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Research Article

The Existence of Cone Critical Point and Common Fixed Point with Applications

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We first establish some new critical point theorems for nonlinear dynamical systems in cone metric spaces or usual metric spaces, and then we present some applications to generalizations of Dancš-Hegedüs-Medvegyev's principle and the existence theorem related with Ekeland's variational principle, Caristi's common fixed point theorem for multivalued maps, Takahashi's nonconvex minimization theorem, and common fuzzy fixed point theorem. We also obtain some fixed point theorems for weakly contractive maps in the setting of cone metric spaces and focus our research on the equivalence between scalar versions and vectorial versions of some results of fixed point and others.

1. Introduction

In 1983, Dancš et al. [1] proved the following interesting existence theorem of critical point (or stationary point) for a nonlinear dynamical system.

Dancš-Hegedüs-Medvegyev's Principle [1]

Let (X, d) be a complete metric space. Let $\Gamma : X \to 2^X$ be a multivalued map with nonempty values. Suppose that the following conditions are satisfied:

(i) for each $x \in X$, we have $x \in \Gamma(x)$, and $\Gamma(x)$ is closed;

(ii) $x, y \in X$ with $y \in \Gamma(x)$ implies $\Gamma(y) \subseteq \Gamma(x)$;

(iii) for each $n \in \mathbb{N}$ and each $x_{n+1} \in \Gamma(x_n)$, we have $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$.

Then there exists $v \in X$ such that $\Gamma(v) = \{v\}$.

Dancš-Hegedüs-Medvegyev's Principle has been popularly investigated and applied in various fields of applied mathematical analysis and nonlinear analysis, see, for example, [2, 3] and references therein. It is well known that the celebrated Ekeland's variational principle can be deduced by the detour of using Dancš-Hegedüs-Medvegyev's principle, and it is equivalent to the Caristi's fixed point theorem, to the Takahashi's nonconvex minimization theorem, to the drop theorem, and to the petal theorem. Many generalizations in various different directions of these results in metric (or quasi-metric) spaces and more general in topological vector spaces have been studied by several authors in the past; for detail, one can refer to [2–12].

Let E be a topological vector space (t.v.s. for short) with its zero vector θ_E. A nonempty subset K of E is called *a convex cone* if $K + K \subseteq K$ and $\lambda K \subseteq K$ for $\lambda \geq 0$. A convex cone K is said to be *pointed* if $K \cap (-K) = \{\theta_E\}$. For a given proper, pointed, and convex cone K in E, we can define a partial ordering $\precsim_K$ with respect to K by

$$x \precsim_K y \Longleftrightarrow y - x \in K. \tag{1.1}$$

$x \prec_K y$ will stand for $x \precsim_K y$ and $x \neq y$, while $x \ll_K y$ will stand for $y - x \in \operatorname{int} K$, where $\operatorname{int} K$ denotes the interior of K.

In the following, unless otherwise specified, we always assume that Y is a locally convex Hausdorff t.v.s. with its zero vector θ, K a proper, closed, convex, and pointed cone in Y with $\operatorname{int} K \neq \emptyset$, $e \in \operatorname{int} K$ and $\precsim_K$ a partial ordering with respect to K. Denote by $\mathbb{R}$ and $\mathbb{N}$ the set of real numbers and the set of positive integers, respectively.

Fixed point theory in K-metric and K-normed spaces was studied and developed by Perov [13], Kvedaras et al. [14], Perov and Kibenko [15], Mukhamadiev and Stetsenko [16], Vandergraft [17], Zabrejko [18], and references therein. In 2007, Huang and Zhang [19] reintroduced such spaces under the name of cone metric spaces and investigated fixed point theorems in such spaces in the same work. Since then, the cone metric fixed point theory is prompted to study by many authors; for detail, see [20–29] and references therein.

Very recently, in order to improve and extend the concept of cone metric space in the sense of Huang and Zhang, Du [23] first introduced the concepts of *TVS-cone metric* and *TVS-cone metric* space as follows.

Definition 1.1 (see [23]). Let X be a nonempty set. A vector-valued function $p: X \times X \to Y$ is said to be a *TVS-cone metric*, if the following conditions hold:

(C1) $\theta \precsim_K p(x, y)$ for all $x, y \in X$ and $p(x, y) = \theta$ if and only if $x = y$;

(C2) $p(x, y) = p(y, x)$ for all $x, y \in X$;

(C3) $p(x, z) \precsim_K p(x, y) + p(y, z)$ for all $x, y, z \in X$.

The pair (X, p) is then called a *TVS-cone metric space*.

Definition 1.2 (see [23]). Let (X, p) be a *TVS-cone metric* space, $x \in X$, and, $\{x_n\}_{n\in\mathbb{N}}$ let be a sequence in X.

(i) $\{x_n\}$ is said to *TVS-cone converge to* x if, for every $c \in Y$ with $\theta \ll_K c$, there exists a natural number $\mathbb{N}_0$ such that $p(x_n, x) \ll_K c$ for all $n \geq \mathbb{N}_0$. We denote this by *cone*-$\lim_{n\to\infty} x_n = x$ or $x_n \xrightarrow{\text{cone}} x$ as $n \to \infty$ and call x the *TVS-cone limit* of $\{x_n\}$.

(ii) $\{x_n\}$ is said to be a *TVS-cone Cauchy sequence* if, for every $c \in Y$ with $\theta \ll_K c$, there is a natural number $\mathbb{N}_0$ such that $p(x_n, x_m) \ll_K c$ for all $n, m \geq \mathbb{N}_0$.

(iii) (X, p) is said to be *TVS-cone complete* if every *TVS*-cone Cauchy sequence in X is *TVS-cone* convergent.

In [23], the author proved the following important results.

Theorem 1.3 (see [23]). *Let (X, p) be a TVS-cone metric spaces. Then $d_p : X \times X \to [0, \infty)$ defined by $d_p := \xi_e \circ p$ is a metric, where the nonlinear scalarization function $\xi_e : Y \to \mathbb{R}$ is defined by*

$$\xi_e(y) = \inf\{r \in \mathbb{R} : y \in re - K\}, \quad \forall y \in Y. \tag{1.2}$$

Example 1.4. Let $X = [0,1]$, $Y = \mathbb{R}^2$, $K = \mathbb{R}^2_+ = \{(x,y) \in \mathbb{R}^2 : x, y \geq 0\}$, $e = (1,1)$, and $\theta = (0,0)$. Define $p : X \times X \to Y$ by

$$p(x,y) = (|x-y|, 5|x-y|). \tag{1.3}$$

Then (X, p) is a *TVS-cone* complete metric space. It is easy to verify that

$$d_p(x,y) = \xi_e(p(x,y)) = \inf\{r \in \mathbb{R} : p(x,y) \in re - K\} = 5|x-y|, \tag{1.4}$$

so d_p is a metric on X, and (X, d_p) is a complete metric space.

Theorem 1.5 (see [23]). *Let (X, p) be a TVS-cone metric space, let $x \in X$, and let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in X. Then the following statements hold.*

(a) *If $\{x_n\}$ TVS-cone converges to (i.e., $x_n \xrightarrow{\text{cone}} x$ as $n \to \infty$), then $d_p(x_n, x) \to 0$ as $n \to \infty$ (i.e., $x_n \xrightarrow{d_p} x$ as $n \to \infty$).*

(b) *If $\{x_n\}$ is a TVS-cone Cauchy sequence in (X, p), then $\{x_n\}$ is a Cauchy sequence (in usual sense) in (X, d_p).*

The paper is organized as follows. In Section 2, we first establish some new critical point theorems for nonlinear dynamical systems in cone metric spaces or usual metric spaces, and then we present some applications to generalizations of Dancš-Hegedüs-Medvegyev's principle and the existence theorem related with Ekeland's variational principle, Caristi's common fixed point theorem for multivalued maps, Takahashi's nonconvex minimization theorem, and the common fuzzy fixed point theorem. Section 3 is dedicated to the study of fixed point theorems for weakly contractive maps in the setting of cone metric spaces. In Section 4, we focus our research on the equivalence between scalar versions and vectorial vesions of some results of fixed point and others.

2. Critical Point Theorems in Cone Metric Spaces

Let X be a nonempty set. A fuzzy set in X is a function of X into $[0,1]$. Let $\mathcal{F}(X)$ be the family of all fuzzy sets in X. A fuzzy map on X is a map from X into $\mathcal{F}(X)$. This enables us to regard each fuzzy map as a two-variable function of $X \times X$ into $[0,1]$. Let F be a fuzzy map on X. An element x of X is said to be a fuzzy fixed point of F if $F(x,x) = 1$ (see, e.g., [4, 5, 30–32]). Let $\Gamma : X \to 2^X$ be a multivalued map. A point $x \in X$ is called to be *a critical point* (or *stationary point*) [1–3, 7, 32] of Γ if $\Gamma(v) = \{v\}$.

Recall that the nonlinear scalarization function $\xi_e : Y \to \mathbb{R}$ is defined by

$$\xi_e(y) = \inf\{r \in \mathbb{R} : y \in re - K\}, \quad \forall y \in Y. \tag{2.1}$$

Lemma 2.1 (see [6, 23, 29, 33]). *For each $r \in \mathbb{R}$ and $y \in Y$, the following statements are satisfied:*

(i) $\xi_e(y) \leq r \Leftrightarrow y \in re - K$,

(ii) $\xi_e(y) > r \Leftrightarrow y \notin re - K$,

(iii) $\xi_e(y) \geq r \Leftrightarrow y \notin re - \operatorname{int} K$,

(iv) $\xi_e(y) < r \Leftrightarrow y \in re - \operatorname{int} K$,

(v) $\xi_e(\cdot)$ *is positively homogeneous and continuous on* Y,

(vi) *if* $y_1 \in y_2 + K$ *(i.e.,* $y_2 \precsim_K y_1$*), then* $\xi_e(y_2) \leq \xi_e(y_1)$,

(vii) $\xi_e(y_1 + y_2) \leq \xi_e(y_1) + \xi_e(y_2)$ *for all* $y_1,\ y_2 \in Y$.

Remark 2.2. Notice that the reverse statement of (vi) in Lemma 2.1 (i.e., $\xi_e(y_2) \leq \xi_e(y_1) \Rightarrow y_1 \in y_2 + K$ or $y_2 \precsim_K y_1$) does not hold in general. For example, let $Y = \mathbb{R}^2$, let $K = \mathbb{R}^2_+ = \{(x, y) \in \mathbb{R}^2 : x, y \geq 0\}$, and let $e = (1, 1)$. Then K is a proper, closed, convex, and pointed cone in Y with $\operatorname{int} K = \{(x, y) \in \mathbb{R}^2 : x, y > 0\} \neq \emptyset$ and $e \in \operatorname{int} K$. For $r = 1$, it is easy to see that $y_1 = (5, -6) \notin re - \operatorname{int} K$, and $y_2 = (0, 0) \in re - \operatorname{int} K$. By applying (iii) and (iv) of Lemma 2.1, we have $\xi_e(y_2) < 1 \leq \xi_e(y_1)$ while $y_1 \notin y_2 + K$.

Definition 2.3. Let A be a nonempty subset of a *TVS-cone metric* space (X, p).

(i) The *TVS-cone* closure of A, denoted tvsc-cl(A), is defined by

$$\text{tvsc-cl}(A) = \left\{ x \in X : \exists \{x_n\} \subset A \text{ such that } x_n \overset{\text{cone}}{\longrightarrow} x \text{ as } n \longrightarrow \infty \right\}. \tag{2.2}$$

Obviously, $A \subseteq$ tvsc-cl(A).

(ii) A is said to be *TVS-cone closed* if $A =$ tvsc-cl(A).

(iii) A is said to be *TVS-cone open* if the complement $X \setminus A$ of A is *TVS-cone* closed.

If $Y = \mathbb{R}$, $K = [0, \infty) \subset \mathbb{R}$ and $e = 1$, then $p \equiv d$ is a metric in usual sense, and the closure of A is denoted by $\mathrm{cl}_d(A)$.

Theorem 2.4. *Let* (X, p) *be a TVS-cone metric space and let*

$$\mathcal{T}_p = \{U \subseteq X : U \text{ is } TVS\text{-cone open in } (X, p)\}. \tag{2.3}$$

Then $\mathcal{T}_p$ *is a topology on* (X, p) *induced by* p.

Proof. Clearly, $\emptyset$ and X are *TVS-cone* closed in (X, p). Thus, X and $\emptyset$ are *TVS-cone* open in (X, p), and hence $\emptyset, X \in \mathcal{T}_p$. Let $U_1, U_2 \in \mathcal{T}_p$. Then $V_1 = X \setminus U_1$ and $V_2 = X \setminus U_2$ are *TVS-cone* closed in (X, p). We claim that $U_1 \cap U_2 \in \mathcal{T}_p$. Let $v \in$ tvsc-cl$(V_1 \cup V_2)$. Then $\exists \{v_n\} \subset V_1 \cup V_2$ such that $v_n \overset{\text{cone}}{\longrightarrow} v$ as $n \to \infty$. Without loss of generality, we may assume that there exists a subsequence $\{v_{n_k}\}$ of $\{v_n\} \cap V_1$. Since $v_{n_k} \overset{\text{cone}}{\longrightarrow} v$ as $k \to \infty$, we have $v \in$ tvsc-cl$(V_1) = V_1 \subseteq V_1 \cup V_2$. So tvsc-cl$(V_1 \cup V_2) \subseteq V_1 \cup V_2$, and; hence, $V_1 \cup V_2$ is *TVS-cone* closed in (X, p). From

$$U_1 \cap U_2 = X \setminus (V_1 \cup V_2), \tag{2.4}$$

we see that $U_1 \cap U_2$ is *TVS-cone* open in (X, p) and $U_1 \cap U_2 \in \mathcal{T}_p$.

Let I be any index set, and let $\{U_i\}_{i\in I} \subset \mathcal{T}_p$. We show that $\bigcup_{i\in I} U_i \in \mathcal{T}_p$. For each $i \in I$, set $V_i = X \setminus U_i$. Thus, V_i is *TVS-cone* closed in X for all $i \in I$. Let $w \in$ tvsc-cl$(\bigcap_{i\in I} V_i)$. Then $\exists \{w_n\} \subset \bigcap_{i\in I} V_i$ such that $w_n \overset{\text{cone}}{\longrightarrow} w$ as $n \to \infty$. For each $i \in I$, since $\{w_n\} \subset \mathcal{V}_i$ and $w_n \overset{\text{cone}}{\longrightarrow} w$, $w \in$ tvsc-cl$(V_i) = V_i$. Hence, $w \in \bigcap_{i\in I} V_i$. So tvsc-cl$(\bigcap_{i\in I} V_i) \subseteq \bigcap_{i\in I} V_i$, and then $\bigcap_{i\in I} V_i$ is *TVS-cone* closed in (X, p). Since

$$\bigcup_{i\in I} U_i = X \setminus \bigcap_{i\in I} V_i, \tag{2.5}$$

$\bigcup_{i\in I} U_i$ is *TVS-cone* open in (X, p), and $\bigcup_{i\in I} U_i \in \mathcal{T}_p$.

Therefore, by above, we prove that $\mathcal{T}_p$ is a topology on (X, p). □

The following result is simple, but it is very useful in this paper.

Lemma 2.5. *Let E be a t.v.s., K a convex cone with* $\operatorname{int} K \neq \emptyset$ *in E, and let a, b, $c \in E$. Then the following statements hold.*

(i) $\operatorname{int} K + K \subseteq \operatorname{int} K$.

(ii) *If $a \precsim_K b$ and $b \precsim_K c$, then $a \precsim_K c$.*

(iii) *If $a \precsim_K b$ and $b \ll_K c$, then $a \ll_K c$.*

(iv) *If $a \ll_K b$ and $b \precsim_K c$, then $a \ll_K c$.*

(v) *If $a \ll_K b$ and $b \ll_K c$, then $a \ll_K c$.*

(vi) *If $a \precsim_K c$ and $b \precsim_K c$, then $a + b \precsim_K 2c$.*

(vii) *If $a \precsim_K c$ and $b \ll_K c$, then $a + b \ll_K 2c$.*

(viii) *If $a \ll_K c$ and $b \ll_K c$, then $a + b \ll_K 2c$.*

Proof. The conclusion (i) follows from the facts that the set int $K + K$ is open in E, and K is a convex cone. By the transitivity of partial ordering $\precsim_K$, we have the conclusion (ii). To see (iii), since $a \precsim_K b \Leftrightarrow b - a \in K$ and $b \ll_k c \Leftrightarrow c - b \in \operatorname{int} K$, it follows from (i) that

$$c - a = (c - b) + (b - a) \in \operatorname{int} K + K \subseteq \operatorname{int} K, \tag{2.6}$$

which means that $a \ll_k c$. The proofs of conclusions (iv)–(viii) are similar to (iii). □

Definition 2.6. Let (X, p) be a *TVS-cone* metric space. A nonempty subset C of X is said to be *TVS-cone compact* if every sequence in C has a *TVS-cone* convergent subsequence whose *TVS-cone* limit is an element of C.

If X is *TVS-cone* compact, then we say that (X, p) is a *TVS-cone* compact metric space.

Theorem 2.7. *Let C be a nonempty subset of a TVS-cone metric space (X, p). Then the following statements hold.*

(a) *If C is a closed set in the metric space (X, d_p), then C is TVS-cone closed in (X, p) and* tvsc-cl$(C) = cl_{d_p}(C)$, *where $d_p := \xi_e \circ p$.*

(b) *If C is TVS-cone compact, then it is TVS-cone closed.*

(c) *If C is TVS-cone closed and (X, p) is TVS-cone complete, then (C, p) is also TVS -cone complete.*

(d) *If C is TVS-cone compact, then (C, p) is TVS-cone complete.*

(e) *If C is TVS-cone compact, then C is (sequentially) compact in the metric space (X, d_p).*

Proof. Applying Theorem 1.3, d_p is a metric on X. Let C be a closed set in the metric space (X, d_p). By Theorem 1.5, we have

$$\begin{aligned} \text{tvsc-cl}(C) &= \left\{ x \in X : \exists \{x_n\} \subset C \text{ such that } x_n \stackrel{\text{cone}}{\longrightarrow} x \text{ as } n \longrightarrow \infty \right\} \\ &\subseteq \left\{ x \in X : \exists \{x_n\} \subset C \text{ such that } x_n \stackrel{d_p}{\longrightarrow} x \text{ as } n \longrightarrow \infty \right\} \\ &= \text{cl}_{d_p}(C) = C, \end{aligned} \tag{2.7}$$

which implies that C is *TVS-cone* closed in (X, p) and tvsc-cl(C) = cl$_{d_p}(C)$. Hence, the conclusion (a) holds.

Next, assume that C is *TVS-cone* compact in (X, p). Let $x \in$ tvsc-cl(C). Then there exists $\{x_n\} \subset C$ such that $x_n \stackrel{\text{cone}}{\longrightarrow} x$ as $n \to \infty$. By the *TVS-cone* compactness of C, there exist $\{x_{n_j}\} \subset \{x_n\}$ and $w \in C$ such that $x_{n_j} \stackrel{\text{cone}}{\longrightarrow} w$ as $j \to \infty$. Applying Theorem 1.5, $d_p(x_n, x) \to 0$ as $n \to \infty$ and $d_p(x_{n_j}, w) \to 0$ as $j \to \infty$. By the uniqueness of limit, $x = w \in C$, and; hence, tvsc-cl$(C) \subseteq C$. So C is *TVS-cone* closed in (X, p), and (b) is proved.

To see (c), let $\{x_n\}$ be a *TVS-cone* Cauchy sequence in $(C, p) \subseteq (X, p)$. Since (X, p) is *TVS-cone* complete, there exists $v \in X$ such that $x_n \stackrel{\text{cone}}{\longrightarrow} v$ as $n \to \infty$. Hence, $v \in$ tvsc-cl(C) = C, which show that (C, p) is *TVS-cone* complete.

Let us verify (d). Given $c \in Y$ with $\theta \ll_K c$, and let $\{z_n\}$ be a *TVS-cone* Cauchy sequence in (C, p). Then there exists $\nu_1 \in \mathbb{N}$ such that $p(z_n, z_m) \ll_K (1/2)c$ for all $m, n \geq \nu_1$. Since C is *TVS-cone* compact, there exists a subsequence $\{z_{n_t}\}$ of $\{z_n\}$, and $\widehat{z} \in C$ such that $z_{n_t} \stackrel{\text{cone}}{\longrightarrow} \widehat{z}$ as $t \to \infty$. For c, there exists $\nu_2 \in \mathbb{N}$ such that $p(z_{n_t}, \widehat{z}) \ll_K (1/2)c$ for all $t \geq \nu_2$. Let $\nu = \max\{\nu_1, \nu_2\}$. For any $n \geq \nu$, since

$$\begin{gathered} p(z_n, \widehat{z}) \precsim_K p(z_n, z_{n_\nu}) + p(z_{n_\nu}, \widehat{z}), \\ p(z_n, z_{n_\nu}) + p(z_{n_\nu}, \widehat{z}) \ll_K c, \end{gathered} \tag{2.8}$$

by (iii) of Lemma 2.5, we have $p(z_n, \widehat{z}) \ll_K c$. So $\{x_n\}$ is *TVS-cone* convergent to x. Therefore, (C, p) is *TVS-cone* complete.

The conclusion (e) is obvious. The proof is completed. □

Let C be a subset of a *TVS-cone* metric space (X, p). We denote

$$\text{diam}_{d_p}(C) := \begin{cases} 0, & \text{if } C = \emptyset, \\ \sup\{d_p(x, y) : x, y \in C\}, & \text{if } C \neq \emptyset. \end{cases} \tag{2.9}$$

It is obvious that $A \subseteq B$ in (X, p) implies diam$_{d_p}(A) \leq$ diam$_{d_p}(B)$.

Now, we first introduce theconcepts of fitting nest.

Definition 2.8. A sequence $\{A_n\}_{n\in\mathbb{N}}$ of subsets of a *TVS-cone* metric space (X,p) is said to be a *fitting nest* if it satisfies the following properties:

(FN1) $A_{n+1} \subseteq A_n$ for each $n \in \mathbb{N}$,

(FN2) for any $c \in Y$ with $\theta \ll_K c$, there exists $n_c \in \mathbb{N}$ such that $p(x,y) \ll_K c$ for all $x, y \in A_{n_c}$.

Remark 2.9. (a) It is easy to observe that if $Y = \mathbb{R}$, $K = [0,\infty) \subset \mathbb{R}$, and $e = 1$, then p is a metric, and Assumption (FN2) is equivalent to $\lim_{n\to\infty} \operatorname{diam}(A_n) = 0$ if Assumption (FN1) holds, where $\operatorname{diam}(A_n)$ is the diameter of A_n. Indeed, "$\lim_{n\to\infty} \operatorname{diam}(A_n) = 0 \Rightarrow$ (FN2)" is obvious. Conversely, if (FN2) holds, then, by (FN1), for any $\varepsilon > 0$, there exists $n_\varepsilon \in \mathbb{N}$ such that $\operatorname{diam}(A_n) \leq \varepsilon$ for all $n \geq n_\varepsilon$. This show that $\lim_{n\to\infty} \operatorname{diam}(A_n) = 0$.

(b) Let (X,d) be a metric space. Then a sequence $\{A_n\}_{n\in\mathbb{N}}$ in (X,d) is a fitting nest $\Leftrightarrow A_{n+1} \subseteq A_n$ for each $n \in \mathbb{N}$ and $\lim_{n\to\infty} \operatorname{diam}(A_n) = 0$.

The following intersection theorem in *TVS-cone metric* spaces is one of the main results of this paper.

Theorem 2.10. *Let $\{A_n\}$ be a fitting nest in a TVS-cone metric space (X,p). Then the following statements hold.*

(a) $\lim_{n\to\infty} \mathit{diam}_{d_p}(A_n) = 0$.

(b) *If X is TVS-cone complete and A_n is TVS-cone closed in (X,p) for all $n \in \mathbb{N}$, then $\bigcap_{n=1}^{\infty} A_n$ contains precisely one point.*

Proof. (a) Let $\varepsilon > 0$ be given. Then $\theta \ll_K \varepsilon e$. By (FN2) and (iv) of Lemma 2.1, there exists $n_0 \in \mathbb{N}$ such that

$$p(x,y) \ll_K \varepsilon e \Longleftrightarrow d_p(x,y) = \xi_e \circ p(x,y) < \varepsilon, \tag{2.10}$$

for all $x, y \in A_{n_0}$, which implies $\operatorname{diam}_{d_p}(A_{n_0}) \leq \varepsilon$. By (FN1), we obtain

$$\operatorname{diam}_{d_p}(A_n) \leq \operatorname{diam}_{d_p}(A_{n_0}) \leq \varepsilon, \quad \forall n \geq n_0. \tag{2.11}$$

Hence, $\lim_{n\to\infty} \operatorname{diam}_{d_p}(A_n) = 0$.

(b) Given $c \in Y$ with $\theta \ll_K c$. By (FN2), there exists $n_c \in \mathbb{N}$ such that $p(a,b) \ll_K c$ for all $a, b \in A_{n_c}$. For each $n \in \mathbb{N}$, choose $x_n \in A_n$. Then, for m, $n \in \mathbb{N}$ with $m \geq n \geq n_c$; since $x_m \in A_m \subseteq A_n \subseteq A_{n_c}$ from (FN1), we have

$$p(x_m, x_n) \ll_K c. \tag{2.12}$$

Hence, $\{x_n\}$ is a *TVS-cone* Cauchy sequence in (X,p). By the *TVS-cone* completeness of (X,p), there exists $w \in X$, such that $\{x_n\}$ *TVS-cone* converges to w. For any $n \in \mathbb{N}$, from the *TVS-cone* closedness of A_n and $x_m \xrightarrow{\text{cone}} w$ as $m \to \infty$, we have

$$w \in \text{tvsc-cl}(A_n) = A_n. \tag{2.13}$$

So $w \in \bigcap_{n=1}^{\infty} A_n$, and hence $\bigcap_{n=1}^{\infty} A_n \neq \emptyset$. Finally, we claim $\bigcap_{n=1}^{\infty} A_n = \{w\}$. For each $z \in \bigcap_{n=1}^{\infty} A_n$, applying (a), we have

$$d_p(z,w) \leq \operatorname{diam}_{d_p}(A_n) \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{2.14}$$

Hence, $d_p(z,w) = 0$ or, equivalency, $z = w$, which gives the required result (b). □

Theorem 2.11. *Let (X,p) be a TVS-cone complete metric space. Then there exists a nonempty proper subset $\mathcal{M}$ of X, such that $\mathcal{M}$ contains infinite points of X, and $(\mathcal{M}, d_p)$ is a complete metric space, where $d_p := \xi_e \circ p$.*

Proof. Let $\{A_n\}$ be a fitting nest in (X,p). Following the same argument as in the proof of (b), we can obtain a sequence $\{x_n\}$ satisfying

(1) $x_n \in A_n$,

(2) $\{x_n\}$ is a *TVS-cone* Cauchy sequence in (X,p),

(3) $\{x_n\}$*TVS-cone* converges to some point w in X.

Applying Theorem 1.5 with (2) and (3), we know that $\{x_n\}$ is a Cauchy sequence in (X,d_p), and $d_p(x_n,w) \to 0$ or $x_n \xrightarrow{d_p} w$ as $n \to \infty$. Let $\mathcal{M} = \{x_n\}_{n\in\mathbb{N}} \cup \{w\}$. Therefore, $(\mathcal{M}, d_p)$ is a complete metric space. □

The celebrated Cantor intersection theorem [2] in metric spaces can be proved by Theorem 2.10 and Remark 2.9.

Corollary 2.12 (Cantor). *Let (X,d) be a metric space, and let $\{A_n\}_{n\in\mathbb{N}}$ be a sequence of closed subsets of X satisfying $A_{n+1} \subseteq A_n$ for each $n \in \mathbb{N}$ and $\lim_{n\to\infty} \operatorname{diam}(A_n) = 0$. Then $\bigcap_{n=1}^{\infty} A_n$ contains precisely one point.*

The following existence theorems relate with critical point and common fuzzy fixed point for a nonlinear dynamical system in *TVS-cone* complete metric spaces or complete metric spaces.

Theorem 2.13. *Let (X,p) be a TVS-cone complete metric space, let $g : X \to X$ a map, and let $\Gamma : X \to 2^X$ be a multivalued map with nonempty values. Let I be any index set. For each $i \in I$, let F_i be a fuzzy map on X. Suppose that the following conditions are satisfied.*

(H1) *For each $x \in X$, $\Gamma(x)$ is TVS-cone closed in (X,p).*

(H2) *$x, y \in X$ with $y \in \Gamma(x)$ implies $g(y) \in \Gamma(x)$ and $\Gamma(y) \subseteq \Gamma(x)$.*

(H3) *For any $\{x_n\} \subset X$ with $g(x_{n+1}) \in \Gamma(x_n)$ for each $n \in \mathbb{N}$, it satisfies the following:*
For any $c \in Y$ with $\theta \ll_K c$, there exists $n_c \in \mathbb{N}$ such that $p(s,t) \ll_K c$ for all $s,t \in \Gamma(x_{n_c})$.

(H4) *For any $(i,x) \in I \times X$, there exists $y_{(i,x)} \in \Gamma(x)$ such that $F_i(x, y_{(i,x)}) = 1$.*

Then there exists $v \in X$ such that

(a) *$F_i(v,v) = 1$ for all $i \in I$.*

(b) *$\Gamma(v) = \{g(v)\} = \{v\}$.*

Proof. Let $u \in X$ be given. Define a sequence $\{x_n\}$ by $x_1 = u$ and $x_{n+1} \in \Gamma(x_n)$ for $n \in \mathbb{N}$. Hence, $g(x_{n+1}) \in \Gamma(x_n)$ for all $n \in \mathbb{N}$ from (H2). For each $n \in \mathbb{N}$, let $A_n = \Gamma(x_n)$. By (H2) and (H3), we know that $\{A_n\}$ is a fitting nest in (X,p). By (H1), A_n is *TVS-cone* closed in (X,p) for all $n \in \mathbb{N}$. Applying Theorem 2.10, there exists $v \in X$ such that $\bigcap_{n=1}^{\infty} \Gamma(x_n) = \bigcap_{n=1}^{\infty} A_n = \{v\}$. Since $v \in \Gamma(x_n)$ for all $n \in \mathbb{N}$, by (H2), we obtain

$$\emptyset \neq \Gamma(v) \subseteq \bigcap_{n=1}^{\infty} \Gamma(x_n) = \{v\}, \tag{2.15}$$

which implies $\Gamma(v) = \{v\}$. For each $i \in I$, by (H4), $F_i(v,v) = 1$. By (H2) again, we have $g(v) \in \Gamma(v) = \{v\}$. Therefore, $\Gamma(v) = \{g(v)\} = \{v\}$. The proof is completed. □

Remark 2.14. Let $Y = \mathbb{R}$, let $K = [0,\infty)$, and let $e = 1$, then p is a metric, and Assumption (H3) in Theorem 2.13 is equivalent to

(H3)$_{\mathbb{R}}$ for any $\{x_n\} \subset X$ with $g(x_{n+1}) \in \Gamma(x_n)$ for each $n \in \mathbb{N}$, we have $\lim_{n\to\infty} \operatorname{diam}(\Gamma(x_n)) = 0$.

The following critical point theorems are immediate from Theorem 2.13.

Theorem 2.15. *Let (X,d) be a complete metric space, let $g : X \to X$ be a map and let $\Gamma : X \to 2^X$ be a multivalued map with nonempty values. Let I be any index set. For each $i \in I$, let F_i be a fuzzy map on X. Suppose that the following conditions are satisfied.*

(H1)$_{\mathbb{R}}$ *For each $x \in X$, $\Gamma(x)$ is closed.*

(H2) *$x, y \in X$ with $y \in \Gamma(x)$ implies $g(y) \in \Gamma(x)$ and $\Gamma(y) \subseteq \Gamma(x)$.*

(H3)$_{\mathbb{R}}$ *For any $\{x_n\} \subset X$ with $g(x_{n+1}) \in \Gamma(x_n)$ for each $n \in \mathbb{N}$, we have* $\lim_{n\to\infty} \operatorname{diam}(\Gamma(x_n)) = 0$.

(H4) *For any $(i,x) \in I \times X$, there exists $y_{(i,\,x)} \in \Gamma(x)$ such that $F_i(x, y_{(i,x)}) = 1$.*

Then there exists $v \in X$ such that

(a) $F_i(v,v) = 1$ *for all $i \in I$.*

(b) $\Gamma(v) = \{g(v)\} = \{v\}$.

Corollary 2.16. *Let (X,p) be a TVS- cone complete metric space, and let $\Gamma : X \to 2^X$ be a multivalued map with nonempty values. Suppose that the following conditions are satisfied.*

(i) *For each $x \in X$, $\Gamma(x)$ is TVS-cone closed in (X,p).*

(ii) *$x, y \in X$ with $y \in \Gamma(x)$ implies $\Gamma(y) \subseteq \Gamma(x)$.*

(iii) *For any $\{x_n\} \subset X$ with $x_{n+1} \in \Gamma(x_n)$, for each $n \in \mathbb{N}$, it satisfies the following:*

For any $c \in Y$ with $\theta \ll_K c$, there exists $n_c \in \mathbb{N}$ such that $p(s,t) \ll_K c$ for all $s, t \in \Gamma(x_{n_c})$.

Then there exists $v \in X$ such that $\Gamma(v) = \{v\}$.

Proof. Let F be a fuzzy map on X defined by $F(x,y) = 1$ for all x, $y \in X$, and let $g \equiv \mathrm{id}$ be an identity map. Therefore, the which gives the required result (b)conclusion follows from Theorem 2.13. □

Corollary 2.17. *Let (X, d) be a complete metric space, and let $\Gamma : X \to 2^X$ be a multivalued map with nonempty values. Suppose that the following conditions are satisfied.*

(i) *For each $x \in X$, $\Gamma(x)$ is closed.*

(ii) *$x, y \in X$ with $y \in \Gamma(x)$ implies $\Gamma(y) \subseteq \Gamma(x)$.*

(iii) *For any $\{x_n\} \subset X$ with $x_{n+1} \in \Gamma(x_n)$, for each $n \in \mathbb{N}$, we have $\lim_{n\to\infty} \operatorname{diam}(\Gamma(x_n)) = 0$.*

Then there exists $v \in X$ such that $\Gamma(v) = \{v\}$.

Theorem 2.18. *Let (X, d) be a complete metric space, and let $\Gamma : X \to 2^X$ be a multivalued map with nonempty values such that $x, y \in X$ with $y \in \Gamma(x)$ implies $\Gamma(y) \subseteq \Gamma(x)$. Then the following statements holds.*

(1) *If a sequence $\{x_n\}$ in X satisfies $x_{n+1} \in \Gamma(x_n)$ for each $n \in \mathbb{N}$ and $\lim_{n\to\infty} \operatorname{diam}(\Gamma(x_n)) = 0$, then $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$.*

(2) *If (X, d) has the following property $(\mathcal{D})$:*

$$\text{for any } \{x_n\} \subset X \text{ with } x_{n+1} \in \Gamma(x_n) \text{ for each } n \in \mathbb{N}, \text{we have } \lim_{n\to\infty} d(x_n, x_{n+1}) = 0, \qquad (\mathcal{D})$$

then there exists a sequence $\{z_n\}$ in X satisfying $z_{n+1} \in \Gamma(z_n)$ for each $n \in \mathbb{N}$ and $\lim_{n\to\infty} \operatorname{diam}(\Gamma(z_n)) = 0$.

Proof. (1) Let $\{x_n\}$ in X with $x_{n+1} \in \Gamma(x_n)$ for each $n \in \mathbb{N}$, and let $\lim_{n\to\infty} \operatorname{diam}(\Gamma(x_n)) = 0$. For any $n \in \mathbb{N}$, by our hypothesis, $\Gamma(x_{n+1}) \subseteq \Gamma(x_n)$, and; hence,

$$x_{m+1} \in \Gamma(x_m) \subseteq \Gamma(x_n), \quad \forall m \in \mathbb{N} \text{ with } m \geq n. \tag{2.16}$$

So $d(x_{n+1}, x_{n+2}) \leq \operatorname{diam}(\Gamma(x_n))$ for each $n \in \mathbb{N}$. Since $\lim_{n\to\infty} \operatorname{diam}(\Gamma(x_n)) = 0$, we have

$$\lim_{n\to\infty} d(x_{n+1}, x_{n+2}) = \lim_{n\to\infty} \operatorname{diam}(\Gamma(x_n)) = 0. \tag{2.17}$$

That is, $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$.

(2) Suppose that (X, d) has the property $(\mathcal{D})$. Define a function $\tau : X \to \mathbb{R}$ by

$$\tau(x) = \sup_{y\in\Gamma(x)} d(x, y). \tag{2.18}$$

We first note that $\tau(w) < \infty$ for some $w \in X$. Indeed, on the contrary, assume that $\tau(x) = \infty$ for all $x \in X$. Let $\widehat{u} \in X$ be given. Set $u_1 = \widehat{u}$. Since $\tau(u_1) = \infty$, $\tau(u_1) = \sup_{y\in\Gamma(u_1)} d(u_1, y) > 1$, and then there exists $u_2 \in \Gamma(u_1)$ such that $d(u_1, u_2) > 1$. Since $\tau(u_2) > 2$, there exists $u_3 \in \Gamma(u_2)$ such that $d(u_2, u_3) > 2$. Continuing in the process, we can obtain a sequence $\{u_n\} \subset X$, such that, for each $n \in \mathbb{N}$,

$(\mathcal{K}_1)$ $u_{n+1} \in \Gamma(u_n)$,

$(\mathcal{K}_2)$ $d(u_n, u_{n+1}) > n$.

By $(\mathcal{P})$, we have $\lim_{n\to\infty} d(u_n, u_{n+1}) = 0$. On the other hand, by $(\mathcal{K}_2)$, we also obtain $\lim_{n\to\infty} d(u_n, u_{n+1}) = \infty$, a contradiction. Therefore, there exists $w \in X$ such that $\tau(w) < \infty$. Let $z_1 = w$. Since

$$\infty > \tau(z_1) = \sup_{y\in\Gamma(z_1)} d(z_1, y) \geq \frac{1}{2}\operatorname{diam}(\Gamma(z_1)), \tag{2.19}$$

we have $0 \leq \operatorname{diam}(\Gamma(z_1)) < \infty$, and there exists $z_2 \in \Gamma(z_1)$ such that

$$d(z_1, z_2) > \frac{1}{2}\operatorname{diam}(\Gamma(z_1)) - \frac{1}{2}. \tag{2.20}$$

Since $\Gamma(z_2) \subseteq \Gamma(z_1)$, we have $\tau(z_2) \leq \tau(z_1) < \infty$ and $0 \leq \operatorname{diam}(\Gamma(z_2)) < \infty$. So there exists $z_3 \in \Gamma(z_2)$ such that

$$d(z_2, z_3) > \frac{1}{2}\operatorname{diam}(\Gamma(z_2)) - \frac{1}{2^2}. \tag{2.21}$$

Continuing in this way, we can construct a sequence $\{z_n\}$ in X satisfying, for each $n \in \mathbb{N}$,

$(\mathcal{K}_3)$ $z_{n+1} \in \Gamma(z_n)$,

$(\mathcal{K}_4)$ $d(z_n, z_{n+1}) > (1/2)\operatorname{diam}(\Gamma(z_n)) - (1/2^n)$.

From $(\mathcal{P})$, we have $\lim_{n\to\infty} d(z_n, z_{n+1}) = 0$. By $(\mathcal{K}_4)$, it follows that $\lim_{n\to\infty}\operatorname{diam}(\Gamma(z_n)) = 0$. □

Remark 2.19. In general, under the same assumptions of Theorem 2.18, $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$ does not always imply $\lim_{n\to\infty}\operatorname{diam}(\Gamma(x_n)) = 0$. For example, let $X = [0, 5]$ with the metric $d(x, y) = |x - y|$. Then (X, d) is a complete metric space. For each $x \in X$, define $\Gamma : X \to 2^X$ by $\Gamma(x) = [x, 5]$. Thus, $x, y \in X$ with $y \in \Gamma(x)$ implies $\Gamma(y) \subseteq \Gamma(x)$. Choose $\{x_n\}$ in X with $x_n = 2$; for all $n \in \mathbb{N}$, we have $\lim_{n\to\infty}\operatorname{diam}(\Gamma(x_n)) = 3 \neq 0$ while $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$.

The following result is also a generalized Dancš-Hegedüs-Medvegyev's principle with common fuzzy fixed point. Notice that we do not assume $x \in \Gamma(x)$ for all $x \in X$.

Theorem 2.20. *Let (X, d) be a complete metric space. Let $\Gamma : X \to 2^X$ be a multivalued map with nonempty values. Let I be any index set. For each $i \in I$, let F_i be a fuzzy map on X. Suppose that the following conditions are satisfied.*

(D1) *For each $x \in X$, $\Gamma(x)$ is closed.*

(D2) *$x, y \in X$ with $y \in \Gamma(x)$ implies $\Gamma(y) \subseteq \Gamma(x)$.*

(D3) *(**Property** $(\mathcal{P})$) for any $\{x_n\} \subset X$ with $x_{n+1} \in \Gamma(x_n)$ for each $n \in \mathbb{N}$, we have $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$.*

(D4) *For any $(i, x) \in I \times X$, thereexists $y_{(i,x)} \in \Gamma(x)$ such that $F_i(x, y_{(i,x)}) = 1$.*

Then there exists $v \in X$ such that

(a) $F_i(v,v) = 1$ *for all* $i \in I$,

(b) $\Gamma(v) = \{v\}$.

Proof. By conclusion (2) of Theorem 2.18, there exists a sequence $\{z_n\}$ in X satisfying $z_{n+1} \in \Gamma(z_n)$ for each $n \in \mathbb{N}$ and $\lim_{n\to\infty} \operatorname{diam}(\Gamma(z_n)) = 0$. For each $n \in \mathbb{N}$, let $A_n = \Gamma(z_n)$. By (D2) and $\lim_{n\to\infty} \operatorname{diam}(\Gamma(z_n)) = 0$, we see that $\{A_n\}$ is a fitting nest in (X,d). Applying Theorem 2.10, there exists $v \in X$ such that $\bigcap_{n=1}^{\infty} \Gamma(z_n) = \bigcap_{n=1}^{\infty} A_n = \{v\}$. Since $v \in \Gamma(z_n)$ for all $n \in \mathbb{N}$, by (D2) again, we obtain

$$\emptyset \neq \Gamma(v) \subseteq \bigcap_{n=1}^{\infty} \Gamma(z_n) = \{v\}, \tag{2.22}$$

which implies $\Gamma(v) = \{v\}$. For each $i \in I$, by (D4), $F_i(v,v) = 1$. The proof is completed. □

The following existence theorem relate with common fixed point for multivalued maps and critical point for a nonlinear dynamical system in *TVS-cone* complete metric spaces.

Theorem 2.21. *Let (X,p), g, and Γ be the same as in Theorem 2.13. Assume that conditions (H1), (H2), and (H3) in Theorem 2.13 hold. Let I be any index set. For each $i \in I$, let $T_i : X \to 2^X$ be a multivalued map with nonempty values. Suppose that, for each $(i,x) \in I \times X$, there exists $y_{(i,x)} \in T_i(x) \cap \Gamma(x)$. Then there exists $v \in X$ such that*

(a) *v is a common fixed point for the family $\{T_i\}_{i\in I}$ (i.e., $v \in T_i(v)$ for all $i \in I$),*

(b) $\Gamma(v) = \{g(v)\} = \{v\}$.

Proof. For each $i \in I$, define a fuzzy map F_i on X by

$$F_i(x,y) = \chi_{T_i(x)}(y), \tag{2.23}$$

where χ_A is the characteristic function for an arbitrary set $A \subset X$. Note that $y \in T_i(x) \Leftrightarrow F_i(x,y) = 1$ for $i \in I$. Then, for any $(i,x) \in I \times X$, there exists $y_x \in \Gamma(x)$ such that $F_i(x,y_x) = 1$. So (H4) in Theorem 2.13 holds. Therefore, the result follows from Theorem 2.13. □

Remark 2.22. (a) Theorems 2.20 and 2.21 all generalize and improve the primitive Dancš-Hegedüs-Medvegyev's principle.

(b) Corollary 2.17 is a special case of Theorem 2.20 or Theorem 2.21.

The following result is a special case of [32, Theorem 4.1], but it can also be proved by applying Theorem 2.20 (please follow a similar argument as in the proof of [32, Theorem 4.1]).

Theorem 2.23. *Let (X,d) be a complete metric space, let $f : X \to (-\infty,\infty]$ be a proper l.s.c. and bounded from below function, and let $\varphi : (-\infty,\infty] \to (0,\infty)$ be a nondecreasing function. Let I be any index set. For each $i \in I$, let F_i be a fuzzy map on X. Suppose that, for each $(i,x) \in I \times X$, there* *exists* $y_{(i,x)} \in X$ *such that* $F_i(x, y_{(i,x)}) = 1$ *and* $d(x, y_{(i,x)}) \leq \varphi(f(x))(f(x) - f(y_{(i,x)}))$. *Then, for each* $u \in X$ *with* $f(u) < \infty$, *there exists* $v \in X$ *such that*

(a) $d(u, v) \leq \varphi(f(u))(f(u) - f(v))$,

(b) $d(v, x) > \varphi(f(v))(f(v) - f(x))$ *for all* $x \in X$ *with* $x \neq v$,

(c) $F_i(v, v) = 1$ *for all* $i \in I$.

Moreover, if further assume that

(H) *for any* $x \in X$ *with* $f(x) > \inf_{z\in X} f(z)$, *there exists* $y \in X$ *with* $y \neq x$ *such that* $d(x, y) \leq \varphi(f(x))(f(x) - f(y))$, *then* $f(v) = \inf_{z\in X} f(z)$.

By using Theorem 2.23, one can immediately obtain the following existence theorem related to generalized Ekeland's variational principle, generalized Takahashi's nonconvex minimization theorem, and generalized Caristi's common fixed point theorem for multivalued maps.

Theorem 2.24. *Let* (X, d), f, *and* φ *be be the same as in Theorem 2.23. Let* I *be any index set. For each* $i \in I$, *let* $T_i : X \to 2^X$ *be a multivalued map with nonempty values such that, for each* $(i, x) \in I \times X$, *there exists* $y_{(i,x)} \in T_i(x)$ *such that* $d(x, y_{(i,x)}) \leq \varphi(f(x))(f(x) - f(y_{(i,x)}))$. *Then, for each* $u \in X$ *with* $f(u) < \infty$, *there exists* $v \in X$ *such that*

(a) $d(u, v) \leq \varphi(f(u))(f(u) - f(v))$.

(b) $d(v, x) > \varphi(f(v))(f(v) - f(x))$ *for all* $x \in X$ *with* $x \neq v$.

(c) v *is a common fixed point for the family* $\{T_i\}_{i\in I}$.

Moreover, if further assume that

(H) *for any* $x \in X$ *with* $f(x) > \inf_{z\in X} f(z)$, *there exists* $y \in X$ *with* $y \neq x$ *such that* $d(x, y) \leq \varphi(f(x))(f(x) - f(y))$, *then* $f(v) = \inf_{z\in X} f(z)$.

3. Fixed Point Theorems in Cone Metric Spaces

In this section, motivated by the recent results of Abbas and Rhoades [21], we will present some generalizations of those in *TVS-cone* complete metric spaces.

Theorem 3.1. *Let* (X, p) *be a TVS-cone complete metric space. Suppose that* $T,\ S : X \to X$ *are two self-maps of* X *satisfying*

$$p(Tx, Sy) \precsim_K \alpha p(x, y) + \beta[p(x, Tx) + p(y, Sy)] + \gamma[p(x, Sy) + p(y, Tx)], \tag{3.1}$$

for all x, $y \in X$, *where* α, β, $\gamma \geq 0$ *and* $\alpha + 2\beta + 2\gamma < 1$. *Then the following statements hold:*

(a) *There exists a nonempty proper subset* W *of* X, *such that* W *contains infinite points of* X *and* (W, d_p) *is a complete metric space, where* $d_p := \xi_e \circ p$.

(b) T *and* S *have a unique common fixed point in* X *(in fact, the unique common fixed point of* T *and* S *belongs to* W*). Moreover, for each* $x_0 \in X$, *the mixed iterative sequence* $\{x_n\}_{n\in\mathbb{N}\cup\{0\}}$, *defined by* $x_{2n+1} = Tx_{2n}$ *and* $x_{2n+2} = Sx_{2n+1}$ *for* $n \in \mathbb{N} \cup \{0\}$, *TVS-cone converges to the common fixed point.*

(c) *Any fixed point of* T *is a fixed point of* S, *and conversely.*

Proof. Since Y is a locally convex Hausdorff's t.v.s. with its zero vector θ, let τ denote the topology of Y, and let $\mathcal{U}_\tau$ be the base at θ consisting of all absolutely convex neighborhood of θ. Let

$$\mathcal{L} = \{\ell : \ell \text{ be a Minkowski functional of } U \text{ for } U \in \mathcal{U}_\tau\}. \tag{3.2}$$

Then $\mathcal{L}$ is a family of seminorms on Y. For each $\ell \in \mathcal{L}$, let

$$V(\ell) = \{y \in Y : \ell(y) < 1\}, \tag{3.3}$$

and let

$$\mathcal{U}_{\mathcal{L}} = \{U : U = r_1 V(\ell_1) \cap r_2 V(\ell_2) \cap \cdots \cap r_n V(\ell_n),\ r_k > 0,\ \ell_k \in \mathcal{L},\ 1 \le k \le n,\ n \in \mathbb{N}\}. \tag{3.4}$$

Then $\mathcal{U}_{\mathcal{L}}$ is a base at θ, and the topology $\Gamma_{\mathcal{L}}$ generated by $\mathcal{U}_{\mathcal{L}}$ is the weakest topology for Y such that all seminorms in $\mathcal{L}$ are continuous and $\tau = \Gamma_{\mathcal{L}}$. Moreover, given any neighborhood $\mathcal{O}_\theta$ of θ, there exists $U \in \mathcal{U}_{\mathcal{L}}$ such that $\theta \in U \subset \mathcal{O}_\theta$ (see, e.g., [34, Theorem 12.4 in II.12, Page 113] or the proofs of [28, Theorem 3.1] and [29, Theorem 2.1]).

Let $x_0 \in X$ be given. First, from our hypothesis, we have

$$0 \le \alpha + \beta + \gamma < 1 - \beta - \gamma. \tag{3.5}$$

If x_0 is a fixed point of T, then, by using (3.1),

$$\begin{aligned} p(x_0, Sx_0) &= p(Tx_0, Sx_0) \\ &\precsim_K \alpha p(x_0, x_0) + \beta\left[p(x_0, Tx_0) + p(x_0, Sx_0)\right] + \gamma\left[p(x_0, Sx_0) + p(x_0, Tx_0)\right], \end{aligned} \tag{3.6}$$

implies that

$$p(x_0, Sx_0) \precsim_K (\beta + \gamma) p(x_0, Sx_0), \tag{3.7}$$

or

$$(1 - \beta - \gamma) p(x_0, Sx_0) \in -K. \tag{3.8}$$

Since $1 - \beta - \gamma > 0$ and K is pointed, we have $p(x_0, Sx_0) \in K \cap (-K) = \{\theta\}$. So $p(x_0, Sx_0) = \theta$, and; hence, $Sx_0 = x_0 = Tx_0$; that is, x_0 is a common fixed point of T and S. Otherwise, if $Tx_0 \neq x_0$, we will define the mixed iterative sequence $\{x_n\}$ by $x_{2n+1} = Tx_{2n}$ and $x_{2n+2} = Sx_{2n+1}$ for $n \in \mathbb{N} \cup \{0\}$. Then $p(x_1, x_0) \neq \theta$. We claim that $\{x_n\}$ is a *TVS-cone* Cauchy's sequence in (X, p). Let $\lambda = (\alpha + \beta + \gamma)/(1 - \beta - \gamma)$.

By (3.5), we know $\lambda \in [0,1)$. For each $n \in \mathbb{N} \cup \{0\}$, we have

$$\begin{aligned}
p(x_{2n+1}, x_{2n+2}) &= p(Tx_{2n}, Sx_{2n+1}) \\
&\precsim_K \alpha p(x_{2n}, x_{2n+1}) + \beta\left[p(x_{2n}, Tx_{2n}) + p(x_{2n+1}, Sx_{2n+1})\right] \\
&\quad + \gamma[d(x_{2n}, Sx_{2n+1}) + d(x_{2n+1}, Tx_{2n})] \\
&\precsim_K (\alpha+\beta+\gamma)p(x_{2n}, x_{2n+1}) + (\beta+\gamma)p(x_{2n+1}, x_{2n+2}),
\end{aligned} \tag{3.9}$$

which implies that

$$p(x_{2n+1}, x_{2n+2}) \precsim_K \lambda p(x_{2n}, x_{2n+1}). \tag{3.10}$$

Similarly, we also obtain

$$p(x_{2n+2}, x_{2n+3}) \precsim_K \lambda p(x_{2n+1}, x_{2n+2}). \tag{3.11}$$

Hence, for each $n \in \mathbb{N}$,

$$p(x_{n+1}, x_n) \precsim_K \lambda p(x_n, x_{n-1}) \precsim_K \cdots \precsim_K \lambda^n p(x_1, x_0). \tag{3.12}$$

Therefore, for $m, n \in \mathbb{N}$ with $m > n$,

$$p(x_m, x_n) \precsim_K \sum_{j=n}^{m-1} p(x_{j+1}, x_j) \precsim_K \frac{\lambda^n}{1-\lambda} p(x_1, x_0). \tag{3.13}$$

Given $c \in Y$ with $\theta \ll_K c$ (i.e., $c \in \operatorname{int} K = \operatorname{int}(\operatorname{int} K)$), there exists a neighborhood N_θ of θ such that $c + N_\theta \subseteq \operatorname{int} K$. Therefore, there exists $U_c \in \mathcal{U}_{\mathcal{L}}$ with $U_c \subseteq N_\theta$ such that $c + U_c \subseteq c + N_\theta \subseteq \operatorname{int} K$, where

$$U_c = r_1 V(\ell_1) \cap r_2 V(\ell_2) \cap \cdots \cap r_s V(\ell_s), \tag{3.14}$$

for some $r_i > 0$, $\ell_i \in \mathcal{L}$ and $1 \le i \le s$. Let $\delta = \min\{r_i : 1 \le i \le s\} > 0$, and let $\rho = \max\{\ell_i(p(x_1, x_0)) : 1 \le i \le s\}$. If $\rho = 0$, since each ℓ_i is a seminorm, we have $\ell_i(p(x_1, x_0)) = 0$ and

$$\ell_i\left(-\frac{\lambda^n}{1-\lambda} p(x_1, x_0)\right) = \frac{\lambda^n}{1-\lambda} \ell_i(p(x_1, x_0)) = 0 < r_i, \tag{3.15}$$

for all $1 \le i \le s$ and all $n \in \mathbb{N}$. If $\rho > 0$, since $\lambda \in [0,1)$, $\lim_{n\to\infty}(\lambda^n/(1-\lambda)) = 0$, and hence there exists $n_0 \in \mathbb{N}$ such that $\lambda^n/(1-\lambda) < \delta/\rho$ for all $n \ge n_0$. So, for each $i \in \{1,2,\ldots,s\}$ and any $n \ge n_0$, we obtain

$$\begin{aligned} \ell_i\left(-\frac{\lambda^n}{1-\lambda}p(x_1,x_0)\right) &= \frac{\lambda^n}{1-\lambda}\ell_i(p(x_1,x_0)) \\ &< \frac{\delta}{\rho}\ell_i(p(x_1,x_0)) \\ &\le \delta \\ &\le r_i. \end{aligned} \tag{3.16}$$

Therefore, for any $n \ge n_0$, $(-\lambda^n/(1-\lambda))p(x_1,x_0) \in r_i V(\ell_i)$ for all $1 \le i \le s$, and hence $(-\lambda^n/(1-\lambda))p(x_1,x_0) \in U_c$. So we obtain

$$c - \frac{\lambda^n}{1-\lambda}p(x_1,x_0) \in c + U_c \subseteq \operatorname{int} K, \tag{3.17}$$

or

$$\frac{\lambda^n}{1-\lambda}p(x_1,x_0) \ll_K c, \tag{3.18}$$

for all $n \ge n_0$. For $m, n \in \mathbb{N}$ with $m > n \ge n_0$, by (3.13), (3.18), and Lemma 2.5, it follows that

$$p(x_m,x_n) \ll_K c \quad \text{for } m,n \in \mathbb{N} \text{ with } m > n \ge n_0. \tag{3.19}$$

Hence, $\{x_n\}$ is a *TVS-cone* Cauchy sequence in (X,p). By the *TVS-cone* completeness of (X,p), there exists $v \in X$, such that $\{x_n\}$ *TVS-cone* converges to v. On the other hand, applying Theorem 1.5, $\{x_n\}$ is a Cauchy sequence in (X,d_p), and $d_p(x_n,v) \to 0$ or $x_n \overset{d_p}{\to} v$ as $n \to \infty$. Let $W = \{x_n\}_{n\in\mathbb{N}} \cup \{v\}$. Then (W,d_p) is a complete metric space, and the conclusion (a) holds.

To see (b), it suffices to show that v is the unique common fixed point of T and S. By (v) and (vi) of Lemma 2.1, the assumption (3.1) implies

$$d_p(Tx,Sy) \le \alpha\, d_p(x,y) + \beta\left[d_p(x,Tx) + d_p(y,Sy)\right] + \gamma\left[d_p(x,Sy) + d_p(y,Tx)\right], \quad \forall x,y \in X. \tag{3.20}$$

For any $n \in \mathbb{N} \cup \{0\}$, by (3.20),

$$\begin{aligned} d_p(v,Sv) &\le d_p(v,x_{2n+1}) + d_p(Tx_{2n},Sv) \\ &\le d_p(v,x_{2n+1}) + \alpha d_p(x_{2n},v) + \beta\left[d_p(x_{2n},x_{2n+1}) + d_p(v,Sv)\right] \\ &\quad + \gamma\left[d_p(x_{2n},Sv) + d_p(v,x_{2n+1})\right] \\ &\le d_p(v,x_{2n+1}) + \alpha\, d_p(x_{2n},v) + \beta\left[d_p(x_{2n},x_{2n+1}) + d_p(v,Sv)\right] \\ &\quad + \gamma\left[d_p(x_{2n},v) + d_p(v,Sv) + d_p(v,x_{2n+1})\right], \end{aligned} \tag{3.21}$$

which implies that

$$d_p(v,Sv) \le \frac{1}{1-\beta-\gamma}\{d_p(v,x_{2n+1}) + \alpha d_p(x_{2n},v) + \beta[d_p(x_{2n},x_{2n+1})] \\ +\gamma[d_p(x_{2n},v) + d_p(v,x_{2n+1})]\}. \tag{3.22}$$

Since d_p is a metric and $x_n \overset{d_p}{\to} v$ as $n \to \infty$, the right-hand side of (3.22) approaches zero as $n \to \infty$. Hence, $d_p(v,Sv) = 0$ or $Sv = v$. Also, since

$$\begin{aligned} d_p(Tv,v) &= d_p(Tv,Sv) \\ &\le \alpha\, d_p(v,v) + \beta[d_p(v,Tv) + d_p(v,Sv)] + \gamma[d_p(v,Sv) + d_p(v,Tv)] \\ &= (\beta+\gamma)d_p(Tv,v), \end{aligned} \tag{3.23}$$

this implies that

$$(1-\beta-\gamma)d_p(Tv,v) \le 0. \tag{3.24}$$

Since $1-\beta-\gamma > 0$, we have $d_p(Tv,v) = 0$ or $Tv = v$. Therefore, v is a common fixed point of T and S. Suppose that there exists $w \in X$ such that $Tw = Sw = w$. Since

$$\begin{aligned} d_p(v,w) &= d_p(Tv,Sw) \\ &\le \alpha\, d_p(v,w) + \beta[d_p(v,Tv) + d_p(w,Sw)] + \gamma[d_p(v,Sw) + d_p(w,Tv)] \\ &= (\alpha+2\gamma)d_p(v,w), \end{aligned} \tag{3.25}$$

and $1-\alpha-2\gamma > 2\beta \ge 0$, it follows that $d_p(v,w) = 0$, and hence $w = v$. So the uniqueness of common fixed point in X of T, and S is proved. Following a similar argument as above, one can verify conclusion (c). The proof is completed. □

The following result is immediate from Theorem 3.1.

Corollary 3.2. *Let (X,d) be a complete metric space. Suppose that T, $S : X \to X$ are two self-maps of X satisfying*

$$d(Tx,Sy) \le \alpha d(x,y) + \beta[d(x,Tx) + d(y,Sy)] + \gamma[d(x,Sy) + d(y,Tx)], \tag{3.26}$$

for all x, $y \in X$, where α, β, $\gamma \ge 0$ and $\alpha + 2\beta + 2\gamma < 1$. Then the following statements hold.

(a) *T and S have a unique common fixed point in X. Moreover, for each $x_0 \in X$, the mixed iterative sequence $\{x_n\}_{n\in\mathbb{N}\cup\{0\}}$, defined by $x_{2n+1} = Tx_{2n}$ and $x_{2n+2} = Sx_{2n+1}$ for $n \in \mathbb{N}\cup\{0\}$, converges to the common fixed point;*

(b) *any fixed point of T is a fixed point of S, and conversely.*

Remark 3.3. [21, Theorem 2.1] is a special case of Theorem 3.1.

The following result improves and extends [23, Theorem 2.3], and it is immediate from Theorem 3.1.

Theorem 3.4. *Let (X,p) be a TVS-cone complete metric space, and let the map $T : X \to X$ be a cone-contraction; that is, T satisfies the contractive condition*

$$p(Tx,Ty) \precsim_K \alpha p(x,y) \tag{3.27}$$

for all $x,\ y \in X$, where $\alpha \in [0,1)$ is a constant. Then the following statements hold:

(a) *there exists a nonempty proper subset M of X, such that M contains infinite points of X, and (M,d_p) is a complete metric space, where $d_p := \xi_e \circ p$;*

(b) *T has a unique fixed point in X (in fact, the unique fixed point of T belongs to M). Moreover, for each $x \in X$, the iterative sequence $\{T^n x\}_{n\in\mathbb{N}}$ TVS -cone converges to the fixed point.*

Proof. Set $S = T$ and $\beta = \gamma = 0$ in Theorem 3.1. □

Remark 3.5. (a) It is obvious that the classical Banach's contraction principle is a special case of Theorem 3.4;

(b) Theorem 3.4 generalizes and improves [19, Theorem 1] and [24, Theorem 2.3].

(c) In fact, following a very similar argument as in the proof of Theorem 3.1 under the assumptions of Theorem 3.4, we can obtain an important fact that there exists a nonempty subset M of X, such that (M,d_p) is a complete metric space and $TM \subseteq M$. Since

$$p(Tx,Ty) \precsim_K \alpha p(x,y) \Longrightarrow d_p(Tx,Ty) \le \alpha\, d_p(x,y), \quad \forall x,y \in X, \tag{3.28}$$

one can apply the Banach's contraction principle to prove that T has a unique fixed point in X. So the classical Banach contraction principle and [23, Theorem 2.3] are equivalent if we are only asked to find the fixed point of T.

(d) Another proof of Theorem 2.11 is given hereunder. Let $\varphi : X \to X$ be any cone-contraction. Take $v_0 \in X$ and let $v_n = \varphi v_{n-1} = \varphi^n v_0$ for $n \in \mathbb{N}$. Following a similar argument as in the proof of Theorem 3.1, there exists $\widehat{v} \in X$, such that $v_n \xrightarrow{d_p} \widehat{v}$ as $n \to \infty$. Let $\mathcal{M} = \{v_n\}_{n\in\mathbb{N}} \cup \{\widehat{v}\}$. Then $(\mathcal{M},d_p)$ is a complete metric space.

Theorem 3.6. *Let (X,p) be a TVS-cone complete metric space, and let the map $T : X \to X$ satisfy*

$$p(Tx,Ty) \precsim_K \beta[p(x,Tx) + p(y,Ty)] \tag{3.29}$$

for all $x,\ y \in X$, where $\beta \in [0,1/2)$ is a constant. Then the following statements hold:

(a) *there exists a nonempty proper subset M of X, such that M contains infinite points of X, and (M,d_p) is a complete metric space, where $d_p := \xi_e \circ p$;*

(b) *T has a unique fixed point in X (in fact, the unique fixed point of T belongs to M). Moreover, for each $x \in X$, the iterative sequence $\{T^n x\}_{n\in\mathbb{N}}$TVS-cone converges to the fixed point.*

Proof. Set $S = T$ and $\alpha = \gamma = 0$ in Theorem 3.1. □

Remark 3.7. Theorem 3.6 generalizes the fixed point theorems of Kannan's type [21, 24, 35].

Theorem 3.8. *Let (X, p) be a TVS-cone complete metric space, and let the map $T : X \to X$ satisfy*

$$p(Tx, Ty) \precsim_K \gamma[p(x, Ty) + p(y, Tx)], \tag{3.30}$$

for all $x, y \in X$, where $\gamma \in [0, 1/2)$ is a constant. Then the following statements hold:

(a) *there exists a nonempty proper subset M of X, such that M contains infinite points of X and (M, d_p) is a complete metric space, where $d_p := \xi_e \circ p$;*

(b) *T has a unique fixed point in X (in fact, the unique fixed point of T belongs to M). Moreover, for each $x \in X$, the iterative sequence $\{T^n x\}_{n\in\mathbb{N}}$ TVS -cone converges to the fixed point.*

Proof. Set $S = T$ and $\alpha = \beta = 0$ in Theorem 3.1. □

Remark 3.9. Theorem 3.8 improves the fixed point theorems of Chatterjea's type [21, 24, 36].

4. Some Equivalences

In this final section, we introduce the following new concepts.

Definition 4.1. Let X be a nonempty set with a *TVS-cone metric* p, $x \in X$, and let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in X.

(i) $\{x_n\}$ *w-cone* converges to x if for any $\varepsilon > 0$, there is a natural number $\mathbb{N}_0$ such that

$$p(x_n, x) \ll_K \varepsilon e, \quad \forall n \geq \mathbb{N}_0. \tag{4.1}$$

We denote this by *w-cone* $\lim_{n\to\infty} x_n = x$ or $x_n \overset{w\text{-cone}}{\longrightarrow} x$ as $n \to \infty$ and call x the *w-cone* limit of $\{x_n\}$;

(ii) $\{x_n\}$ is a *w-cone* Cauchy's sequence; if for any $\varepsilon > 0$, there is a natural number $\mathbb{N}_0$ such that $p(x_n, x_m) \ll_K \varepsilon e$ for all $n, m \geq \mathbb{N}_0$;

(iii) (X, p) is *w-cone* complete if every *w-cone* Cauchy sequence in X is w-cone convergent.

We establish the following crucial and useful properties.

Theorem 4.2. *Let X be a nonempty set with a TVS-cone metric p, $x \in X$, and let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in X. Then the following statements hold.*

(a) *$\{x_n\}$ w-cone converges to x if and only if $d_p(x_n, x) \to 0$ as $n \to \infty$.*

(b) *$\{x_n\}$ is a w-cone Cauchy sequence in (X, p) if and only if $\{x_n\}$ is a Cauchy sequence (in usual sense) in (X, d_p).*

(c) *(X, p) is w-cone complete if and only if (X, d_p) is a complete metric space.*

Proof. Let $\varepsilon > 0$ be given. If $\{x_n\}$ w-cone converges to x, then, from (iv) of Lemma 2.1, there exists $n_0 \in \mathbb{N}$ such that

$$p(x_n, x) \ll_K \varepsilon e \Longleftrightarrow d_p(x_n, x) = \xi_e \circ p(x_n, x) < \varepsilon \tag{4.2}$$

for all $n \geq n_0$. Hence, $d_p(x_n, x) \to 0$ as $n \to \infty$. Conversely, if $d_p(x_n, x) \to 0$ as $n \to \infty$, then, by (4.2), we show that $\{x_n\}$ w-cone converges to x. Therefore, (a) holds.

To see (b), let $\{x_n\}$ be a w-cone Cauchy sequence in (X, p). Then there exists $n_1 \in \mathbb{N}$ such that

$$p(x_n, x_m) \ll_K \varepsilon e \Longleftrightarrow d_p(x_n, x_m) < \varepsilon, \tag{4.3}$$

for all $n, m \geq n_1$. So $\{x_n\}$ is a Cauchy sequence in (X, d_p). The converse holds obviously.

Conclusion (c) is immediate from conclusions (a) and (b). □

Theorem 4.3. *Let (X, p) be a w-cone complete metric space. Suppose that $T, S : X \to X$ are two self-maps of X satisfying*

$$p(Tx, Sy) \precsim_K \alpha p(x, y) + \beta[p(x, Tx) + p(y, Sy)] + \gamma[p(x, Sy) + p(y, Tx)] \tag{4.4}$$

for all $x, y \in X$, where $\alpha, \beta, \gamma \geq 0$ and $\alpha + 2\beta + 2\gamma < 1$. Then the following statements hold:

(a) *T and S have a unique common fixed point in X. Moreover, for each $x_0 \in X$, the mixed iterative sequence $\{x_n\}_{n \in \mathbb{N} \cup \{0\}}$, defined by $x_{2n+1} = Tx_{2n}$ and $x_{2n+2} = Sx_{2n+1}$ for $n \in \mathbb{N} \cup \{0\}$, w-cone converges to the common fixed point;*

(b) *any fixed point of T is a fixed point of S, and conversely.*

Proof. Applying (c) of Theorem 4.2, (X, d_p) is a complete metric space. By Lemma 2.1, the assumption (4.4) implies

$$d_p(Tx, Sy) \leq \alpha\, d_p(x, y) + \beta[d_p(x, Tx) + d_p(y, Sy)] + \gamma[d_p(x, Sy) + d_p(y, Tx)], \quad \forall x, y \in X. \tag{4.5}$$

Therefore, the conclusion follows from Corollary 3.2 and Theorem 4.2. □

It is obvious that Theorem 4.3 implies Corollary 3.2, so we obtain the following equivalence between Theorem 4.3 and Corollary 3.2.

Theorem 4.4. *Theorem 4.3 and Corollary 3.2 are equivalent.*

Remark 4.5. Using the same argument as above, we can prove the equivalences between scalar version and vectorial version of the Banach contraction principle, Kannan's fixed point theorem, Chatterjea's fixed point theorem, and others (e.g., [20, 21, 24, 25]) in w-cone complete metric spaces.

The following result tell us the relationship between the *TVS-cone* completeness and the *w-cone* completeness.

Theorem 4.6. *If (X, p) is TVS-cone complete, then there exists a nonempty proper subset M of X, such that M contains infinite points of X, and (M, p) is a w-cone complete metric space.*

Proof. Applying Theorem 2.11, there exists a nonempty proper subset M of X, such that M contains infinite points of X, and (M, d_p) is a complete metric space. By using (c) of Theorem 4.2, (M, p) is a *w-cone* complete metric space. □

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