Hindawi Publishing Corporation
Journal of Applied Mathematics
Volume 2011, Article ID 513467, 15 pages
doi:10.1155/2011/513467

Research Article

On the Cauchy Problem for the b-Family Equations with a Strong Dispersive Term

Xuefei Liu,[1] Mingxuan Zhu,[2] and Zaihong Jiang[2]

[1] *College of Mathematics and Computer Science, Chongqing Three Gorges University, Wanzhou 404000, Chongqing, China*
[2] *Department of Mathematics, Zhejiang Normal University, Jinhua 321004, Zhejiang, China*

Correspondence should be addressed to Zaihong Jiang, jzhong790405@gmail.com

Received 2 June 2011; Accepted 23 August 2011

Academic Editor: J. Biazar

In this paper, we consider b-family equations with a strong dispersive term. First, we present a criterion on blow-up. Then global existence and persistence property of the solution are also established. Finally, we discuss infinite propagation speed of this equation.

1. Introduction

Recently, Holm and Staley [1] studied the exchange of stability in the dynamics of solitary wave solutions under changes in the nonlinear balance in a 1 + 1 evolutionary partial differential equation related both to shallow water waves and to turbulence. They derived the following equations (the b-family equations):

$$y_t + uy_x + bu_x y = 0, \quad t > 0, \ x \in \mathbb{R}, \tag{1.1}$$

where $u(x)$ denotes the velocity field and $y(x,t) = u - u_{xx}$.

Detailed description of the corresponding strong solutions to (1.1) with u_0 being its initial data was given by Zhou [2]. He established a sufficient condition in profile on the initial data for blow-up in finite time. The necessary and sufficient condition for blow-up is still a challenging problem for us at present. More precious, Theorem 3.1 in [2] means that no matter what the profile of the compactly supported initial datum $u_0(x)$ is (no matter whether it is positive or negative), for any $t > 0$ in its lifespan, the solution $u(x,t)$ is positive at infinity and negative at negative infinity; it is really a very nice property for the b-family equations.

The famous Camassa-Holm equation [3] and Degasperis-Procesi equation [4] are the special cases with $b = 2$ and $b = 3$, respectively. Many papers [5–12] are devoted to their study.

In this paper, we consider the following b-family equations with a strong dispersive term:

$$y_t + uy_x + bu_x y + \lambda y_x = 0, \quad t > 0, \ x \in \mathbb{R}, \tag{1.2}$$

where $y = u - u_{xx}$, $\lambda > 0$, and $\lambda y_x = \lambda(u_x - u_{xxx})$ is the strong dispersive term.

Let $\Lambda = (1 - \partial_x^2)^{1/2}$; then, the operator Λ^{-2} can be expressed by its associated Green's function $G = (1/2)e^{-|x|}$ as $\Lambda^{-2} f(x) = G * f(x) = (1/2)\int_{\mathbb{R}} e^{-|x-y|} f(y)dy$. So (1.2) is equivalent to the following equation:

$$u_t + uu_x + \partial_x G * \left(\frac{b}{2}u^2 + \frac{3-b}{2}u_x^2\right) + \lambda u_x = 0. \tag{1.3}$$

Similar to the Camassa-Holm equation [5], it is easy to establish the following local well-posedness theorem for (1.2).

Theorem 1.1. *Given $u_0 \in H^s(\mathbb{R})$, $s > 3/2$, then there exist a T and a unique solution u to* (1.3) *such that*

$$u(x,t) \in C([0,T); H^s(\mathbb{R})) \cap C^1\Big([0,T); H^{s-1}(\mathbb{R})\Big). \tag{1.4}$$

To make the paper concise, we would like to omit the detailed proof.

The paper is organized as follows. In Section 2, we get a criterion on blow-up. A condition for global existence is found in Section 3. Persistence property is considered in Section 4. In Section 5, the infinite propagation speed will be established analogous to the b-family equation.

2. Blow-Up

The maximum value of T in Theorem 1.1 is called the lifespan of the solution, in general. If $T < \infty$, that is, $\lim_{t\to T^-}\|u(\cdot,t)\|_{H^s} = \infty$, we say the solution blows up in finite time.

The following lemma tells us that the solution blows up if and only if the first-order derivative blows up.

Lemma 2.1. *Assume that $u_0 \in H^s(\mathbb{R})$, $s > 2$. If $b = 1/2$, then the solution of* (1.2) *will exist globally in time. If $b > 1/2$, then the solution blows up if and only if u_x becomes unbounded from below in finite time. If $b < 1/2$, the solution blows up in finite time if and only if u_x becomes unbounded from above in finite time.*

Proof. By direct computation, one has

$$\|y\|_{L^2}^2 = \int_{\mathbb{R}} (u - u_{xx})^2 dx = \int_{\mathbb{R}} u^2 + 2u_x^2 + u_{xx}^2 dx. \tag{2.1}$$

Hence,

$$\|u\|_{H^2}^2 \le \|y\|_{L^2}^2 \le 2\|u\|_{H^2}^2. \tag{2.2}$$

Applying y on (1.2) and integration by parts, we obtain

$$\frac{d}{dt}\int_{\mathbb{R}} y^2 dx = \int_{\mathbb{R}} 2yy_t dx = -2\int_{\mathbb{R}} y(uy_x + bu_x y + \lambda y_x)dx = (1-2b)\int_{\mathbb{R}} u_x y^2 dx. \tag{2.3}$$

If $b = 1/2$, then $(d/dt)\int_{\mathbb{R}} y^2 dx = 0$. Hence,

$$\|u\|_{H^2}^2 \le \|y\|_{L^2}^2 = \|y_0\|_{L^2}^2. \tag{2.4}$$

Equation (2.4) implies the corresponding solution exists globally.

If $b > 1/2$, due to the Gronwall inequality, it is clear that, from (2.3), u_x is bounded from below on $[0,T)$ and then the H_2-norm of the solution is also bounded on $[0,T)$. On the other hand,

$$u(x,t) = \left(1-\partial_x^2\right)^{-1} y(x,t) = \int_{\mathbb{R}} G(x-\xi)y(\xi)d\xi. \tag{2.5}$$

Therefore

$$\|u_x\|_{L^\infty} \le \left|\int_{\mathbb{R}} G_x(x-\xi)y(\xi)d\xi\right| \le \|G_x\|_{L^2}\|y\|_{L^2} = \frac{1}{2}\|y\|_{L^2} \le \|u\|_{H^2}, \tag{2.6}$$

where we use (2.2). Hence, (2.6) tells us if H_2-norm of the solution is bounded, then the L_∞-norm of the first derivative is bounded.

By the same argument, we can get the similar result for $b < 1/2$.

This completes the proof. □

Motivated by Mckean's deep observation for the Camassa-Holm equation [7], we can do the similar particle trajectory as

$$\begin{gathered} q_t = u(q,t) + \lambda, \quad 0 < t < T, \ x \in \mathbb{R}, \\ q(x,0) = x, \quad x \in \mathbb{R}, \end{gathered} \tag{2.7}$$

where T is the lifespan of the solution; then, q is a diffeomorphism of the line. Differentiating the first equation in (2.7) with respect to x, one has

$$\frac{dq_t}{dx} = q_{xt} = u_x(q,t), \quad t \in (0,T). \tag{2.8}$$

Hence

$$q_x(x,t) = \exp\left\{\int_0^t u_x(q,s)\,ds\right\}, \quad q_x(x,0) = 1. \tag{2.9}$$

Since

$$\frac{d}{dt}\left(y(q)q_x^b\right) = \left[y_t(q) + (u(q,t)+\lambda)y_x(q) - bu_x(q,t)y(q)\right]q_x^b = 0, \tag{2.10}$$

it follows that

$$y(q)q_x^b = y_0(x). \tag{2.11}$$

Then we establish sufficient condition on the initial data to guarantee blow-up for (1.2).

Theorem 2.2. *Let $b \geq 2$. Suppose that $u_0 \in H^2(\mathbb{R})$ and there exists an $x_0 \in \mathbb{R}$ such that $y_0(x_0) = (1-\partial_x^2)u_0(x_0) = 0$,*

$$y_0 \geq 0(\not\equiv 0) \text{ for } x \in (-\infty, x_0), \qquad y_0 \leq 0(\not\equiv 0) \text{ for } x \in (x_0, \infty). \tag{2.12}$$

Then the corresponding solution $u(x,t)$ to (1.2) *with u_0 as the initial datum blows up in finite time.*

Proof. Suppose that the solution exists globally. Due to (2.11) and the initial condition (2.12), we have $y(q(x_0,t),t) = 0$, and

$$\begin{aligned} y(q(x,t),t) &\geq 0(\not\equiv 0), \quad \text{for } x \in (-\infty, x_0), \\ y(q(x,t),t) &\leq 0(\not\equiv 0), \quad \text{for } x \in (x_0, \infty), \end{aligned} \tag{2.13}$$

for all $t \geq 0$. Since $u(x,t) = G * y(x,t)$, one can write $u(x,t)$ and $u_x(x,t)$ as

$$\begin{aligned} u(x,t) &= \frac{1}{2}e^{-x}\int_{-\infty}^x e^{\xi}y(\xi,t)\,d\xi + \frac{1}{2}e^x\int_x^\infty e^{-\xi}y(\xi,t)\,d\xi, \\ u_x(x,t) &= -\frac{1}{2}e^{-x}\int_{-\infty}^x e^{\xi}y(\xi,t)\,d\xi + \frac{1}{2}e^x\int_x^\infty e^{-\xi}y(\xi,t)\,d\xi. \end{aligned} \tag{2.14}$$

Consequently,

$$u_x^2(x,t) - u^2(x,t) = -\int_{-\infty}^x e^{\xi}y(\xi,t)\,d\xi\int_x^\infty e^{-\xi}y(\xi,t)\,d\xi, \tag{2.15}$$

for all $t > 0$.

For any fixed t, if $x \leq q(x_0, t)$, then

$$\begin{aligned}
u_x^2(x,t) - u^2(x,t) &= -\left(\int_{-\infty}^{q(x_0,t)} e^{\xi} y(\xi,t)\,d\xi - \int_{x}^{q(x_0,t)} e^{\xi} y(\xi,t)\,d\xi\right) \\
&\quad \times \left(\int_{q(x_0,t)}^{\infty} e^{-\xi} y(\xi,t)\,d\xi + \int_{x}^{q(x_0,t)} e^{-\xi} y(\xi,t)\,d\xi\right) \\
&= u_x^2(q(x_0,t),t) - u^2(q(x_0,t),t) - \int_{-\infty}^{x} e^{\xi} y(\xi,t)\,d\xi \int_{x}^{q(x_0,t)} e^{-\xi} y(\xi,t)\,d\xi \\
&\quad + \int_{x}^{q(x_0,t)} e^{\xi} y(\xi,t)\,d\xi \int_{q(x_0,t)}^{\infty} e^{-\xi} y(\xi,t)\,d\xi \\
&\leq u_x^2(q(x_0,t),t) - u^2(q(x_0,t),t).
\end{aligned} \tag{2.16}$$

Similarly, for $x \geq q(x_0, t)$, we also have

$$u_x^2(x,t) - u^2(x,t) \leq u_x^2(q(x_0,t),t) - u^2(q(x_0,t),t). \tag{2.17}$$

Combining (2.16) and (2.17) together, we get that for any fixed t,

$$u_x^2(x,t) - u^2(x,t) \leq u_x^2(q(x_0,t),t) - u^2(q(x_0,t),t), \tag{2.18}$$

for all $x \in \mathbb{R}$.

Differentiating (1.3), we get

$$u_{tx} + uu_{xx} - \frac{b}{2}u^2 - \frac{1-b}{2}u_x^2 + G * \left(\frac{b}{2}u^2 + \frac{3-b}{2}u_x^2\right) + \lambda u_{xx} = 0. \tag{2.19}$$

Differentiating $u_x(q(x_0,t),t)$ with respect to t, where q is the diffeomorphism defined in (2.7),

$$\begin{aligned}
&\partial_t u_x(q(x_0,t),t) \\
&\quad = u_{xt}(q(x_0,t),t) + u_{xx}(q(x_0,t),t) q_t(q(x_0,t),t) \\
&\quad = \frac{b}{2}u^2(q(x_0,t),t) + \frac{1-b}{2}u_x^2(q(x_0,t),t) - G * \left(\frac{b}{2}u^2(x,t) + \frac{3-b}{2}u_x^2(x,t)\right) \\
&\quad = G * \left(\frac{b}{2}u^2(q(x_0,t),t) + \frac{1-b}{2}u_x^2(q(x_0,t),t) - \frac{b}{2}u^2(x,t) - \frac{3-b}{2}u_x^2(x,t)\right)
\end{aligned}$$

$$
\begin{aligned}
&= G * \left(\frac{b-2}{2} \left(u^2(q(x_0,t),t) - u_x^2(q(x_0,t),t) - u^2(x,t) + u_x^2(x,t) \right) \right) \\
&\quad + G * \left(u^2(q(x_0,t),t) - \frac{1}{2} u_x^2(q(x_0,t),t) - u^2(x,t) - \frac{1}{2} u_x^2(x,t) \right) \\
&\leq \frac{1}{2} u^2(q(x_0,t),t) - \frac{1}{2} u_x^2(q(x_0,t),t),
\end{aligned}
\tag{2.20}
$$

where we use (2.18) and the following inequality: $G * (u^2(x,t) + (1/2)u_x^2(x,t)) \geq (1/2)u^2$.

Claim 1. $u_x(q(x_0),t) < 0$ is decreasing and $u^2(q(x_0,t),t) < u_x^2(q(x_0,t),t)$ for all $t \geq 0$.

Suppose not; that is, there exists a t_0 such that $u^2(q(x_0,t),t) < u_x^2(q(x_0,t),t)$ on $[0,t)$ and $u^2(q(x_0,t),t) = u_x^2(q(x_0,t),t)$. Now, let

$$
\begin{aligned}
I(t) &:= \frac{1}{2} e^{-q(x_0,t)} \int_{-\infty}^{q(x_0,t)} e^{\xi} y(\xi,t) d\xi, \\
II(t) &:= \frac{1}{2} e^{q(x_0,t)} \int_{q(x_0,t)}^{\infty} e^{-\xi} y(\xi,t) d\xi.
\end{aligned}
\tag{2.21}
$$

Firstly, differentiating $I(t)$, we have

$$
\begin{aligned}
\frac{dI(t)}{dt} &= -\frac{1}{2} (u(q(x_0,t),t) + \lambda) e^{-q(x_0,t)} \int_{-\infty}^{q(x_0,t)} e^{\xi} y(\xi,t) d\xi \\
&\quad + \frac{1}{2} e^{-q(x_0,t)} \int_{-\infty}^{q(x_0,t)} e^{\xi} y_t(\xi,t) d\xi \\
&= \frac{1}{2} (u+\lambda)(u_x - u)(q(x_0,t),t) - \frac{1}{2} e^{-q(x_0,t)} \\
&\quad \times \int_{-\infty}^{q(x_0,t)} e^{\xi} \left(u y_x + 2 u_x y + \frac{b-2}{2} \left(u^2 - u_x^2 \right)_x (q(x_0,t),t) + \lambda y_x \right) d\xi \\
&\geq \frac{1}{2} (u+\lambda)(u_x - u)(q(x_0,t),t) + \frac{1}{4} \left(u^2 + u_x^2 - 2 u u_x \right)(q(x_0,t),t) \\
&\quad - \frac{\lambda}{2} (u_x - u)(q(x_0,t),t) \\
&= \frac{1}{4} \left(u_x^2 - u^2 \right)(q(x_0,t),t) > 0, \quad \text{on } [0,t_0).
\end{aligned}
\tag{2.22}
$$

Secondly, by the same argument, we get

$$\begin{aligned}
\frac{dII(t)}{dt} &= \frac{1}{2}\left(u(q(x_0,t),t)+\lambda\right)e^{q(x_0,t)}\int_{q(x_0,t)}^{\infty} e^{-\xi}y(\xi,t)d\xi \\
&\quad+\frac{1}{2}e^{q(x_0,t)}\int_{q(x_0,t)}^{\infty} e^{-\xi}y_t(\xi,t)d\xi \\
&= \frac{1}{2}(u+\lambda)(u_x+u)(q(x_0,t),t)-\frac{1}{2}e^{q(x_0,t)} \\
&\quad\times\int_{q(x_0,t)}^{\infty} e^{-\xi}\left(uy_x+2u_xy+\frac{b-2}{2}\left(u^2-u_x^2\right)_x(q(x_0,t),t)+\lambda y_x\right)d\xi \\
&\le \frac{1}{2}(u+\lambda)(u_x+u)(q(x_0,t),t)-\frac{1}{4}\left(u^2+u_x^2+2uu_x\right)(q(x_0,t),t) \\
&\quad-\frac{\lambda}{2}(u_x+u)(q(x_0,t),t) \\
&= -\frac{1}{4}\left(u_x^2-u^2\right)(q(x_0,t),t)<0, \quad \text{on } [0,t_0).
\end{aligned} \tag{2.23}$$

Hence, it follows from (2.22), (2.23), and the continuity property of ODEs that

$$\left(u_x^2-u^2\right)(q(x_0,t),t) = -4I(t)II(t) > -4I(0)II(0) > 0. \tag{2.24}$$

Moreover, due to (2.22) and (2.23) again, we have the following equation for $(u_x^2 - u^2)(q(x_0,t),t)$:

$$\begin{aligned}
\frac{d}{dt}\left(u_x^2-u^2\right)(q(x_0,t),t) &= -\frac{d}{dt}\left(\int_{-\infty}^{q(x_0,t)} e^{\xi}y(\xi,t)d\xi\int_{q(x_0,t)}^{\infty} e^{-\xi}y(\xi,t)d\xi\right) \\
&= -\frac{d}{dt}\left(e^{-q(x_0,t)}\int_{-\infty}^{q(x_0,t)} e^{\xi}y(\xi,t)d\xi\right)e^{q(x_0,t)}\int_{q(x_0,t)}^{\infty} e^{-\xi}y(\xi,t)d\xi \\
&\quad-e^{-q(x_0,t)}\int_{-\infty}^{q(x_0,t)} e^{\xi}y(\xi,t)d\xi\frac{d}{dt}\left(e^{q(x_0,t)}\int_{q(x_0,t)}^{\infty} e^{-\xi}y(\xi,t)d\xi\right) \\
&\ge -\frac{1}{2}\left(u_x^2-u^2\right)(q(x_0,t),t)e^{q(x_0,t)}\int_{q(x_0,t)}^{\infty} e^{-\xi}y(\xi,t)d\xi \\
&\quad+\frac{1}{2}\left(u_x^2-u^2\right)(q(x_0,t),t)e^{-q(x_0,t)}\int_{-\infty}^{q(x_0,t)} e^{\xi}y(\xi,t)d\xi \\
&= -u_x(q(x_0,t),t)\left(u_x^2-u^2\right)(q(x_0,t),t).
\end{aligned} \tag{2.25}$$

Now, substituting (2.20) into (2.25), it yields

$$\frac{d}{dt}\left(u_x^2 - u^2\right)(q(x_0,t),t) \geq \frac{1}{2}\left(u_x^2 - u^2\right)(q(x_0,t),t)\left(\int_0^t \left(u_x^2 - u^2\right)(q(x_0,\tau),\tau)d\tau - 2u_{0x}(x_0)\right). \tag{2.26}$$

Before completing the proof, we need the following technical lemma.

Lemma 2.3. *Suppose that $\Psi(t)$ is twice continuously differential satisfying*

$$\begin{gathered}\Psi''(t) \geq C_0 \Psi'(t)\Psi(t), \quad t > 0,\ C_0 > 0, \\ \Psi(t) > 0, \quad \Psi'(t) > 0.\end{gathered} \tag{2.27}$$

Then $\psi(t)$ blows up in finite time. Moreover, the blow-up time can be estimated in terms of the initial datum as

$$T \leq \max\left\{\frac{2}{C_0\Psi(0)}, \frac{\Psi(0)}{\Psi'(0)}\right\}. \tag{2.28}$$

Let $\Psi(t) = \int_0^t (u_x^2 - u^2)(q(x_0,\tau),\tau)d\tau - 2u_{0x}(x_0)$; then, (2.26) is an equation of type (2.27) with $C_0 = 1/2$. The proof is complete by applying Lemma 2.3. ☐

Remark 2.4. Mckean got the necessary and sufficient condition for the Camassa-Holm equation in [7]. It is worth pointing out that Zhou and his collaborators [13] gave a new proof to Mckean's theorem. However, the necessary and sufficient condition for (1.2) is still a challenging problem for us at present.

3. Global Existence

In this section, a global existence result is proved.

Theorem 3.1. *Supposing that $u_0 \in H^3$, $y_0 = (1 - \partial_x^2)u_0$ is one sign. Then the corresponding solution to* (1.2) *exists globally.*

Proof. We can assume that $y_0 \geq 0$. It is sufficient to prove $u_x(x,t)$ has a lower and supper bound for all t. In fact,

$$u_x(x,t) = -\frac{1}{2}e^{-x}\int_{-\infty}^{x} e^{\xi}y(\xi,t)d\xi + \frac{1}{2}e^{x}\int_{x}^{\infty} e^{-\xi}y(\xi,t)d\xi, \tag{3.1}$$

so

$$
\begin{aligned}
u_x(x,t) &\geq -\frac{1}{2}e^{-x}\int_{-\infty}^{x} e^{\xi}y(\xi,t)d\xi \geq -\frac{1}{2}\int_{-\infty}^{x} y(\xi,t)d\xi \\
&\geq -\frac{1}{2}\int_{-\infty}^{\infty} y(\xi,t)d\xi = -\frac{1}{2}\int_{-\infty}^{\infty} y_0(\xi,t)d\xi, \\
u_x(x,t) &\leq \frac{1}{2}e^{x}\int_{x}^{\infty} e^{-\xi}y(\xi,t)d\xi \leq \frac{1}{2}\int_{x}^{\infty} y(\xi,t)d\xi \\
&\leq \frac{1}{2}\int_{-\infty}^{\infty} y(\xi,t)d\xi = \frac{1}{2}\int_{-\infty}^{\infty} y_0(\xi,t)d\xi.
\end{aligned}
\tag{3.2}
$$

This completes the proof. □

4. Persistence Property

Now, we will investigate the following property for the strong solutions to (1.2) in L^∞-space which asymptotically exponentially decay at infinity as their initial profiles. The main idea comes from a recent work of Zhou and his collaborators [6] for the standard Camassa-Holm equation (for slower decay rate, we refer to [14]).

Theorem 4.1. *Assume that for some $T > 0$ and $s > 5/2$, $u \in C([0,T]; H^s(\mathbb{R}))$ is a strong solution of (1.2) and that $u_0(x) = u(x,0)$ satisfies that for some $\theta \in (0,1)$,*

$$|u_0(x)|, |u_{0x}(x)| \sim O\left(e^{-\theta x}\right). \tag{4.1}$$

Then

$$|u(x,t)|, |u_x(x,t)| \sim O\left(e^{-\theta x}\right) \tag{4.2}$$

uniformly in the time interval $[0,T]$.

Proof. First, we will introduce the weight function to get the desired result. This function $\varphi_N(x)$ with $N \in \mathbb{Z}^+$ is independent on t as follows:

$$
\varphi_N(x) = \begin{cases} 1, & x \leq 0, \\ e^{\theta x}, & x \in (0,N), \\ e^{\theta N}, & x \geq N, \end{cases} \tag{4.3}
$$

which implies that

$$0 \leq \varphi_N'(x) \leq \varphi_N(x). \tag{4.4}$$

From (1.3), we can get

$$\partial_t(u\varphi_N) + (u\varphi_N)u_x + \varphi_N\partial_x G * \left(\frac{b}{2}u^2 + \frac{3-b}{2}u_x^2\right) + \lambda\varphi_N u_x = 0. \tag{4.5}$$

Multiplying (4.5) by $(u\varphi_N)^{2p-1}$ with $p \in \mathbb{Z}^+$ and integrating the result in the x-variable, we get

$$\begin{aligned}&\int_{-\infty}^{+\infty} \partial_t(u\varphi_N)(u\varphi_N)^{2p-1}dx + \int_{-\infty}^{+\infty}(u\varphi_N)u_x(u\varphi_N)^{2p-1}dx \\ &\quad+ \int_{-\infty}^{+\infty}\varphi_N\partial_x G * \left(\frac{b}{2}u^2 + \frac{3-b}{2}u_x^2\right)(u\varphi_N)^{2p-1} + \int_{-\infty}^{+\infty}\lambda\varphi_N u_x(u\varphi_N)^{2p-1}dx = 0,\end{aligned} \tag{4.6}$$

from which we can deduce that

$$\frac{d}{dt}\|u\varphi_N\|_{L^{2p}} \le \|u_x\|_{L^\infty}\|u\varphi_N\|_{L^{2p}} + \left\|\varphi_N\left(\partial_x G * \left(\frac{b}{2}u^2 + \frac{3-b}{2}u_x^2\right) + \lambda u_x\right)\right\|_{L^{2p}}. \tag{4.7}$$

Denoting $M = \sup_{t\in[0,T]}\|u(t)\|_{H^s}$ and by Gronwall's inequality, we obtain

$$\|u\varphi_N\|_{L^{2p}} \le \left(\|u_0\varphi_N\|_{L^{2p}} + \int_0^t \left\|\varphi_N\left(\partial_x G * \left(\frac{b}{2}u^2 + \frac{3-b}{2}u_x^2\right) + \lambda u_x\right)\right\|_{L^{2p}} d\tau\right)e^{Mt}. \tag{4.8}$$

Taking the limits in (4.8), we get

$$\|u\varphi_N\|_{L^\infty} \le \left(\|u_0\varphi_N\|_{L^\infty} + \int_0^t \left\|\varphi_N\left(\partial_x G * \left(\frac{b}{2}u^2 + \frac{3-b}{2}u_x^2\right) + \lambda u_x\right)\right\|_{L^\infty} d\tau\right)e^{Mt}. \tag{4.9}$$

Next differentiating (1.3) in the x-variable produces the equation

$$u_{tx} + uu_{xx} + u_x^2 + \partial_x^2 G * \left(\frac{b}{2}u^2 + \frac{3-b}{2}u_x^2\right) + \lambda u_{xx} = 0. \tag{4.10}$$

Using the weight function, we can rewrite (4.10) as

$$\partial_x(u_x\varphi_N) + uu_{xx}\varphi_N + (u_x\varphi_N)u_x + \varphi_N\partial_x^2 G * \left(\frac{b}{2}u^2 + \frac{3-b}{2}u_x^2\right) + \lambda\varphi_N u_{xx} = 0. \tag{4.11}$$

Multiplying (4.11) by $(u_x\varphi_N)^{2p-1}$ with $p \in \mathbb{Z}^+$ and integrating the result in the x-variable, it follows that

$$\begin{aligned}
&\int_{-\infty}^{+\infty} \partial_t(u_x\varphi_N)(u_x\varphi_N)^{2p-1}dx + \int_{-\infty}^{+\infty} uu_{xx}\varphi_N(u_x\varphi_N)^{2p-1}dx \\
&\quad + \int_{-\infty}^{+\infty}(u_x\varphi_N)u_x(u_x\varphi_N)^{2p-1}dx + \int_{-\infty}^{+\infty}\varphi_N\partial_x^2 G * \left(\frac{b}{2}u^2 + \frac{3-b}{2}u_x^2\right)(u_x\varphi_N)^{2p-1}dx \\
&\quad + \int_{-\infty}^{+\infty}\lambda\varphi_N u_{xx}(u_x\varphi_N)^{2p-1}dx = 0.
\end{aligned} \tag{4.12}$$

For the second term on the right side of (4.12), we know

$$\left|\int_{-\infty}^{+\infty}(u+\lambda)u_{xx}\varphi_N(u_x\varphi_N)^{2p-1}dx\right| \le 2(\|u\|_{L^\infty} + \|u_x\|_{L^\infty})\|u_x\varphi_N\|_{L^{2p}}^{2p}. \tag{4.13}$$

Using the above estimate and the Hölder inequality, we deduce that

$$\frac{d}{dt}\|u_x\varphi_N\|_{L^{2p}} \le 2M\|u_x\varphi_N\|_{L^{2p}} + \left\|\varphi_N\partial_x^2 G * \left(\frac{b}{2}u^2 + \frac{3-b}{2}u_x^2\right)\right\|_{L^{2p}}. \tag{4.14}$$

Thanks to Gronwall's inequality, it holds that

$$\|u_x\varphi_N\|_{L^{2p}} \le \left(\|u_{0x}\varphi_N\|_{L^{2p}} + \left\|\varphi_N\partial_x^2 G * \left(\frac{b}{2}u^2 + \frac{3-b}{2}u_x^2\right)\right\|_{L^{2p}}\right)e^{2Mt}. \tag{4.15}$$

Taking the limits in (4.15), we have

$$\|u_x\varphi_N\|_{L^\infty} \le \left(\|u_{0x}\varphi_N\|_{L^\infty} + \int_0^t\left\|\varphi_N\partial_x^2 G * \left(\frac{b}{2}u^2 + \frac{3-b}{2}u_x^2\right)\right\|_{L^\infty}d\tau\right)e^{2Mt}. \tag{4.16}$$

Combining (4.9) and (4.16) together, it follows that

$$\begin{aligned}
\|u\varphi_N\|_{L^\infty} + \|u_x\varphi_N\|_{L^\infty} \le{}& (\|u_0\varphi_N\|_{L^\infty} + \|u_{0x}\varphi_N\|_{L^\infty})e^{2Mt} \\
&+ e^{2Mt}\left(\int_0^t\left\|\varphi_N\left(\partial_x G * \left(\frac{b}{2}u^2 + \frac{3-b}{2}u_x^2\right) + \lambda u_x\right)\right\|_{L^\infty}\right. \\
&\left.+\left\|\varphi_N\partial_x^2 G * \left(\frac{b}{2}u^2 + \frac{3-b}{2}u_x^2\right)\right\|_{L^\infty}d\tau\right).
\end{aligned} \tag{4.17}$$

A simple calculation shows that there exists $c_0 > 0$, depending only on $\theta \in (0,1)$, such that for any $N \in \mathbb{Z}^+$,

$$\varphi_N(x)\int_{-\infty}^{\infty}e^{-|x-y|}\frac{1}{\varphi_N(y)}d_y \le c_0 = \frac{4}{1-\theta}. \tag{4.18}$$

Thus, for any appropriate function f and g, one sees that

$$
\begin{aligned}
\left|\varphi_N G * f(x)g(x)\right| &= \left|\frac{1}{2}\varphi_N(x)\int_{-\infty}^{\infty} e^{-|x-y|} f(y)g(y)dy\right| \\
&\leq \frac{1}{2}\varphi_N(x)\int_{-\infty}^{\infty} e^{-|x-y|}\frac{1}{\varphi_N(y)}\varphi_N(y)f(y)g(y)dy \\
&\leq \frac{1}{2}\left(\varphi_N(x)\int_{-\infty}^{\infty} e^{-|x-y|}\frac{1}{\varphi_N(y)}dy\right)\|\varphi_N f\|_{L^\infty}\|g\|_{L^\infty} \\
&\leq c_0\left\|\varphi_N f\right\|_{L^\infty}\left\|g\right\|_{L^\infty}.
\end{aligned}
\tag{4.19}
$$

Similarly, we can get

$$
\begin{aligned}
&\left|\varphi_N \partial_x G * f(x)g(x)\right| \leq c_0\left\|\varphi_N f\right\|_{L^\infty}\left\|g\right\|_{L^\infty}, \\
&\left|\varphi_N \partial_x^2 G * f(x)g(x)\right| \leq c_0\left\|\varphi_N f\right\|_{L^\infty}\left\|g\right\|_{L^\infty}.
\end{aligned}
\tag{4.20}
$$

Thus, inserting the above estimates into (4.17), there exists a constant $\widetilde{c} = \widetilde{c}(M,T) \geq 0$ such that

$$
\begin{aligned}
\left\|u\varphi_N\right\|_{L^\infty} + \left\|u_x\varphi_N\right\|_{L^\infty} &\leq \widetilde{c}\int_0^t (\|u\|_{L^\infty} + \|u_x\|_{L^\infty} + \lambda)(\left\|u\varphi_N\right\|_{L^\infty} + \left\|u_x\varphi_N\right\|_{L^\infty})d\tau \\
&\quad + \widetilde{c}(\left\|u_0\varphi_N\right\|_{L^\infty} + \left\|u_{0x}\varphi_N\right\|_{L^\infty}) \\
&\leq \widetilde{c}(\left\|u_0\varphi_N\right\|_{L^\infty} + \left\|u_{0x}\varphi_N\right\|_{L^\infty}) + \widetilde{c}\int_0^t (\left\|u\varphi_N\right\|_{L^\infty} + \left\|u_x\varphi_N\right\|_{L^\infty})d\tau.
\end{aligned}
\tag{4.21}
$$

Hence, for any $t \in \mathbb{Z}^+$ and any $t \in [0,T]$, we have

$$
\begin{aligned}
\left\|u\varphi_N\right\|_{L^\infty} + \left\|u_x\varphi_N\right\|_{L^\infty} &\leq \widetilde{c}(\left\|u_0\varphi_N\right\|_{L^\infty} + \left\|u_{0x}\varphi_N\right\|_{L^\infty}) \\
&\leq \widetilde{c}\left(\left\|u_0 \max\left(1, e^{\theta x}\right)\right\|_{L^\infty} + \left\|u_{0x} \max\left(1, e^{\theta x}\right)\right\|_{L^\infty}\right).
\end{aligned}
\tag{4.22}
$$

Finally, taking the limit as N goes to infinity in (4.22), we find that for any $t \in [0,T]$

$$
\left\|ue^{\theta x}\right\|_{L^\infty} + \left\|u_x e^{\theta x}\right\|_{L^\infty} \leq \widetilde{c}\left(\left\|u_0 \max\left(1, e^{\theta x}\right)\right\|_{L^\infty} + \left\|u_{0x} \max\left(1, e^{\theta x}\right)\right\|_{L^\infty}\right),
\tag{4.23}
$$

which completes the proof of the theorem. □

5. Infinite Propagation Speed

Recently, Himonas and his collaborators established infinite propagation speed for the Camassa-Holm equation in [6]. Later, Guo [15, 16] considered a similar problem on the weakly dissipative Camassa-Holm equation and the weakly dissipative Degasperis-Procesi equation. Recently, infinite propagation speed for a class of nonlocal dispersive θ-equations was established in [17]. The purpose of this section is to give a more detailed description on the corresponding strong solution $u(x,t)$ to (1.2) in its life span with initial data $u_0(x)$ being compactly supported. The main theorem is as follows.

Theorem 5.1. *Let $0 \leq b \leq 3$. Assume that for some $T \geq 0$ and $s \geq 5/2$, $u \in C([0,T]; H^s(\mathbb{R}))$ is a strong solution of* (1.2). *If $u_0(x) = u(x,0)$ has compact support $[a,c]$, then for $t \in (0,T]$, one has*

$$u(x,t) = \begin{cases} f_+(t)e^{-x}, & \text{for } x > q(c,t), \\ f_-(t)e^{-x}, & \text{for } x < q(a,t), \end{cases} \tag{5.1}$$

where $f_+(t)$ and $f_-(t)$ denote continuous nonvanishing functions, with $f_+(t) > 0$ and $f_-(t) < 0$ for $t \in (0,T]$. Furthermore, $f_+(t)$ is strictly increasing function, while $f_-(t)$ is strictly decreasing function.

Proof. Since u_0 has compact support in x in the interval $[a,c]$, from (2.11), so does $y(x,t)$ in the interval $[q(a,t), q(c,t)]$ in its lifespan. Hence the following functions are well defined:

$$E(t) = \int_{\mathbb{R}} e^{x} y(x,t)dx, \qquad F(t) = \int_{\mathbb{R}} e^{-x} y(x,t)dx, \tag{5.2}$$

with

$$E_0 = \int_{\mathbb{R}} e^{x} y_0(x)dx = 0, \qquad F_0 = \int_{\mathbb{R}} e^{-x} y_0(x)dx = 0. \tag{5.3}$$

Then for $x > q(c,t)$, we have

$$u(x,t) = \frac{1}{2}e^{-|x|} * y(x,t) = \frac{1}{2}e^{-x}\int_{q(a,t)}^{q(b,t)} e^{\tau} y(\tau,t)d\tau = \frac{1}{2}e^{-x}E(t). \tag{5.4}$$

Similarly, when $x < q(a,t)$, we get

$$u(x,t) = \frac{1}{2}e^{-|x|} * y(x,t) = \frac{1}{2}e^{x}\int_{q(a,t)}^{q(b,t)} e^{-\tau} y(\tau,t)d\tau = \frac{1}{2}e^{x}F(t). \tag{5.5}$$

Hence, as consequences of (5.4) and (5.5), we have

$$
\begin{aligned}
u(x,t) = -u_x(x,t) = u_{xx}(x,t) &= \frac{1}{2}e^{-x}E(t), \quad \text{as } x > q(c,t), \\
u(x,t) = u_x(x,t) = u_{xx}(x,t) &= \frac{1}{2}e^{x}F(t), \quad \text{as } x < q(a,t).
\end{aligned} \tag{5.6}
$$

On the other hand,

$$
\frac{dE(t)}{dt} = \int_{\mathbb{R}} e^x y_t(x,t)dx. \tag{5.7}
$$

It is easy to get

$$
y_t = -uu_x + (uu_x)_{xx} - \partial_x\left(\frac{b}{2}u^2 + \frac{3-b}{2}u_x^2\right) - \lambda u_x + \lambda u_{xxx}. \tag{5.8}
$$

Substituting identity (5.8) into $dE(t)/dt$, we obtain

$$
\begin{aligned}
\frac{dE(t)}{dt} &= \int_{\mathbb{R}} e^x\left(-uu_x + (uu_x)_{xx} - \partial_x\left(\frac{b}{2}u^2 + \frac{3-b}{2}u_x^2\right)\right)dx \\
&\quad + \int_{\mathbb{R}} e^x(-\lambda u_x + \lambda u_{xxx})dx \\
&= \int_{\mathbb{R}} e^x\left(\frac{b}{2}u^2 + \frac{3-b}{2}u_x^2\right)dx,
\end{aligned} \tag{5.9}
$$

where we use (5.6).

Therefore, in the lifespan of the solution, we have

$$
E(t) = \int_0^t\int_{\mathbb{R}} e^x\left(\frac{b}{2}u^2 + \frac{3-b}{2}u_x^2\right)(x,\tau)dx d\tau > 0. \tag{5.10}
$$

By the same argument, one can check that the following identity for $F(t)$ is true:

$$
F(t) = -\int_0^t\int_{\mathbb{R}} e^{-x}\left(\frac{b}{2}u^2 + \frac{3-b}{2}u_x^2\right)(x,\tau)dx\, d\tau < 0. \tag{5.11}
$$

In order to complete the proof, it is sufficient to let $f_+(t) = (1/2)E(t)$ and $f_-(t) = (1/2)F(t)$. □

Remark 5.2. The main result in [18] is that any nontrivial classical solution of the b-family equation with dispersive term will not have compact support if its initial data has this property. But Theorem 4.1 means that no matter what the profile of the compactly supported initial datum $u_0(x)$ is (no matter whether it is positive or negative), for any $t > 0$ in its lifespan, the solution $u(x,t)$ is positive at infinity and negative at negative infinity. So Theorem 4.1 is an improvement of that in [18].

Acknowledgment

This work is partially supported by the Zhejiang Innovation Project (T200905), ZJNSF (Grant no. R6090109), and NSFC (Grant no. 11101376).

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