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Research Article

On Properties of the Choquet Integral of Interval-Valued Functions

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Based on the concept of an interval-valued function which is motivated by the goal to represent an uncertain function, we define the Choquet integral with respect to a fuzzy measure of interval-valued functions. We also discuss convergence in the (C) mean and convergence in a fuzzy measure of sequences of measurable interval-valued functions. In particular, we investigate the convergence theorem for the Choquet integral of measurable interval-valued functions.

1. Introduction

Wang [1], Pedrycz et al. [2], Ha and Wu [3], and T. Murofushi et al. [4, 5] defined the concepts of various convergence of sequences of measurable functions and discussed its theoretical underpinnings along with related interpretation issues. Many researchers [2, 4–11] also have been studying the Choquet integral which is regarded as one of aggregation operator being used in the decision making and information theory.

The main idea of this study is the concept of interval-valued functions which is associated with the representation of uncertain functions. In the past decade, it has been suggested to use intervals in order to represent uncertainty, for examples, closed set-valued functions [4, 7–13], interval-valued probability [14], fuzzy set-valued measures [13], and economic uncertainty [14].

In Section 2, we list definitions and basic properties of a fuzzy measure, the Choquet integral, and various convergences of sequences of measurable functions. In Section 3, we provide the new definitions of the Choquet integral with respect to a fuzzy measure of measurable interval-valued functions as well as various convergences of sequences of measurable interval-valued functions and investigate their properties. We also discuss convergence in the (C) mean and convergence in a fuzzy measure of sequences of measurable interval-valued functions. In particular, we prove the convergence theorem for the Choquet integral of measurable interval-valued functions. In Section 4, we give a brief summary results and some conclusions.

2. Preliminaries and Definitions

Let $(X, \mathfrak{B})$ be a measurable space, where X denote a nonempty set, and $\mathfrak{B}$ stands for a σ-algebra of subsets of X. Denote $\mathfrak{F}$ by the set of all nonnegative measurable functions on $(X, \mathfrak{B})$, $R^+ = [0, \infty)$, and $R^+ = [0, \infty]$.

Definition 2.1 (see [1–5]). (1) A set function $\mu : \mathfrak{B} \rightarrow R^+$ is called a fuzzy measure if

(FM1) $\mu(\emptyset) = 0$ (vanishes on $\emptyset$);

(FM2) $A, B \in \mathfrak{B}$ and $A \subset B \Rightarrow \mu(A) \leq \mu(B)$ (monotonicity);

(FM3) $A_1 \subset A_2 \subset \cdots \subset \cdots, A_n \in \mathfrak{B}$ $(n = 1, 2, \ldots) \Rightarrow \lim_{n\to\infty}\mu(A_n) = \mu(\cup_{n=1}^{\infty} A_n)$ (continuity from below);

(FM4) $A_1 \supset A_2 \supset \cdots \supset A_n \supset \cdots, A_n \in \mathfrak{B}$ $(n = 1, 2, \ldots)$ and $\mu(A_1) < \infty \Rightarrow \lim_{n\to\infty}\mu(A_n) = \mu(\cap_{n=1}^{\infty} A_n)$ (continuity from above).

Remark that a fuzzy measure is known to be the generalization of a classical measure satisfying (FM3) and (FM4) where additivity is replaced by the weaker condition of monotonicity.

Definition 2.2 (see [5]). (1) A fuzzy measure μ is said to be autocontinuous from above (resp., below) if $A \in \mathfrak{F}$, $\{B_n\} \subset \mathfrak{F}$, and $\lim_{n\to\infty}\mu(B_n) = 0 \Rightarrow \lim_{n\to\infty}\mu(A \cup B_n) = \mu(A)$ (resp., $\lim_{n\to\infty}\mu(A \setminus B_n) = \mu(A)$).

(2) If μ is autocontinuous both from above and from below, it is said to be autocontinuous.

Definition 2.3 (see [1, 2, 4, 5]). (1) Let $A \in \mathfrak{B}$ and $f \in \mathfrak{F}$. The Choquet integral of f with respect to a fuzzy measure μ is defined by

$$(C)\int_A f\,d\mu = \int_0^{\infty} \mu(G_f(\alpha) \cap A)\,d\alpha, \tag{2.1}$$

where the integral on the right-hand side is the Lebesgue integral and $G_f(\alpha) = \{x \mid f(x) \geq \alpha\}$.

(2) A measurable function f is said to be (C) integrable if the Choquet integral of f on X exists and its value is finite.

Instead of $(C)\int_X f\,d\mu$, we write $(C)\int f\,d\mu$. Note that if we take $\mathrm{X} = \{x_1, x_2, \ldots, x_n\}$ and $\mathfrak{B} = \wp(X)$ is the power set of X and f is a measurable function on X, then

$$(C)\int_A f\,d\mu = \sum_{i=1}^{n} f(x_{(i)})\left[\mu(A_{(i)}) - \mu(A_{(i+1)})\right], \tag{2.2}$$

where $(\cdot)$ is a permutation on $\{1,2,\dots,n\}$ such that $0 = f(x_{(0)}) \le f(x_{(1)}) \le \cdots \le f(x_{(n)})$ and $A_{(i)} = \{x_{(i)}, x_{(i+1)}, \dots, x_{(n)}\}$ and $A_{(n+1)} = \emptyset$. From (2.2), clearly we have

$$(\mathrm{C})\int_A f d\mu = \sum_{i=1}^{n}\left(f\left(x_{(i)}\right) - f\left(x_{(i-1)}\right)\right)\mu\left(A_{(i)}\right). \tag{2.3}$$

Definition 2.4 (see [1, 2]). Let $\{f_n\} \subset \mathfrak{F}$ and $f \in \mathfrak{F}$. A sequence $\{f_n\}$ converges in the (C) mean to f if

$$\lim_{n\to\infty}(\mathrm{C})\int_A \left|f_n - f\right| d\mu = 0. \tag{2.4}$$

Definition 2.5 (see [1, 2]). Let $A \in \mathfrak{B}$. A sequence $\{f_n\}$ is called equally (C) integrable on A if for any given $\varepsilon > 0$, there exists $N(\varepsilon) > 0$ such that

$$(\mathrm{C})\int_A f_n d\mu \le \int_0^N \mu\left(G_{f_n}(\alpha) \cap A\right) d\alpha + \varepsilon \tag{2.5}$$

for all $n = 1, 2, \dots$.

It is easy to see that if there exists a (C) integrable function g such that $|f_n| \le g$ for all $n = 1, 2, \dots$, then $\{f_n\}$ is equally (C) integrable.

Definition 2.6 (see [2]). Let $A \in \mathfrak{B}$, $\{f_n\} \subset \mathfrak{F}$, and $f \in \mathfrak{F}$. We say that $\{f_n\}$ converges in μ to f on A if for any given $\varepsilon > 0$,

$$\lim_{n\to\infty}\mu\left(\left\{x \mid \left|f_n(x) - f(x)\right| \ge \varepsilon\right\} \cap A\right) = 0. \tag{2.6}$$

Definition 2.7 (see [2, 4, 5]). Let $f, g \in \mathfrak{F}$. f and g are comonotonic if for every pair $x, y \in X$,

$$f(x) < f(y) \Longrightarrow g(x) \le g(y). \tag{2.7}$$

Theorem 2.8 (see [1–5]). *Let $A \in \mathfrak{B}$, $f, g \in \mathfrak{F}$, and μ be a fuzzy measure.*

(1) If $f \le g$, then $(\mathrm{C})\int_A f d\mu \le (\mathrm{C})\int_A g d\mu$.

(2) If a and b are nonnegative real numbers, then

$$(\mathrm{C})\int_A (af + b) d\mu = a(\mathrm{C})\int_A f d\mu + b\mu(A). \tag{2.8}$$

(3) If f and g are comonotonic, then

$$(\mathrm{C})\int_A (f + g) d\mu = (\mathrm{C})\int_A f d\mu + (\mathrm{C})\int_A g d\mu. \tag{2.9}$$

(4) If we define $(f \vee g)(x) = f(x) \vee g(x)$ *for all* $x \in X$, *then*

$$(C)\int_A f \vee g d\mu \geq (C)\int_A f d\mu \vee (C)\int_A g d\mu. \tag{2.10}$$

(5) If we define $(f \wedge g)(x) = f(x) \wedge g(x)$ *for all* $x \in X$, *then*

$$(C)\int_A f \wedge g d\mu \leq (C)\int_A f d\mu \wedge (C)\int_A g d\mu. \tag{2.11}$$

Theorem 2.9 (see [1, 2]). *Let* $A \in \mathfrak{B}$ *and* $\{f_n\}$ *be equally* (C) *integrable. If* $\{f_n\}$ *converges in the* (C) *mean to* f *and* μ *is autocontinuous, then*

$$\lim_{n\to\infty}(C)\int_A f_n d\mu = (C)\int_A f d\mu. \tag{2.12}$$

Note that if $\{f_n\}$ satisfies (2.12), then it is said to be Choquet weak converge to f.

3. Interval-Valued Functions and the Choquet Integral

Let $C(R^+)$ be the set of all closed subsets in $\mathbb{R}^+$ and $I(\mathbb{R})$ the set of all bounded closed intervals (intervals, for short) in $\mathbb{R} = (-\infty, \infty)$, that is,

$$I(R) = \left\{\left[a^l, a^r\right] \mid a^l, a^r \in \mathbb{R},\ a^l \leq a^r\right\}. \tag{3.1}$$

For any $a \in \mathbb{R}$, we define $a = [a, a]$. Obviously, $a \in I(\mathbb{R})$ (see [9, 10, 14, 15]).

Definition 3.1. If $\overline{a} = [a^l, a^r]$, $\overline{b} = [b^l, b^r] \in I(\mathbb{R}^+)$, and $k \in \mathbb{R}^+$, then we define arithmetic, maximum, minimum, order, and inclusion operations as follows:

(1) $\overline{a} + \overline{b} = [a^l + b^l, a^r + b^r]$,

(2) $k\overline{a} = [ka^l, ka^r]$,

(3) $\overline{a}\overline{b} = [a^l b^l, a^r b^r]$,

(4) $\overline{a} \vee \overline{b} = [a^l \vee b^l, a^r \vee b^r]$,

(5) $\overline{a} \wedge \overline{b} = [a^l \wedge b^l, a^r \wedge b^r]$,

(6) $\overline{a} \leq \overline{b}$ if and only if $a^l \leq b^l$ and $a^r \leq b^r$,

(7) $\overline{a} < \overline{b}$ if and only if $\overline{a} \leq \overline{b}$ and $\overline{a} \neq \overline{b}$, and

(8) $\overline{a} \subset \overline{b}$ if and only if $b^l \leq a^l$ and $a^r \leq b^r$.

Let $\mathfrak{F}_c$ be the set of all measurable closed set-valued functions on $(X, \mathfrak{B})$ and $\mathfrak{F}_i$ the set of all measurable interval-valued functions. Recall that a closed set-valued function $F : X \to C(\mathbb{R}^+)$ is said to be measurable if for any open set $O \subset \mathbb{R}^+$,

$$F^{-1}(O) = \{x \in \mathbb{R}^+ \mid F(x) \cap O \neq \emptyset\} \in \mathfrak{B}. \tag{3.2}$$

Then, we introduce the Choquet integral of measurable interval-valued functions (see [7–11]).

Definition 3.2 (see [11]). (1) Let $A \in \mathfrak{B}$, $F \in \mathfrak{F}_c$ and μ be a fuzzy measure. The Choquet integral of F with respect to μ on A is defined by

$$(C)\int_A F d\mu = \left\{(C)\int_A f d\mu \mid f \in S_c(F)\right\}, \tag{3.3}$$

where $S_c(F)$ is the family of measurable selections of F, that is,

$$S_c(F) = \left\{f : X \to \mathbb{R}^+ \mid (C)\int_A f d\mu < \infty,\ f(x) \in F(x)\ \mu - \text{a.e.}\right\}. \tag{3.4}$$

(2) F is said to be (C) integrable if $(C)\int F d\mu \neq \emptyset$.
(3) F is said to be (C) integrably function f such that

$$\|F(x)\| = \sup_{r \in F(x)} r \leq g(x) \quad \forall\, x \in X. \tag{3.5}$$

We note that μ-a.e. means almost everywhere in a fuzzy measure μ. Then, we obtain the following theorem which is a useful tool to investigate various convergences of interval-valued functions.

Theorem 3.3 (see [11, Theorem 3.16(iii)]). *Let $A \in \mathfrak{B}$ and μ be a fuzzy measure. If $F = [f^l, f^r] : X \to I(R^+)$ is a (C) integrably bounded interval-valued function, then*

$$(C)\int F d\mu = \left[(C)\int f^l d\mu, (C)\int f^r d\mu\right]. \tag{3.6}$$

Now, we define convergence in the (C) mean, equally (C) integrable, and convergence in a fuzzy measure and discuss their properties.

Definition 3.4. Let $\{F_n\} \subset \mathfrak{F}_i$ and $F \in \mathfrak{F}_i$. A sequence $\{F_n\}$ converges in the (C) mean to F if

$$\lim_{n\to\infty}(C)\int d_H(F_n(x), F(x))d\mu = 0, \tag{3.7}$$

where d_H is the Hausdorrf metric on $C(\mathbb{R}^+)$.

Recall that for each pair $A, B \in C(\mathbb{R}^+)$,

$$d_H(A, B) = \max\left\{\sup_{y\in B}\inf_{x\in A}|x - y|, \sup_{x\in A}\inf_{y\in B}|x - y|\right\}. \tag{3.8}$$

Then, it is easy to see that for each pair $\overline{a} = [a^l, a^r]$, $\overline{b} = [b^l, b^r] \in I(\mathbb{R}^+)$,

$$d_H\left(\overline{a}, \overline{b}\right) = \max\left\{\left|a^l - b^l\right|, \left|a^r - b^r\right|\right\}. \tag{3.9}$$

By (3.7) and (3.9), we obtain the following theorem.

Theorem 3.5. *Let $\{F_n = [f_n^l, f_n^r]\} \subset \mathfrak{F}_i$ and $F = [f^l, f^r] \in \mathfrak{F}$. If a sequence $\{F_n\}$ converges in the (C) mean to F, then a sequence $\{f_n^l\}$ (resp., $\{f_n^r\}$) converges in the (C) mean to f^l (resp., f^r).*

Proof. By (3.7) and Theorem 2.8 (4), we have

$$\begin{aligned}
&(C)\int d_H(F_n(x), F(x))d\mu(x) \\
&\quad = (C)\int \max\left\{\left|f_n^l(x) - f^l(x)\right|, \left|f_n^r(x) - f^r(x)\right|\right\}d\mu \\
&\quad \geq (C)\int\left|f_n^l(x) - f^l(x)\right|d\mu(x) \vee (C)\int\left|f_n^r(x) - f^r(x)\right|d\mu(x).
\end{aligned} \tag{3.10}$$

From (3.10), we can derive the followings:

$$\lim_{n\to\infty}(C)\left|f_n^l - f^l\right|d\mu = 0, \qquad \lim_{n\to\infty}(C)\left|f_n^r - f^r\right|d\mu = 0. \tag{3.11}$$

Thus, the proof is complete. □

Definition 3.6. Let $A \in \mathfrak{B}$. A sequence $\{F_n\}$ is called equally (C) integrable on A if for any given $\varepsilon > 0$, there exists $N(\varepsilon) > 0$ such that

$$d_H\left((C)\int_A F_n d\mu, \left[\int_0^N \mu\left(G_{f_n^l}(\alpha) \cap A\right)d\alpha, \int_0^N \mu\left(G_{f_n^r}(\alpha) \cap A\right)d\alpha\right]\right) \leq \varepsilon, \tag{3.12}$$

for all $n = 1, 2, \ldots$.

Theorem 3.7. *Let $A \in \mathfrak{B}$ and μ be a fuzzy measure. If a sequence $\{F_n\}$ is equally (C) integrable on A, then sequences $\{f_n^l\}$ and $\{f_n^r\}$ are equally (C) integrable on A.*

Proof. By (3.9) and (3.12), we have

$$\begin{aligned}
&d_H\left((C)\int_A F_n d\mu, \left[\int_0^N \mu\left(G_{f_n^l}(\alpha) \cap A\right)d\alpha, \int_0^N \mu\left(G_{f_n^r}(\alpha) \cap A\right)d\alpha\right]\right) \\
&\quad = \max\left\{\left|(C)\int_A f^l d\mu - \int_0^N \mu\left(G_{f_n^l}(\alpha) \cap A\right)d\alpha\right|, \left|(C)\int_A f^r d\mu - \int_0^N \mu\left(G_{f_n^r}(\alpha) \cap A\right)d\alpha\right|\right\}.
\end{aligned} \tag{3.13}$$

By the assumption and (3.13), given $\varepsilon > 0$, there exists $N(\varepsilon) > 0$ such that

$$\max\left\{\left|(C)\int_A f^l d\mu - \int_0^N \mu\left(G_{f_n^l}(\alpha)\cap A\right)d\alpha\right|, \left|(C)\int_A f^r d\mu - \int_0^N \mu\left(G_{f_n^r}(\alpha)\cap A\right)d\alpha\right|\right\} \le \varepsilon. \tag{3.14}$$

Thus we have

$$\begin{gathered}(C)\int_A f^l d\mu - \int_0^N \mu\left(G_{f_n^l}(\alpha)\cap A\right)d\alpha \le \varepsilon,\\ (C)\int_A f^r d\mu - \int_0^N \mu\left(G_{f_n^r}(\alpha)\cap A\right)d\alpha \le \varepsilon.\end{gathered} \tag{3.15}$$

Thus, the proof is complete. □

Definition 3.8. Let $A \in \mathfrak{B}$, $\{F_n\} \subset \mathfrak{F}_i$, and $F \in \mathfrak{F}_i$. We say that $\{F_n\}$ converges in μ to F on A if for any given $\varepsilon > 0$,

$$\lim_{n\to\infty}\mu(\{x \mid d_H(F_n(x), F(x)) \ge \varepsilon\}\cap A) = 0. \tag{3.16}$$

Theorem 3.9. *Let $A \in \mathfrak{B}$, $\{F_n = [f_n^l, f_n^r]\} \subset \mathfrak{F}_i$, and $F = [f^l, f^r] \in \mathfrak{F}_i$. If $\{F_n\}$ converges in μ to F on A, then a sequence $\{f_n^l\}$ (resp., $\{f_n^r\}$) converges in μ to f^l (resp., f^r) on A.*

Proof. By (3.9), we have

$$d_H(F_n(x), F(x)) = \max\left\{\left|f_n^l(x) - f^l(x)\right|, \left|f_n^r(x) - f^r(x)\right|\right\}, \quad \forall x \in X. \tag{3.17}$$

Since $\{x \mid |f_n^l(x) - f^l(x)| \ge \varepsilon\} \subset \{x \mid d_H(F_n(x), F(x)) \ge \varepsilon\}$ and $\{x | f_n^r(x) - f^r(x)| \ge \varepsilon\} \subset \{x \mid d_H(F_n(x), F(x)) \ge \varepsilon\}$, by (3.16), we have

$$\begin{gathered}\lim_{n\to\infty}\mu\left(\left\{x \mid \left|f_n^l(x) - f^l(x)\right| \ge \varepsilon\right\}\cap A\right) = 0,\\ \lim_{n\to\infty}\mu(\{x \mid |f_n^r(x) - f^r(x)| \ge \varepsilon\}\cap A) = 0.\end{gathered} \tag{3.18}$$

Thus, the proof is complete. □

Definition 3.10. Let $F, G \in \mathfrak{F}_i$. F and G are comonotonic if and only if for every pair $x, y \in X$,

$$F(x) < F(y) \Longrightarrow F(x) \le G(y). \tag{3.19}$$

Theorem 3.11. *Let $F = [f^l, f^r]$, $G = [g^l, g^r] \in \mathfrak{F}_i$. If F and G are comonotonic, then f^l and g^l are comonotonic, and f^r and g^r are comonotonic.*

Proof. If $x, y \in X$ and $f^l(x) < f^l(y)$ and $f^r(x) < f^r(y)$, then we have $F(x) < G(y)$. Since F and G are comonotonic, $G(x) \leq G(y)$. Then, we have $g^l(x) \leq g^l(y)$ and $g^r(x) \leq g^r(y)$. Thus, the proof is complete. □

Then, we discuss some properties of the Choquet integral with respect to a fuzzy measure of measurable interval-valued functions.

Theorem 3.12. *Let $A \in \mathfrak{B}$, $F, G \in \mathfrak{F}_i$, and let μ be a fuzzy measure.*

(1) If $F \leq G$, then $(C)\int_A F d\mu \leq (C)\int_A G d\mu$.

(2) If a and b are nonnegative real numbers, then

$$(C)\int_A \int (aF + b) d\mu = a(C)\int_A F d\mu + b\mu(A). \tag{3.20}$$

(3) If F and G are comonotonic, then

$$(C)\int_A (F + G) d\mu = (C)\int_A F d\mu + (C)\int_A G d\mu. \tag{3.21}$$

(4) If we define $(F \vee G)(x) = F(x) \vee G(x)$ for all $x \in X$, then

$$(C)\int_A F \vee G d\mu \geq (C)\int_A F d\mu \vee (C)\int_A G d\mu. \tag{3.22}$$

(5) If we define $(F \wedge G)(x) = F(x) \wedge G(x)$ for all $x \in X$, then

$$(C)\int_A F \wedge G d\mu \leq (C)\int_A F d\mu \wedge (C)\int_A G d\mu. \tag{3.23}$$

Proof. (1) If $F = [f^l, f^r] \leq G = [g^l, g^r]$, then $f^l \leq g^l$ and $f^r \leq g^r$. By Theorem 2.8(1), $(C)\int f^l d\mu \leq (C)\int g^l d\mu$ and $(C)\int f^r d\mu \leq (C)\int g^r d\mu$. By Definition 3.1(6) and (3.6), we have

$$(C)\int F d\mu = \left[(C)\int f^l d\mu, (C)\int f^r d\mu\right] \leq \left[(C)\int g^l d\mu, (C)\int g^r d\mu\right] = (C)\int G d\mu. \tag{3.24}$$

(2) By the same method of (1) and Theorem 2.8(2), we have

$$\begin{aligned}(C)\int [aF + b] d\mu &= \left[(C)\int a f^l + b d\mu, (C)\int a f^r + b d\mu\right] \\ &= \left[a(C)\int f^l d\mu + b, a(C)\int f^r d\mu + b\right] = a(C)\int F d\mu + b.\end{aligned} \tag{3.25}$$

(3) The proof is similar to (2).

(4) Since $F \vee G = [f^l \vee g^l, f^r \vee g^r]$, by Theorem 2.8(4),

$$\begin{aligned}(C)\int F \vee G d\mu &= \left[(C)\int f^l \vee g^l d\mu, (C)\int f^r \vee g^r d\mu\right] \\ &\geq \left[(C)\int f^l d\mu \vee (C)\int g^l d\mu, (C)\int f^r d\mu \vee (C)\int g^r d\mu\right] \\ &= \left[(C)\int f^l d\mu, (C)\int f^r d\mu\right] \vee (C)\left[\int g^l d\mu, (C)\int g^r d\mu\right] \\ &= (C)\int F d\mu \vee (C)\int G d\mu.\end{aligned} \tag{3.26}$$

(5) The proof is similar to (4). □

Finally, we obtain the main result which is the following convergence theorem for the Choquet integral with respect to an autocontinuous fuzzy measure of a measurable interval-valued function.

Theorem 3.13. *If $A \in \mathfrak{B}$ and $\{F_n\}$ is equally (C) integrable and If $\{F_n\}$ converges in the (C) to F and μ is autocontinuous, then*

$$\lim_{n\to\infty} d_H\left((C)\int_A F_n d\mu, (C)\int_A F d\mu\right) = 0. \tag{3.27}$$

Proof. Since $\{F_n\}$ is equally (C) integrable on A, by Theorem 3.7, $\{f_n^l\}$ and $\{f_n^r\}$ are equally (C) integrable on A. By Theorem 2.9,

$$\begin{aligned}\lim_{n\to\infty}(C)\int_A f_n^l d\mu &= (C)\int_A f^l d\mu, \\ \lim_{n\to\infty}(C)\int_A f_n^r d\mu &= (C)\int_A f^r d\mu.\end{aligned} \tag{3.28}$$

Thus, for any given $\varepsilon > 0$, there exists K such that

$$\begin{aligned}\left|(C)\int_A f_n^l d\mu - (C)\int_A f^l d\mu\right| &< \varepsilon, \\ \left|(C)\int_A f_n^r d\mu - (C)\int_A f^r d\mu\right| &< \varepsilon,\end{aligned} \tag{3.29}$$

for all $n \geq K$. Then, we have

$$d_H\left((C)\int_A F_n d\mu, (C)\int_A F d\mu\right) < \varepsilon, \tag{3.30}$$

for all $n \geq K$. Thus, the proof is complete. □

4. Conclusions

In this paper, we consider the new concept of the Choquet integral of a measurable interval-valued function which generalizes the Choquet integral of a measurable function mentioned in the papers [2–6, 11]. From Theorems 3.12 and 3.13, we established fundamental properties of the Choquet integral of interval-valued functions and the convergence theorem for the Choquet integral with respect to an autocontinuous fuzzy measure of measurable interval-valued functions.

In the future, by using these results of this paper, we can develop various problems, for example, the Choquet weak convergence of uncertain random sets and the weak convergence theorems for the Aumann integral of measurable interval-valued functions, and so forth.

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