{n,i} = 0, \tag{4.9}$$

$$H_{n,i}' = p_{n,i} - \lambda_{n,i} = 0. \tag{4.10}$$

The equations of motion can be written as total differential equations in many variables as follows:

$$dq_{n,j} = dq_{n,j}, \tag{4.11}$$

$$dq_{n,m-1} = \frac{\partial H_0'}{\partial p_{n,m-1}} dt, \tag{4.12}$$

$$\begin{aligned} dp_{n,j} &= -\frac{\partial H_0'}{\partial q_{n,j}} dt, \\ dp_{n,m-1} &= -\frac{\partial H_0'}{\partial q_{n,m-1}} dt, \\ d\lambda_{n,j} &= d\lambda_{n,j}, \\ d\pi_{n,j} &= -\frac{\partial H_0'}{\partial \lambda_{n,j}} dt + dq_{n,j}, \quad j = 0, 1, \ldots, m-2. \end{aligned} \tag{4.13}$$

The total differential equations are integrable if and only if

$$\begin{aligned} dH_0' &= dp_0 - dH_0 = 0, \\ dH_{n,j}' &= dp_{n,j} - d\lambda_{n,j} = 0, \\ d\Phi_{n,j}' &= d\pi_{n,j} = 0. \end{aligned} \tag{4.14}$$

5. The Canonical Path Integral Quantization

If the coordinates $t,\ q_{n,i},\ \lambda_{n,i}$ are denoted by t_α, that is,

$$t_\alpha = t, q_{n,i}, \lambda_{n,i}, \tag{5.1}$$

then the set of primary constraints (4.8), (4.9), and (4.10) can be written in a compact form as

$$H'_\alpha = 0, \tag{5.2}$$

where

$$H'_\alpha = H'_0, H'_{n,i}, \Phi'_{n,i}. \tag{5.3}$$

Making use of [7], the canonical path integral for the extended Lagrangians can be obtained as

$$\begin{aligned} &K\left(q'_{n,m-1}, q'_{n,i}, \lambda'_{n,i}, t'; q_{n,m-1}, q_{n,i}, \lambda_{n,i}, t\right) \\ &\quad = \int_{q_{n,m-1}}^{q'_{n,m-1}} \prod_{n=1}^{N} \left(Dq_{n,m-1} Dp_{n,m-1}\right) \exp\left[\frac{i}{\hbar}\int_{t_\alpha}^{t'_\alpha}\left(-\overline{H}_\alpha + p_{n,m-1}\frac{\partial \overline{H'_\alpha}}{\partial p_{n,m-1}}\right)dt_\alpha\right], \\ &\qquad\qquad n = 1,\dots,N,\ i = 0,\dots,m-2. \end{aligned} \tag{5.4}$$

Note that (4.12) gives

$$\frac{\partial H'_\alpha}{\partial p_{n,m-1}}dt_\alpha = \frac{\partial H'_0}{\partial p_{n,m-1}}dt + \frac{\partial \Phi'_{n,i}}{\partial p_{n,m-1}}d\lambda_{n,i} + \frac{\partial H'_{n,i}}{\partial p_{n,m-1}}dq_{n,i} = dq_{n,m-1}. \tag{5.5}$$

Therefore, (5.4) can be written as

$$\begin{aligned} &K\left(q'_{n,m-1}, q'_{n,i}, \lambda'_{n,i}, t'; q_{n,m-1}, q_{n,i}, \lambda_{n,i}, t\right) \\ &\quad = \int_{q_{n,m-1}}^{q'_{n,m-1}} \prod_{n=1}^{N} \left(Dq_{n,m-1} Dp_{n,m-1}\right) \exp\left[\frac{i}{\hbar}\int_{t_\alpha}^{t'_\alpha}\left(-\overline{H}_\alpha dt_\alpha + p_{n,m-1}dq_{n,m-1}\right)\right]. \end{aligned} \tag{5.6}$$

However, according to (4.6) and (4.7), we get

$$\begin{aligned} H_{n,i} &= -\lambda_{n,i} \\ \Phi_{n,i} &= 0, \end{aligned} \tag{5.7}$$

so, it can bee found that

$$H_\alpha dt_\alpha = H_0 dt + H_{n,i}dq_{n,i} + \Phi_{n,i}d\lambda_{n,i} = H_0 dt - \lambda_{n,i}dq_{n,i}. \tag{5.8}$$

Then the transition amplitude can be written in the final form as

$$K\left(q'_{n,m-1}, q'_{n,i}, \lambda'_{n,i}, t'; q_{n,m-1}, q_{n,i}, \lambda_{n,i}, t\right) = \int_{q_{n,m-1}}^{q'_{n,m-1}} \prod_{n=1}^{N} \left(Dq_{n,m-1} Dp_{n,m-1}\right) \exp\left[\frac{i}{\hbar}\int_{t_\alpha}^{t'_\alpha}\left(-\overline{H}_0 dt + \lambda_{n,i} dq_{n,i} + p_{n,m-1} dq_{n,m-1}\right)\right]. \tag{5.9}$$

Equation (5.9) represents the canonical path integral quantization of higher-order regular Lagrangians as first-order singular Lagrangians.

6. Examples

In this section, the procedure described throughout this paper will be illustrated by the following two examples.

6.1. Example 1

As a first example, let us consider a one-dimensional second-order regular lagrangian of the form:

$$L_0 = \frac{1}{2}\left(\ddot{q}_1^2 - \dot{q}_1^2 - q_1^2\right). \tag{6.1}$$

If (4.2) is used, we can write

$$\begin{aligned} q_1^{(0)} &= q_{10}, \\ \dot{q}_1 &= q_{11}, \\ \ddot{q}_1 &= \dot{q}_{11}, \end{aligned} \tag{6.2}$$

Hence, the Lagrangian (6.1) becomes

$$L_0 = \frac{1}{2}\left(\dot{q}_{11}^2 - q_{11}^2 - q_{10}^2\right). \tag{6.3}$$

Upon using (4.1), the recursion relation is $\dot{q}_{10} = q_{11}$. And with the aid of (4.3), the extended Lagrangian is simply

$$L_T = \frac{1}{2}\left(\dot{q}_{11}^2 - q_{11}^2 - q_{10}^2\right) + \lambda_{10}\left(\dot{q}_{10} - q_{11}\right). \tag{6.4}$$

The conjugate momenta can be obtained as

$$P_{11} = \frac{\partial L_T}{\partial \dot{q}_{11}} = \dot{q}_{11}, \qquad P_{10} = \frac{\partial L_T}{\partial \dot{q}_{10}} = \lambda_{10}, \qquad \pi_{10} = \frac{\partial L_T}{\partial \dot{\lambda}_{10}} 0. \tag{6.5}$$

It is obvious that the second and third equations are constraints. Therefore, the coordinates q_{10} and λ_{10} represent the restricted coordinates.

Using (4.4), the canonical Hamiltonian takes the form:

$$H_0 = \frac{1}{2}\left(P_{11}^2 + q_{10}^2 + q_{11}^2\right) + \lambda_{10} q_{11}. \tag{6.6}$$

Accordingly, the set of HJPDE's can be written as

$$\begin{aligned} H_0' &= P_0 + H_0 = 0, \\ H_{10}' &= P_{10} - \lambda_{10} = 0, \\ \Phi_{10}' &= \pi_{10} = 0. \end{aligned} \tag{6.7}$$

From (5.9), the canonical path integral quantization for this system is

$$K\left(q_{11}', q_{10}', \lambda_{10}', t'; q_{11}, q_{10}, \lambda_{10}, t\right) = \int Dq_{11} Dp_{11} \exp\left[\frac{i}{\hbar}\int \left(-H_0 dt + p_{11} dq_{11} + \lambda_{10} dq_{10}\right)\right], \tag{6.8}$$

where $Dq_{11} = \lim_{k\to\infty}\prod_{j=1}^{k-1} dq_{11j};\ Dp_{11} = \lim_{k\to\infty}\prod_{j=0}^{k-1}(dp_{11}/2\pi\hbar)$.

$$K = \int Dq_{11} Dp_{11} \exp\left[\frac{i}{\hbar}\int\left(\left(-\frac{1}{2}\left(P_{11}^2 + q_{10}^2 + q_{11}^2\right) - \lambda_{10} q_{11}\right) dt + p_{11} dq_{11} + \lambda_{10} dq_{10}\right)\right]. \tag{6.9}$$

Equation (6.9) can be written in a compact form as

$$K = \int Dq_{11} Dp_{11} \exp\left[\frac{i}{\hbar}\int\left(-\frac{1}{2}\left(P_{11}^2 + q_{10}^2 + q_{11}^2\right) - \lambda_{10} q_{11} + p_{11}\dot{q}_{11} + \lambda_{10}\dot{q}_{10}\right) dt\right]. \tag{6.10}$$

Upon changing the integration over dt to summation, we have

$$K = \int Dq_{11} \prod_{j=0}^{k-1}\left(\frac{dp_{11j}}{2\pi\hbar}\right) \exp\left[\frac{i\varepsilon}{\hbar}\sum_{j=0}^{k-1}\left(-\frac{p_{11j}^2}{2} - \frac{q_{10j}^2}{2} - \frac{q_{11j}^2}{2} + p_{11j}\dot{q}_{11j} + \lambda_{10j}\left(\dot{q}_{10j} - q_{11j}\right)\right)\right]. \tag{6.11}$$

The p_{11j}-integration can be performed using the Gaussian integral:

$$\begin{aligned} K &= \int Dq_{11} \frac{1}{(2\pi\hbar)^k}\left(\frac{2\pi\hbar}{i\varepsilon}\right)^{k/2} \exp\left[\frac{i\varepsilon}{\hbar}\sum_{j=0}^{k-1}\left(\frac{\dot{q}_{11j}^2}{2} - \frac{q_{10j}^2}{2} - \frac{q_{11j}^2}{2} + \lambda_{10j}\left(\dot{q}_{10j} - q_{11j}\right)\right)\right] \\ &= \left(\frac{1}{2\pi\hbar i\varepsilon}\right)^{k/2} \int Dq_{11} \exp\left[\frac{i}{\hbar}\int\left(\frac{\dot{q}_{11}^2}{2} - \frac{q_{10}^2}{2} - \frac{q_{11}^2}{2} + \lambda_{10}\left(\dot{q}_{10} - q_{11}\right)\right) dt\right] \\ &= \left(\frac{1}{2\pi\hbar i\varepsilon}\right)^{k/2} \int Dq_{11} \exp\left[\frac{i}{\hbar}\int L_T dt\right]. \end{aligned} \tag{6.12}$$

6.2. Example 2

As a second example, consider the three-dimensional second-order regular lagrangian:

$$L_0 = \frac{1}{2}\left(\ddot{q}_1^2 + \ddot{q}_2^2 + \ddot{q}_3^2\right) - \frac{1}{2}\left(\dot{q}_1^2 + \dot{q}_3^2\right). \tag{6.13}$$

If we put

$$\begin{aligned} &q_1^{(0)} = q_{10}, \qquad q_2^{(0)} = q_{20}, \qquad q_3^{(0)} = q_{30}, \\ &\dot{q}_1 = q_{11}, \qquad \dot{q}_2 = q_{21}, \qquad \dot{q}_3 = q_{31}, \\ &\ddot{q}_1 = \dot{q}_{11}, \qquad \ddot{q}_2 = \dot{q}_{21}, \qquad \ddot{q}_3 = \dot{q}_{31}. \end{aligned} \tag{6.14}$$

then the above Lagrangian can be written as

$$L_0 = \frac{1}{2}\left(\dot{q}_{11}^2 + \dot{q}_{21}^2 + \dot{q}_{31}^2\right) - \frac{1}{2}\left(q_{11}^2 + q_{31}^2\right). \tag{6.15}$$

Here the recursion relations are

$$\dot{q}_{10} = q_{11}; \qquad \dot{q}_{20} = q_{21}, \qquad \dot{q}_{30} = q_{31}. \tag{6.16}$$

Accordingly, the extended Lagrangian can be given as:

$$L_T = \frac{1}{2}\left(\dot{q}_{11}^2 + \dot{q}_{21}^2 + \dot{q}_{31}^2\right) - \frac{1}{2}\left(q_{11}^2 + q_{31}^2\right) + \lambda_{10}(\dot{q}_{10} - q_{11}) + \lambda_{20}(\dot{q}_{20} - q_{21}) + \lambda_{30}(\dot{q}_{30} - q_{31}). \tag{6.17}$$

The corresponding momenta are calculated as

$$\begin{aligned} &p_{11} = \dot{q}_{11}, \qquad p_{10} = \lambda_{10}, \qquad \pi_{10} = 0, \\ &p_{21} = \dot{q}_{21}, \qquad p_{20} = \lambda_{20}, \qquad \pi_{20} = 0, \\ &p_{31} = \dot{q}_{31}, \qquad p_{30} = \lambda_{30}, \qquad \pi_{30} = 0 \end{aligned} \tag{6.18}$$

Therefore, the canonical Hamiltonian reads

$$H_0 = \frac{p_{11}^2}{2} + \frac{p_{21}^2}{2} + \frac{p_{31}^2}{2} + \frac{1}{2}\left(q_{11}^2 + q_{31}^2\right) + \lambda_{10} q_{11} + \lambda_{20} q_{21} + \lambda_{30} q_{31}. \tag{6.19}$$

Thus, the set of HJPDE's can be written as

$$
\begin{aligned}
H_0' &= P_0 + H_0 = 0, \\
\Phi_{10}' &= \pi_{10} = 0, \\
\Phi_{20}' &= \pi_{20} = 0, \\
\Phi_{30}' &= \pi_{30} = 0, \\
H_{10}' &= p_{10} - \lambda_{10} = 0, \\
H_{20}' &= p_{20} - \lambda_{20} = 0, \\
H_{30}' &= p_{30} - \lambda_{30} = 0.
\end{aligned}
\tag{6.20}
$$

Then, the canonical path integral quantization for this system is constructed as

$$
\begin{aligned}
&K\left(q_{n1}', q_{n0}', \lambda_{n0}', t'; q_{n1}, q_{n0}, \lambda_{n0}, t\right) \\
&\quad = \int \prod_{n=1}^{3} \left(Dq_{n1} Dp_{n1}\right) \exp\left[\frac{i}{\hbar} \int \left(-H_0 dt + \lambda_{n0} dq_{n0} + p_{n1} dq_{n1}\right)\right],
\end{aligned}
\tag{6.21}
$$

where $n = 1, 2, 3$.

$$
K = \int \prod_{n=1}^{3} \left(Dq_{n1} Dp_{n1}\right) \exp\left[\frac{i}{\hbar} \int \left(-\frac{p_{n1}^2}{2} - \frac{q_{11}^2}{2} - \frac{q_{31}^2}{2} + \lambda_{n0}\left(\dot{q}_{n0} - q_{n1}\right) + p_{n1}\dot{q}_{n1}\right) dt\right].
\tag{6.22}
$$

Changing the integration over dt to summation and integrating over p_{11},p_{21} and p_{31} k times we get

$$
\begin{aligned}
K &= \left(\frac{1}{2\pi\hbar i\varepsilon}\right)^{3k/2} \int \prod_{n=1}^{3} Dq_{n1} \exp\left[\frac{i}{\hbar} \int \left(\frac{\dot{q}_{n1}^2}{2} - \frac{q_{11}^2}{2} - \frac{q_{31}^2}{2} + \lambda_{n0}\left(\dot{q}_{n0} - q_{n1}\right)\right) dt\right] \\
&= \left(\frac{1}{2\pi\hbar i\varepsilon}\right)^{3k/2} \int Dq_{11} Dq_{21} Dq_{31} \exp\left[\frac{i}{\hbar} \int L_T dt\right].
\end{aligned}
\tag{6.23}
$$

7. Conclusion

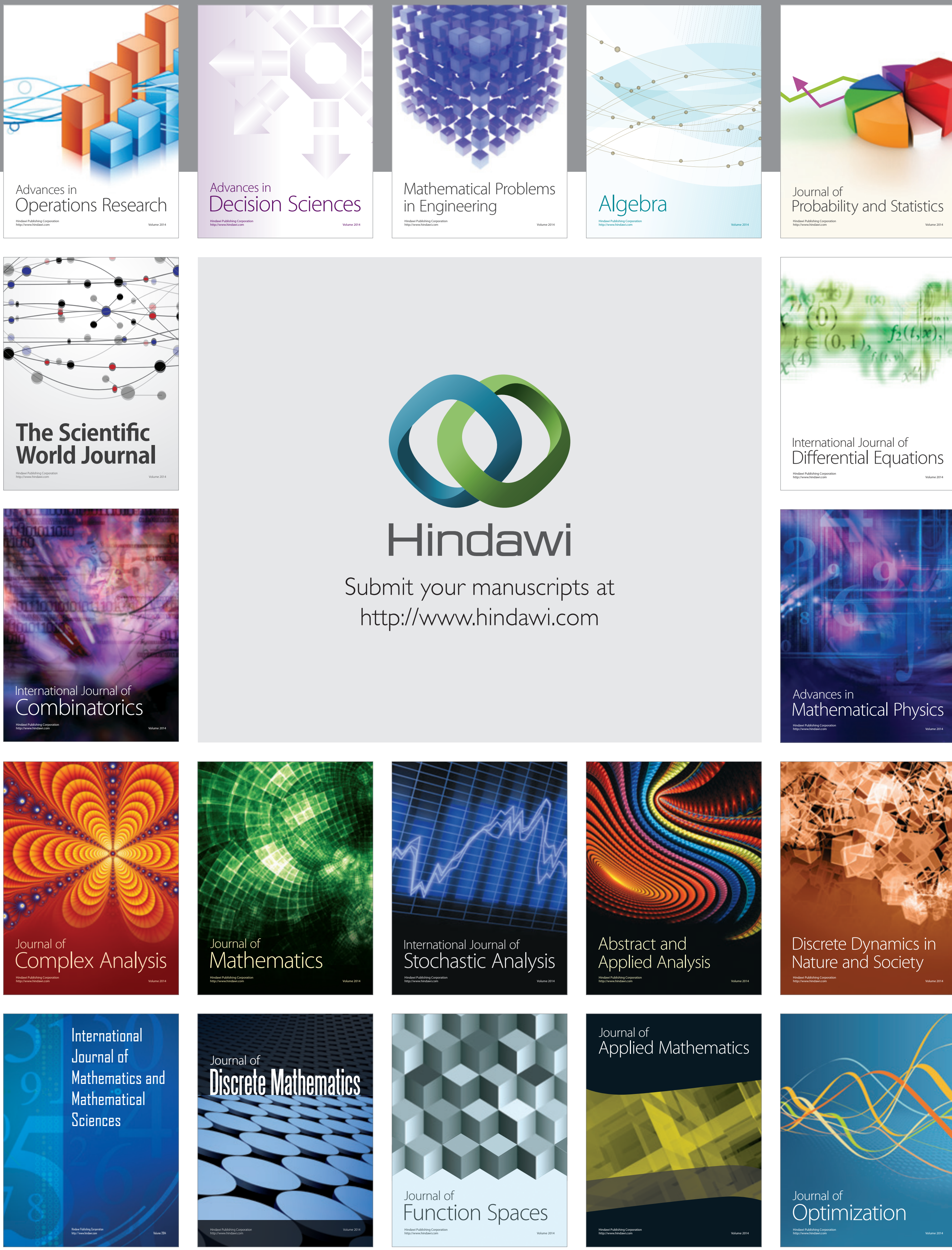
In this work, we have investigated the canonical path integral quantization of higher-order regular Lagrangians. Where the higher-order regular Lagrangians are first treated as first-order singular Lagrangians, this means that each velocity term $\dot{q}_{n,i}$ is replaced by a new function $q_{n,i+1}$, which is led to a constraint equation, $q_{n,i+1} - \dot{q}_{n,i} = 0$, that is added to the original Lagrangian. The same procedure is repeated for the second and other higher order terms of velocities. Every time, a new constraint is obtained and added to the original Lagrangian. As a result to this procedure, the new constructed Lagrangian is the extended first-order Lagrangian.

Once the extended Lagrangian is obtained, it is treated using the well-known Hamilton-Jacobi method which enables us to obtain the equations of motion. Besides, the action integral can be derived and the quantization of the system may be investigated using the canonical path integral approximation.

In this treatment, we believe that the local structure of phase space and its local simplistic geometry is more transparent than in Ostrogradski's approach. In Ostrogradski's approach, the structure of phase space leads to confusion when considering canonical path integral quantization.

References

[1] P. A. M. Dirac, "Generalized Hamiltonian dynamics," *Canadian Journal of Mathematics*, vol. 2, pp. 129–148, 1950.
[2] P. A. M. Dirac, *Lectures on Quantum Mechanics*, vol. 2, Belfer Graduate School of Science, Yeshiva University, New York, NY, USA, 1967.
[3] E. M. Rabei and Y. Guler, "Hamilton-Jacobi treatment of second-class constraints," *Physical Review A*, vol. 46, no. 6, pp. 3513–3515, 1992.
[4] C. Carathéodory, *Calculus of Variations and Partial Differential Equations of the First Order*, Holden-Day, San Francisco, Calif, USA, 1967.
[5] B. M. Pimentel and R. G. Teixeira, "Hamilton-Jacobi formulation for singular systems with second-order Lagrangians," *Il Nuovo Cimento B*, vol. 111, no. 7, pp. 841–854, 1996.
[6] B. M. Pimentel and R. G. Teixeira, "Generalization of the Hamilton-Jacobi approach for higher-order singular systems," *Il Nuovo Cimento B*, vol. 113, no. 6, pp. 805–817, 1998.
[7] S. Muslih and Y. Güler, "The Feynman path integral quantization of constrained systems," *Il Nuovo Cimento B*, vol. 112, no. 1, pp. 97–107, 1997.
[8] E. M. Rabei, "On the quantization of constrained systems using path integral techniques," *Il Nuovo Cimento B*, vol. 115, no. 10, pp. 1159–1165, 2000.
[9] S. I. Muslih, "Quantization of singular systems with second-order Lagrangians," *Modern Physics Letters A*, vol. 17, no. 36, pp. 2383–2391, 2002.
[10] S. I. Muslih, "Path integral formulation of constrained systems with singular-higher order Lagrangians," *Hadronic Journal*, vol. 24, no. 6, pp. 713–720, 2001.
[11] E. M. Rabei, K. I. Nawafleh, and H. B. Ghassib, "Quantization of constrained systems using the WKB approximation," *Physical Review A*, vol. 66, no. 2, pp. 024101/1–024101/4, 2002.
[12] M. V. Ostrogradski, *Mémoires de l'Académie de Saint Peters Bourg*, vol. 6, p. 385, 1850.
[13] J. Govaerts, *Hamiltonian Quantization and Constrained Dynamics*, Leuven University Press, Leuven, Belgium, 1991.
[14] J. M. Pons, "Ostrogradski's theorem for higher-order singular Lagrangians," *Letters in Mathematical Physics*, vol. 17, no. 3, pp. 181–189, 1989.
[15] K. I. Nawafleh and A. A. Alsoub, "Quantization of higher order regular lagrangians as first order singular lagrangians using path integral approach," *Jordan Journal of Physics*, vol. 1, no. 2, pp. 73–78, 2008.


Advances in
Operations Research
Advances in
Decision Sciences
Mathematical Problems
in Engineering
Algebra
Journal of
Probability and Statistics
The Scientific
World Journal
International Journal of
Differential Equations
Hindawi
Submit your manuscripts at
http://www.hindawi.com
International Journal of
Combinatorics
Advances in
Mathematical Physics
Journal of
Complex Analysis
Journal of
Mathematics
International Journal of
Stochastic Analysis
Abstract and
Applied Analysis
Discrete Dynamics in
Nature and Society
International
Journal of
Mathematics and
Mathematical
Sciences
Journal of
Discrete Mathematics
Journal of
Function Spaces
Journal of
Applied Mathematics
Journal of
Optimization