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Research Article

Global Well-Posedness for a Family of MHD-Alpha-Like Models

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Global well-posedness is proved for a family of n-dimensional MHD-alpha-like models.

1. Introduction

In this paper, we consider a family of MHD-alpha-like models:

$$\partial_t v + (-\Delta)^{\theta_2} v + u \cdot \nabla u + \nabla \left(p + \frac{1}{2} b^2 \right) = b \cdot \nabla b, \tag{1.1}$$

$$\partial_t H + (-\Delta)^{\theta_2} H + u \cdot \nabla b - b \cdot \nabla u = 0, \tag{1.2}$$

$$v = \left[1 + \left(-\alpha^2 \Delta \right)^{\theta_1} \right] u, \quad H = \left[1 + \left(-\alpha_M^2 \Delta \right)^{\theta_1} \right] b, \quad \alpha > 0, \alpha_M > 0, \tag{1.3}$$

$$\operatorname{div} v = \operatorname{div} u = \operatorname{div} H = \operatorname{div} b = 0, \tag{1.4}$$

$$(v, H)(0) = (v_0, H_0) \quad \text{in } \mathbb{R}^n (n \geq 3), \tag{1.5}$$

where v is the fluid velocity field, u is the "filtered" fluid velocity, p is the pressure, H is the magnetic field, and b is the "filtered" magnetic field. $\alpha > 0$ and $\alpha_M > 0$ are the length scales and for simplicity we will take $\alpha = \alpha_M = 1$. The parameter $\theta_1 \geq 0$ affects the strength of the nonlinear term and $\theta_2 \geq 0$ represents the degree of viscous dissipation satisfying

$$3\theta_1 + 2\theta_2 = \frac{n+2}{2}. \tag{1.6}$$

When $\theta_1 = \theta_2 = 1$ and $n = 3$, a global well-posedness is proved in [1]. The aim of this paper is to prove a global well-posedness theorem under (1.6). We will prove the following theorem.

Theorem 1.1. *Let* $(u_0, b_0) \in H^s$ *with* $s \geq 1$, $\operatorname{div} v_0 = \operatorname{div} u_0 = \operatorname{div} H_0 = \operatorname{div} b_0 = 0$ *in* $\mathbb{R}^n$, *and* (1.6) *holding true. Then for any* $T > 0$, *there exists a unique strong solution* (u, b) *satisfying*

$$(u, b) \in L^\infty\left(0, T; H^{s+\theta_1}\right) \cap L^2\left(0, T; H^{s+\theta_1+\theta_2}\right). \tag{1.7}$$

Remark 1.2. For studies on some standard MHD-α or Leray-α models, we refer to [2–7] and references therein.

2. Proof of Theorem 1.1

Since it is easy to prove that the problem (1.1)–(1.5) has a unique local smooth solution, we only need to establish the a priori estimates.

Testing (1.1) by u, using (1.3) and (1.4), and letting $\Lambda := (-\Delta)^{1/2}$, we see that

$$\frac{1}{2}\frac{d}{dt}\int u^2 + \left|\Lambda^{\theta_1} u\right|^2 dx + \int \left|\Lambda^{\theta_2} u\right|^2 + \left|\Lambda^{\theta_1+\theta_2} u\right|^2 dx = \int (b \cdot \nabla) b \cdot u dx. \tag{2.1}$$

Testing (1.2) by b and using (1.3) and (1.4), we find that

$$\frac{1}{2}\frac{d}{dt}\int b^2 + \left|\Lambda^{\theta_1} b\right|^2 dx + \int \left|\Lambda^{\theta_2} b\right|^2 + \left|\Lambda^{\theta_1+\theta_2} b\right|^2 dx = \int (b \cdot \nabla) u \cdot b dx. \tag{2.2}$$

Summing up (2.1) and (2.2), thanks to the cancellation of the right-hand side of (2.1) and (2.2), we infer that

$$\frac{1}{2}\frac{d}{dt}\int (u,b)^2 + \left|\Lambda^{\theta_1}(u,b)\right|^2 dx + \int \left|\Lambda^{\theta_2}(u,b)\right|^2 + \left|\Lambda^{\theta_1+\theta_2}(u,b)\right|^2 dx = 0, \tag{2.3}$$

whence

$$\|(u,b)\|_{L^2(0,T;H^{\theta_1+\theta_2})} \leq C. \tag{2.4}$$

Case 1. $\theta_1 + \theta_2 > 1$.

In the following calculations, we will use the following commutator estimates due to Kato and Ponce [8]:

$$\|\Lambda^s(fg) - f\Lambda^s g\|_{L^p} \le C\left(\|\nabla f\|_{L^{p_1}} \left\|\Lambda^{s-1} g\right\|_{L^{q_1}} + \|\Lambda^s f\|_{L^{p_2}} \|g\|_{L^{q_2}}\right), \tag{2.5}$$

with $s > 0$ and $1/p = 1/p_1 + 1/q_1 = 1/p_2 + 1/q_2$.
We will also use the Sobolev inequality:

$$\|\nabla u\|_{L^p} \le C\left\|\Lambda^{\theta_1+\theta_2} u\right\|_{L^2}\left(1 - \frac{n}{p} = \theta_1 + \theta_2 - \frac{n}{2}\right), \tag{2.6}$$

and the Gagliardo-Nirenberg inequality:

$$\|\Lambda^s u\|_{L^{2p/p-1}}^2 \le C\left\|\Lambda^{s+\theta_1} u\right\|_{L^2}\left\|\Lambda^{s+\theta_1+\theta_2} u\right\|_{L^2}. \tag{2.7}$$

Taking Λ^s to (1.1), testing by $\Lambda^s u$, and using (1.3) and (1.4), we infer that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int |\Lambda^s u|^2 + \left|\Lambda^{s+\theta_1} u\right|^2 dx + \int \left|\Lambda^{s+\theta_2} u\right|^2 + \left|\Lambda^{s+\theta_1+\theta_2} u\right|^2 dx \\
&\quad = -\int [\Lambda^s(u\cdot\nabla u) - u\cdot\nabla\Lambda^s u]\Lambda^s u\,dx + \int [\Lambda^s(b\cdot\nabla b) - b\cdot\nabla\Lambda^s b]\Lambda^s u\,dx \\
&\qquad + \int b\cdot\nabla\Lambda^s b\cdot\Lambda^s u\,dx.
\end{aligned} \tag{2.8}$$

Taking Λ^s to (1.2), testing by $\Lambda^s b$, and using (1.3) and (1.4), we deduce that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int |\Lambda^s b|^2 + \left|\Lambda^{s+\theta_1} b\right|^2 dx + \int \left|\Lambda^{s+\theta_2} b\right|^2 + \left|\Lambda^{s+\theta_1+\theta_2} b\right|^2 dx \\
&\quad = -\int [\Lambda^s(u\cdot\nabla b) - u\cdot\nabla\Lambda^s b]\Lambda^s b\,dx + \int [\Lambda^s(b\cdot\nabla u) - b\cdot\nabla\Lambda^s u]\Lambda^s b\,dx \\
&\qquad + \int b\cdot\nabla\Lambda^s u\cdot\Lambda^s b\,dx.
\end{aligned} \tag{2.9}$$

Summing up (2.8) and (2.9), thanks to the cancellation of the right-hand side of (2.8) and (2.9), and using (2.5), (2.6) and (2.7), we conclude that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int |\Lambda^s(u,b)|^2 + \left|\Lambda^{s+\theta_1}(u,b)\right|^2 dx + \int \left|\Lambda^{s+\theta_2}(u,b)\right|^2 + \left|\Lambda^{s+\theta_1+\theta_2}(u,b)\right|^2 dx \\
&\quad \le C\|\nabla u\|_{L^p}\|\Lambda^s u\|^2_{L^{2p/p-1}} + C\|\nabla b\|_{L^p}\|\Lambda^s b\|_{L^{2p/p-1}}\|\Lambda^s u\|_{L^{2p/p-1}} + C\|\nabla u\|_{L^p}\|\Lambda^s b\|^2_{L^{2p/p-1}} \\
&\quad \le C\|\nabla(u,b)\|_{L^p}\|\Lambda^s(u,b)\|^2_{L^{2p/p-1}} \\
&\quad \le C\left\|\Lambda^{\theta_1+\theta_2}(u,b)\right\|_{L^2}\left\|\Lambda^{s+\theta_1}(u,b)\right\|_{L^2}\left\|\Lambda^{s+\theta_1+\theta_2}(u,b)\right\|_{L^2} \\
&\quad \le \frac{1}{2}\left\|\Lambda^{s+\theta_1+\theta_2}(u,b)\right\|^2_{L^2} + C\left\|\Lambda^{\theta_1+\theta_2}(u,b)\right\|^2_{L^2}\left\|\Lambda^{s+\theta_1}(u,b)\right\|^2_{L^2},
\end{aligned} \tag{2.10}$$

which implies (1.7).

Case 2. $0 < \theta_1 + \theta_2 \le 1$ only when $n = 3$.

Testing (1.1) by v, using (1.4), we see that

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int v^2 dx + \int \left|\Lambda^{\theta_2} v\right|^2 dx &= \int (b\cdot\nabla b - u\cdot\nabla u) v\,dx \\
&\le \left(\|b\|_{L^{p_1}}\|\nabla b\|_{L^{2p_1/p_1-2}} + \|u\|_{L^{p_1}}\|\nabla u\|_{L^{2p_1/p_1-2}}\right)\|v\|_{L^2} \\
&\le \|(u,b)\|_{L^{p_1}}\|\nabla(u,b)\|_{L^{2p_1/p_1-2}}\|v\|_{L^2} \\
&\le C\|(u,b)\|_{H^{\theta_1+\theta_2}}\left\|\Lambda^{\theta_2}(v,H)\right\|_{L^2}\|v\|_{L^2}.
\end{aligned} \tag{2.11}$$

Here we have used the Sobolev inequalities

$$\begin{aligned}
&\|(u,b)\|_{L^{p_1}} \le C\|(u,b)\|_{H^{\theta_1+\theta_2}}\left(-\frac{3}{p_1} = \theta_1 + \theta_2 - \frac{3}{2}\right), \\
&\|\nabla(u,b)\|_{L^{2p_1/p_1-2}} \le C\left\|\Lambda^{\theta_2}(v,H)\right\|_{L^2}\left(1 - \frac{3(p_1-2)}{2p_1} = \theta_2 + 2\theta_1 - \frac{3}{2}\right).
\end{aligned} \tag{2.12}$$

Similarly, testing (1.2) by H and using (1.4) and (2.12), we find that

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int H^2 dx + \int \left|\Lambda^{\theta_2} H\right|^2 dx &= \int (b\cdot\nabla u - u\cdot\nabla b) H\,dx \\
&\le \|(u,b)\|_{L^{p_1}}\|\nabla(u,b)\|_{L^{2p_1/p_1-2}}\|H\|_{L^2} \\
&\le C\|(u,b)\|_{H^{\theta_1+\theta_2}}\left\|\Lambda^{\theta_2}(v,H)\right\|_{L^2}\|H\|_{L^2}.
\end{aligned} \tag{2.13}$$

Combining (2.11) and (2.13) and using (2.4) and the Gronwall inequality, we have

$$\|(u,b)\|_{L^2(0,T;H^{\theta_2+2\theta_1})} \le C. \tag{2.14}$$

Similarly to (2.10), we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int |\Lambda^s(u,b)|^2 + \left|\Lambda^{s+\theta_1}(u,b)\right|^2 dx + \int \left|\Lambda^{s+\theta_2}(u,b)\right|^2 + \left|\Lambda^{s+\theta_1+\theta_2}(u,b)\right|^2 dx \\
&\quad \le C\|\nabla(u,b)\|_{L^{p_2}}\|\Lambda^s(u,b)\|^2_{L^{2p_2/p_2-1}} \\
&\quad \le C\|(u,b)\|_{H^{\theta_2+2\theta_1}}\left\|\Lambda^{s+\theta_1}(u,b)\right\|^{2(1-\alpha_1)}_{L^2}\left\|\Lambda^{s+\theta_1+\theta_2}(u,b)\right\|^{2\alpha_1}_{L^2} \\
&\quad \le \frac{1}{2}\left\|\Lambda^{s+\theta_1+\theta_2}(u,b)\right\|^2_{L^2} + C\|(u,b)\|^{1/1-\alpha_1}_{H^{\theta_2+2\theta_1}}\left\|\Lambda^{s+\theta_1}(u,b)\right\|^2_{L^2},
\end{aligned} \tag{2.15}$$

which implies (1.7) by $1/(1-\alpha_1) \le 2$. Here we have used the Sobolev inequality:

$$\|\nabla(u,b)\|_{L^{p_2}} \le C\|(u,b)\|_{H^{\theta_2+2\theta_1}}\left(1-\frac{n}{p_2} < \theta_2 + 2\theta_1 - \frac{n}{2}\right) \tag{2.16}$$

and the Gagliardo-Nirenberg inequality:

$$\|\Lambda^s(u,b)\|_{L^{2p_2/(p_2-1)}} \le C\left\|\Lambda^{s+\theta_1}(u,b)\right\|^{1-\alpha_1}_{L^2}\left\|\Lambda^{s+\theta_1+\theta_2}(u,b)\right\|^{\alpha_1}_{L^2}, \tag{2.17}$$

with $-((p_2-1)/2p_2)n = \alpha_1\theta_2 + \theta_1 - n/2$ and $p_2 \ge 2 \ge 3/(2\theta_1+\theta_2)$. This completes the proof.

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