

Research Article

Optimal Inequalities between Harmonic, Geometric, Logarithmic, and Arithmetic-Geometric Means

Yu-Ming Chu and Miao-Kun Wang

Department of Mathematics, Huzhou Teachers College, Huzhou 313000, China

Correspondence should be addressed to Yu-Ming Chu, chuyuming2005@yahoo.com.cn

Received 19 July 2011; Accepted 4 September 2011

Academic Editor: Laurent Gosse

Copyright © 2011 Y.-M. Chu and M.-K. Wang. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

We find the least values p , q , and s in $(0, 1/2)$ such that the inequalities $H(pa + (1 - p)b, pb + (1 - p)a) > AG(a, b)$, $G(qa + (1 - q)b, qb + (1 - q)a) > AG(a, b)$, and $L(sa + (1 - s)b, sb + (1 - s)a) > AG(a, b)$ hold for all $a, b > 0$ with $a \neq b$, respectively. Here $AG(a, b)$, $H(a, b)$, $G(a, b)$, and $L(a, b)$ denote the arithmetic-geometric, harmonic, geometric, and logarithmic means of two positive numbers a and b , respectively.

1. Introduction

The classical arithmetic-geometric mean $AG(a, b)$ of two positive real numbers a and b is defined as the common limit of sequences $\{a_n\}$ and $\{b_n\}$, which are given by

$$\begin{aligned} a_0 &= a, & b_0 &= b, \\ a_{n+1} &= \frac{a_n + b_n}{2}, & b_{n+1} &= \sqrt{a_n b_n}. \end{aligned} \tag{1.1}$$

Let $H(a, b) = 2ab/(a + b)$, $G(a, b) = \sqrt{ab}$, $L(a, b) = (a - b)/(\log a - \log b)$, $I(a, b) = (1/e)(b^b/a^a)^{1/(b-a)}$, $A(a, b) = (a + b)/2$, and $M_p(a, b) = [(a^p + b^p)/2]^{1/p}$ ($p \neq 0$) and $M_0(a, b) = \sqrt{ab}$ be the harmonic, geometric, logarithmic, identric, arithmetic, and p -th

power means of two positive numbers a and b with $a \neq b$, respectively. Then it is well known that

$$\begin{aligned} \min\{a, b\} < H(a, b) = M_{-1}(a, b) < G(a, b) = M_0(a, b) < L(a, b) \\ < I(a, b) < A(a, b) = M_{-1}(a, b) < \max\{a, b\} \end{aligned} \quad (1.2)$$

for all $a, b > 0$ with $a \neq b$.

Recently, the inequalities for means have been the subject of intensive research. In particular, many remarkable inequalities for arithmetic-geometric mean can be found in the literature [1–9].

Carlson and Vuorinen [2], and Bracken [9] proved that

$$L(a, b) < AG(a, b) \quad (1.3)$$

for all $a, b > 0$ with $a \neq b$.

In [3], Vamanamurthy and Vuorinen established the following inequalities:

$$\begin{aligned} AG(a, b) < I(a, b) < A(a, b), \\ AG(a, b) < M_{1/2}(a, b), \\ AG(a, b) < \frac{\pi}{2}L(a, b), \\ M_1(a, b) < \frac{AG(a^2, b^2)}{AG(a, b)} < M_2(a, b) \end{aligned} \quad (1.4)$$

for all $a, b > 0$ with $a \neq b$.

We recall the Gauss identity [6, 7]

$$AG(1, r')\mathcal{K}(r) = \frac{\pi}{2} \quad (1.5)$$

for $r \in [0, 1)$ and $r' = \sqrt{1 - r^2}$. As usual, \mathcal{K} and \mathcal{E} denote the complete elliptic integrals [8] given by

$$\begin{aligned} \mathcal{K}(r) &= \int_0^{\pi/2} (1 - r^2 \sin^2 \theta)^{-1/2} d\theta = \frac{\pi}{2} \sum_{n=0}^{\infty} \frac{(1/2, n)^2}{(n!)^2} r^{2n}, \quad \mathcal{K}'(r) = \mathcal{K}(r'), \\ \mathcal{E}(r) &= \int_0^{\pi/2} (1 - r^2 \sin^2 \theta)^{1/2} d\theta = \frac{\pi}{2} \sum_{n=0}^{\infty} \frac{(-1/2, n)(1/2, n)}{(n!)^2} r^{2n}, \quad \mathcal{E}'(r) = \mathcal{E}(r'), \end{aligned} \quad (1.6)$$

where $(a, 0) = 1$ for $a \neq 0$, and $(a, n) = \prod_{k=0}^{n-1} (a + k)$.

For fixed $a, b > 0$ with $a \neq b$ and $x \in [0, 1/2]$, let

$$f_1(x) = H(xa + (1-x)b, xb + (1-x)a), \quad (1.7)$$

$$f_2(x) = G(xa + (1-x)b, xb + (1-x)a), \quad (1.8)$$

$$f_3(x) = L(xa + (1-x)b, xb + (1-x)a). \quad (1.9)$$

Then it is not difficult to verify that $f_1(x)$, $f_2(x)$, and $f_3(x)$ are continuous and strictly increasing in $[0, 1/2]$, respectively. Note that $f_1(0) = H(a, b) < AG(a, b) < f_1(1/2) = A(a, b)$, $f_2(0) = G(a, b) < AG(a, b) < f_2(1/2) = A(a, b)$ and $f_3(0) = L(a, b) < AG(a, b) < f_3(1/2) = A(a, b)$.

Therefore, it is natural to ask what are the least values p , q , and s in $(0, 1/2)$ such that the inequalities $H(pa + (1-p)b, pb + (1-p)a) > AG(a, b)$, $G(qa + (1-q)b, qb + (1-q)a) > AG(a, b)$, and $L(sa + (1-s)b, sb + (1-s)a) > AG(a, b)$ hold for all $a, b > 0$ with $a \neq b$, respectively. The main purpose of this paper is to answer these questions. Our main results are Theorems 1.1–1.3.

Theorem 1.1. *If $p \in (0, 1/2)$, then inequality*

$$H(pa + (1-p)b, pb + (1-p)a) > AG(a, b) \quad (1.10)$$

holds for all $a, b > 0$ with $a \neq b$ if and only if $p \geq 1/4$.

Theorem 1.2. *If $q \in (0, 1/2)$, then inequality*

$$G(qa + (1-q)b, qb + (1-q)a) > AG(a, b) \quad (1.11)$$

holds for all $a, b > 0$ with $a \neq b$ if and only if $q \geq 1/2 - \sqrt{2}/4$.

Theorem 1.3. *If $s \in (0, 1/2)$, then inequality*

$$L(sa + (1-s)b, sb + (1-s)a) > AG(a, b) \quad (1.12)$$

holds for all $a, b > 0$ with $a \neq b$ if and only if $s \geq 1/2 - \sqrt{3}/4$.

2. Lemmas

In order to establish our main results we need several formulas and lemmas, which we present in this section.

For $0 < r < 1$, the following derivative formulas were presented in [6, Appendix E, pp. 474-475]:

$$\begin{aligned} \frac{d\mathcal{K}}{dr} &= \frac{\mathcal{E} - r'^2 \mathcal{K}}{rr'^2}, & \frac{d\mathcal{E}}{dr} &= \frac{\mathcal{E} - \mathcal{K}}{r}, \\ \frac{d(\mathcal{E} - r'^2 \mathcal{K})}{dr} &= r\mathcal{K}, & \frac{d(\mathcal{K} - \mathcal{E})}{dr} &= \frac{r\mathcal{E}}{r'^2}, \end{aligned} \quad (2.1)$$

$$\mathcal{K}\left(\frac{2\sqrt{r}}{1+r}\right) = (1+r)\mathcal{K}(r). \quad (2.2)$$

The following Lemma 2.1 can be found in [6, Theorem 3.21(7) and Exercise 3.43(4)].

Lemma 2.1. (1) $(1 + r'^2)\mathcal{E}(r) - 2r'^2 \mathcal{K}(r)$ is strictly increasing from $(0, 1)$ onto $(0, 1)$;
 (2) $\mathcal{E}(r)/r'^{1/2}$ is strictly increasing from $(0, 1)$ onto $(\pi/2, +\infty)$.

Lemma 2.2. *Inequality*

$$\frac{2}{\pi} \mathcal{K}(r) \sqrt{1 - \frac{1}{2}r^2} > 1 \quad (2.3)$$

holds for all $r \in (0, 1)$.

Proof. Let

$$f(r) = \log \left[\frac{2}{\pi} \mathcal{K}(r) \sqrt{1 - \frac{1}{2}r^2} \right]. \quad (2.4)$$

Then simple computations lead to

$$f(0) = 0, \quad (2.5)$$

$$f'(r) = \frac{\mathcal{E}(r) - r'^2 \mathcal{K}(r)}{rr'^2 \mathcal{K}(r)} - \frac{r}{2 - r^2} = \frac{(1 + r'^2)\mathcal{E}(r) - 2r'^2 \mathcal{K}(r)}{rr'^2(2 - r^2)\mathcal{K}(r)}. \quad (2.6)$$

It follows from Lemma 2.1 (1) and (2.6) that $f'(r) > 0$ for $r \in (0, 1)$, which implies that $f(r)$ is strictly increasing in $(0, 1)$.

Therefore, inequality (2.3) follows from (2.4) and (2.5) together with the monotonicity of $f(r)$. \square

Lemma 2.3. *Inequality*

$$\frac{2\sqrt{3}}{\pi} r \mathcal{K}(r) > \log \left(\frac{2 + \sqrt{3}r}{2 - \sqrt{3}r} \right) \quad (2.7)$$

holds for all $r \in (0, 1)$.

Proof. Let

$$g(r) = \frac{2\sqrt{3}}{\pi} r \mathcal{K}(r) - \log\left(\frac{2 + \sqrt{3}r}{2 - \sqrt{3}r}\right). \quad (2.8)$$

Then simple computations lead to

$$g(0) = 0, \quad (2.9)$$

$$g'(r) = \frac{2\sqrt{3}}{\pi} \left(\mathcal{K}(r) + r \frac{\xi(r) - r'^2 \mathcal{K}(r)}{r r'^2} \right) - \frac{4\sqrt{3}}{4 - 3r^2} = \frac{2\sqrt{3}}{\pi(1 + 3r'^2)} \left(\frac{1 + 3r'^2}{r'^{3/2}} \frac{\xi(r)}{r'^{1/2}} - 2\pi \right). \quad (2.10)$$

Clearly the function $r \rightarrow (1 + 3r^2)/r^{3/2}$ is strictly decreasing from $(0, 1)$ onto $(4, +\infty)$. Then (2.10) and Lemma 2.1 (2) lead to the conclusion that $g'(r) > 0$ for $r \in (0, 1)$. Thus, $g(r)$ is strictly increasing in $(0, 1)$.

Therefore, inequality (2.7) follows from (2.8) and (2.9) together with the monotonicity of $g(r)$. \square

3. Proof of Theorems 1.1–1.3

Proof of Theorem 1.1. Let $\lambda = 1/4$, then from the monotonicity of the function $f_1(x) = H(xa + (1-x)b, xb + (1-x)a)$ in $[0, 1/2]$ we know that to prove inequality (1.10) we only need to prove that

$$AG(a, b) < H(\lambda a + (1-\lambda)b, \lambda b + (1-\lambda)a) \quad (3.1)$$

for all $a, b > 0$ with $a \neq b$.

From (1.1) and (1.7) we clearly see that both $AG(a, b)$ and $H(\lambda a + (1-\lambda)b, \lambda b + (1-\lambda)a)$ are symmetric and homogeneous of degree 1. Without loss of generality, we can assume that $a = 1 > b$. Let $t = b \in (0, 1)$ and $r = (1-t)/(1+t)$, then from (1.5) we have

$$H(\lambda a + (1-\lambda)b, \lambda b + (1-\lambda)a) - AG(a, b) = \frac{(t+3)(3t+1)}{8(1+t)} - \frac{\pi}{2\mathcal{K}(t)}. \quad (3.2)$$

Let

$$F(t) = \frac{(t+3)(3t+1)}{8(1+t)} - \frac{\pi}{2\mathcal{K}(t)}. \quad (3.3)$$

Then making use of (2.2) we get

$$F(t) = \frac{(2+r)(2-r)}{4(1+r)} - \frac{\pi}{2(1+r)\mathcal{K}(r)} = \frac{\pi}{8(1+r)\mathcal{K}(r)} F_1(r), \quad (3.4)$$

where $F_1(r) = (2/\pi)(4 - r^2)\mathcal{K}(r) - 4$. Note that

$$\begin{aligned} F_1(r) &= \sum_{n=0}^{\infty} \frac{(1/2, n)^2}{(n!)^2} r^{2n} (4 - r^2) - 4 \\ &= 4r^2 \sum_{n=0}^{\infty} \frac{(1/2, n+1)^2}{[(n+1)!]^2} r^{2n} - r^2 \sum_{n=0}^{\infty} \frac{(1/2, n)^2}{(n!)^2} r^{2n} \\ &= \sum_{n=0}^{\infty} \frac{(1/2, n)^2}{[(n+1)!]^2} (3n^2 + 2n) r^{2(n+1)} > 0. \end{aligned} \quad (3.5)$$

Therefore, inequality (3.1) follows from (3.2)–(3.5). \square

Next, we prove that the parameter $p = \lambda = 1/4$ is the best possible parameter in $(0, 1/2)$ such that inequality (1.10) holds for all $a, b > 0$ with $a \neq b$.

Since for $0 < p < 1/2$ and small $x > 0$,

$$\text{AG}(1, 1-x) = \frac{\pi}{2\mathcal{K}(\sqrt{2x-x^2})} = 1 - \frac{1}{2}x - \frac{1}{16}x^2 + o(x^3), \quad (3.6)$$

$$H(p(1-x) + 1-p, (1-p)(1-x) + p) = 1 - \frac{1}{2}x + \left(-p^2 + p - \frac{1}{4}\right)x^2 + o(x^3). \quad (3.7)$$

It follows from (3.6) and (3.7) that inequality $\text{AG}(1, 1-x) \leq H(p(1-x) + 1-p, (1-p)(1-x) + p)$ holds for small x only $p \geq 1/4$.

Remark 3.1. For $0 < p < 1/2$ and $x > 0$, one has

$$\lim_{x \rightarrow 0} \frac{H(px + 1-p, (1-p)x + p)}{\text{AG}(1, x)} = \lim_{x \rightarrow 0} \frac{4[px + 1-p][(1-p)x + p]}{(1+x)\pi} \mathcal{K}(x') = +\infty. \quad (3.8)$$

Equation (3.8) implies that there does not exist $p \in (0, 1/2)$ such that $\text{AG}(1, x) > H(px + 1-p, (1-p)x + p)$ for all $x \in (0, 1)$.

Proof of Theorem 1.2. Let $\mu = 1/2 - \sqrt{2}/4$, then from the monotonicity of the function $f_2(x) = G(xa + (1-x)b, xb + (1-x)a)$ in $[0, 1/2]$ we know that to prove inequality (1.11) we only need to prove that

$$\text{AG}(a, b) < G(\mu a + (1-\mu)b, \mu b + (1-\mu)a) \quad (3.9)$$

for all $a, b > 0$ with $a \neq b$.

From (1.1) and (1.8) we clearly see that both $AG(a, b)$ and $G(\mu a + (1 - \mu)b, \mu b + (1 - \mu)a)$ are symmetric and homogeneous of degree 1. Without loss of generality, we can assume that $a = 1 > b$. Let $t = b \in (0, 1)$ and $r = (1 - t)/(1 + t)$, then from (1.5) we have

$$G(\mu a + (1 - \mu)b, \mu b + (1 - \mu)a) - AG(a, b) = \sqrt{[\mu + (1 - \mu)t][\mu t + (1 - \mu)]} - \frac{\pi}{2\mathcal{K}(t)}. \tag{3.10}$$

Let

$$G(t) = \sqrt{[\mu + (1 - \mu)t][\mu t + (1 - \mu)]} - \frac{\pi}{2\mathcal{K}(t)}. \tag{3.11}$$

Then making use of (2.2) we have

$$G(t) = \frac{\pi}{2(1+r)\mathcal{K}(r)} \left[\frac{2}{\pi} \mathcal{K}(r) \sqrt{1 - \frac{1}{2}r^2 - 1} \right]. \tag{3.12}$$

Therefore, inequality (3.9) follows from (3.10)–(3.12) together with Lemma 2.2. \square

Next, we prove that the parameter $q = \mu = 1/2 - \sqrt{2}/4$ is the best possible parameter in $(0, 1/2)$ such that inequality (1.11) holds for all $a, b > 0$ with $a \neq b$.

Since for $0 < q < 1/2$ and small $x > 0$,

$$G(q(1 - x) + 1 - q, (1 - q)(1 - x) + q) = 1 - \frac{1}{2}x + \frac{1}{8}(-4q^2 + 4q - 1)x^2 + o(x^3). \tag{3.13}$$

It follows from (3.6) and (3.13) that inequality $AG(1, 1 - x) \leq G(q(1 - x) + 1 - q, (1 - q)(1 - x) + q)$ holds for small x only $q \geq 1/2 - \sqrt{2}/4$.

Remark 3.2. For $0 < q < 1/2$ and $x > 0$, one has

$$\lim_{x \rightarrow 0} \frac{G(qx + 1 - q, (1 - q)x + q)}{AG(1, x)} = \lim_{x \rightarrow 0} \frac{2}{\pi} \sqrt{[qx + 1 - q][(1 - q)x + q]} \mathcal{K}(x') = +\infty. \tag{3.14}$$

Equation (3.14) implies that there does not exist $q \in (0, 1/2)$ such that $AG(1, x) > G(qx + 1 - q, (1 - q)x + q)$ for all $x \in (0, 1)$.

Proof of Theorem 1.3. Let $\beta = 1/2 - \sqrt{3}/4$, then from the monotonicity of $f_3(x) = L(xa + (1 - x)b, xb + (1 - x)a)$ in $[0, 1/2]$ we know that to prove inequality (1.12) we only need to prove that

$$AG(a, b) < L(\beta a + (1 - \beta)b, \beta b + (1 - \beta)a) \tag{3.15}$$

for all $a, b > 0$ with $a \neq b$.

From (1.1) and (1.9) we clearly see that both $AG(a, b)$ and $L(\beta a + (1 - \beta)b, \beta b + (1 - \beta)a)$ are symmetric and homogeneous of degree 1. Without loss of generality, we can assume that $a = 1 > b$. Let $t = b \in (0, 1)$ and $r = (1 - t)/(1 + t)$, then from (1.5) one has

$$\begin{aligned} & L(\beta a + (1 - \beta)b, \beta b + (1 - \beta)a) - AG(a, b) \\ &= \frac{\sqrt{3}(1 - t)}{2 \log \left[\left(\frac{(2 - \sqrt{3})t + 2 + \sqrt{3}}{(2 + \sqrt{3})t + 2 - \sqrt{3}} \right) \right]} - \frac{\pi}{2\mathcal{K}(t)}. \end{aligned} \quad (3.16)$$

Let

$$J(t) = \frac{\sqrt{3}(1 - t)}{2 \log \left[\left(\frac{(2 - \sqrt{3})t + 2 + \sqrt{3}}{(2 + \sqrt{3})t + 2 - \sqrt{3}} \right) \right]} - \frac{\pi}{2\mathcal{K}(t)}. \quad (3.17)$$

Then from (2.2) we get

$$J(t) = \frac{\pi}{2(1 + r)\mathcal{K}(r) \log \left(\frac{(2 + \sqrt{3}r)}{(2 - \sqrt{3}r)} \right)} g(r), \quad (3.18)$$

where $g(r)$ is defined as in Lemma 2.3.

Therefore, inequality (3.15) follows from (3.16)–(3.18) together with Lemma 2.3. \square

Next, we prove that the parameter $s = \beta = 1/2 - \sqrt{3}/4$ is the best possible parameter in $(0, 1/2)$ such that inequality (1.12) holds for all $a, b > 0$ with $a \neq b$.

Since for $0 < s < 1/2$ and small $x > 0$,

$$L(s(1 - x) + 1 - s, (1 - s)(1 - x) + s) = 1 - \frac{1}{2}x + \frac{1}{12}(-4s^2 + 4s - 1)x^2 + o(x^3). \quad (3.19)$$

It follows from (3.6) and (3.19) that inequality $AG(1, 1 - x) \leq L(s(1 - x) + 1 - s, (1 - s)(1 - x) + s)$ holds for small x only $s \geq 1/2 - \sqrt{3}/4$.

Remark 3.3. For $0 < s < 1/2$ and $x > 0$, one has

$$\lim_{x \rightarrow 0} \frac{L(sx + 1 - s, (1 - s)x + s)}{AG(1, x)} = \lim_{x \rightarrow 0} \frac{2}{\pi} \mathcal{K}(x') \frac{(1 - 2s)(1 - x)}{\log[(sx + 1 - s)/((1 - s)x + s)]} = +\infty. \quad (3.20)$$

Equation (3.20) implies that there exist no values $s \in (0, 1/2)$ such that $AG(1, x) > L(sx + 1 - s, (1 - s)x + s)$ for all $x \in (0, 1)$.

Acknowledgments

The authors wish to thank the anonymous referees for their careful reading of the paper and their fruitful comments and suggestions. This research was supported by the Natural Science Foundation of China under Grant 11071069, and Innovation Team Foundation of the Department of Education of Zhejiang Province under Grant T200924.

References

- [1] J. M. Borwein and P. B. Borwein, *Pi and the AGM*, John Wiley and Sons, New York, NY, USA, 1987.
- [2] B. C. Carlson and M. Vuorinen, "Inequality of the AGM and the logarithmic mean," *SIAM Review*, vol. 33, no. 4, p. 655, 1991.
- [3] M. K. Vamanamurthy and M. Vuorinen, "Inequalities for means," *Journal of Mathematical Analysis and Applications*, vol. 183, no. 1, pp. 155–166, 1994.
- [4] T. Hayashi, "Non-commutative arithmetic-geometric mean inequality," *Proceedings of the American Mathematical Society*, vol. 137, no. 10, pp. 3399–3406, 2009.
- [5] L. Knockaert, "Best upper bounds based on the arithmetic-geometric mean inequality," *Archives of Inequalities and Applications*, vol. 1, no. 1, pp. 85–90, 2003.
- [6] G. D. Anderson, M. K. Vamanamurthy, and M. K. Vuorinen, *Conformal Invariants, Inequalities, and Quasiconformal Maps*, John Wiley and Sons, New York, NY, USA, 1997.
- [7] S.-L. Qiu and M. Vuorinen, "Special functions in geometric function theory," in *Handbook of Complex Analysis: Geometric Function Theory*, vol. 2, pp. 621–659, Elsevier Science Publishers B.V., Amsterdam, The Netherlands, 2005.
- [8] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions with Formulas, Graphs and Mathematical Tables*, Dover, New York, NY, USA, 1965.
- [9] P. Bracken, "An arithmetic-geometric mean inequality," *Expositiones Mathematicae*, vol. 19, no. 3, pp. 273–279, 2001.



Hindawi

Submit your manuscripts at
<http://www.hindawi.com>

