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Research Article

Global Dynamical Systems Involving Generalized f-Projection Operators and Set-Valued Perturbation in Banach Spaces

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A new class of generalized dynamical systems involving generalized f-projection operators is introduced and studied in Banach spaces. By using the fixed-point theorem due to Nadler, the equilibrium points set of this class of generalized global dynamical systems is proved to be nonempty and closed under some suitable conditions. Moreover, the solutions set of the systems with set-valued perturbation is showed to be continuous with respect to the initial value.

1. Introduction

It is well known that dynamics system has long time been an interest of many researchers. This is largely due to its extremely wide applications in a huge variety of scientific fields, for instance, mechanics, optimization and control, economics, transportation, equilibrium, and so on. For details, we refer readers to references [1–10] and the references therein.

In 1994, Friesz et al. [3] introduced a class of dynamics named global projective dynamics based on projection operators. Recently, Xia and Wang [7] analyzed the global asymptotic stability of the dynamical system proposed by Friesz as follows:

$$\frac{dx}{dt} = P_K\left(x - \rho N(x)\right) - x, \tag{1.1}$$

where $N : \mathbb{R}^n \to \mathbb{R}^n$ is a single-valued function, $\rho > 0$ is a constant, $P_K x$ denotes the projection of the point x on K; here $K \subset \mathbb{R}^n$ is a nonempty, closed, and convex subset.

Later, in 2006, Zou et al. [9] studied a class of global set-valued projected dynamical systems as follows:

$$\frac{dx(t)}{dt} \in P_K\big(g(x(t)) - \rho N(x(t)) - g(x(t))\big), \quad \text{for a.a. } t \in [0, J],$$
$$x(0) = b, \tag{1.2}$$

where $N : \mathbb{R}^n \to 2^{\mathbb{R}^n}$ is a set-valued function, $g : \mathbb{R}^n \to \mathbb{R}^n$ is a single-valued function, $\rho > 0$ is a constant, $P_K x$ denotes the projection of the point x on K, b is a given point in $\mathbb{R}^n$.

The concept of generalized f-projection operator was first introduced by Wu and Huang [11] in 2006. They also proved that the generalized f-projection operator is an extension of the projection operator P_K in R^n and it owns some nice properties as P_K does; see [12, 13]. Some applications of generalized f-projection operator are also given in [11–13]. Very recently, Li et al. [14] studied the stability of the generalized f-projection operator with an application in Banach spaces. We would like to point out that Cojocaru [15] introduced and studied the projected dynamical systems on infinite Hilbert spaces in 2002.

To explore further dynamic systems in infinite dimensional spaces in more general forms has been one of our major motivations and efforts recently, and this paper is a response to those efforts. In this paper, we introduce and study a new class of generalized dynamical systems involving generalized f-projection operators. By using the fixed-point theorem due to Nadler [16], we prove that the equilibrium points set of this class of generalized global dynamical systems is nonempty and closed. We also show that the solutions set of the systems with set-valued perturbation is continuous with respect to the initial value. The results presented in this paper generalize many existing results in recent literatures.

2. Preliminaries

Let $\mathcal{X}$ be a Banach space and let $K \subset \mathcal{X}$ be a closed convex set, let $N : \mathcal{X} \to 2^{\mathcal{X}}$ be a set-valued mapping, and let $g : \mathcal{X} \to \mathcal{X}$ be a single-valued mapping. The normalized duality mapping J from $\mathcal{X}$ to $\mathcal{X}^*$ is defined by

$$J(x) = \left\{x^* \in \mathcal{X}^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\right\}, \tag{2.1}$$

for $x \in X$. For convenience, we list some properties of $J(\cdot)$ as follows.X is a smooth Banach space, $J(\cdot)$ is single valued and hemicontinuous; that is, J is continuous from the strong topology of $\mathcal{X}$ to the $weak^*$ topology of $\mathcal{X}^*$.

Let $C(\mathcal{X})$ denote the family of all nonempty compact subsets of $\mathcal{X}$ and let $\mathcal{H}(\cdot,\cdot)$ denote the Hausdorff metric on $C(\mathcal{X})$ defined by

$$\mathcal{H}(A, B) = \max\left\{\sup_{a\in A}\inf_{b\in B} d(a, b), \sup_{b\in B}\inf_{a\in A} d(a, b)\right\}, \quad \forall A, B \in C(\mathcal{X}). \tag{2.2}$$

In this paper, we consider a new class of generalized set-valued dynamical system, that is, to find those absolutely continuous functions $x(\cdot)$ from $[0,h] \to \mathcal{X}$ such that

$$\begin{gathered}\frac{dx(t)}{dt} \in \Pi_K^f\left(g(x(t)) - \rho N(x(t))\right) - g(x(t)), \quad \text{for a.a. } t \in [0,h], \\ x(0) = b,\end{gathered} \tag{2.3}$$

where $b \in \mathcal{X}$, $\rho > 0$ is a constant and $f : K \to R \cup \{+\infty\}$ is proper, convex, and lower semicontinuous and $\Pi_K^f : \mathcal{X} \to 2^K$ is a generalized f-projection operator denoted by

$$\Pi_K^f x = \left\{u \in K : G(J(x), u) = \inf_{\xi \in K} G(J(x), \xi)\right\}, \quad \forall x \in \mathcal{X}. \tag{2.4}$$

It is well known that many problems arising in the economics, physical equilibrium analysis, optimization and control, transportation equilibrium, and linear and nonlinear mathematics programming problems can be formulated as projected dynamical systems (see, e.g., [1–10, 15, 17] and the references therein). We also would like to point out that problem (2.3) includes the problems considered in Friesz et al. [3], Xia and Wang [7], and Zou et al. [9] as special cases. Therefore, it is important and interesting to study the generalized projected dynamical system (2.3).

Definition 2.1. A point x^* is said to be an equilibrium point of global dynamical system (2.3), if x^* satisfies the following inclusion:

$$0 \in \Pi_K^f\left(g(x) - \rho N(x)\right) - g(x). \tag{2.5}$$

Definition 2.2. A mapping $N : \mathcal{X} \to \mathcal{X}$ is said to be

(i) α-strongly accretive if there exists some $\alpha > 0$ such that

$$\left(N(x) - N(y), J(x-y)\right) \geq \alpha\|x-y\|^2, \quad \forall x, y \in K; \tag{2.6}$$

(ii) ξ-Lipschitz continuous if there exists a constant $\xi \geq 0$ such that

$$\|N(x) - N(y)\| \leq \xi\|x-y\|, \quad \forall x, y \in K. \tag{2.7}$$

Definition 2.3. A set-valued mapping $T : \mathcal{X} \to \mathcal{X}$ is said to be ξ-Lipschitz continuous if there exists a constant $\xi > 0$ such that

$$\mathcal{H}\left(T(x), T(y)\right) \leq \xi\|x-y\|, \quad \forall x, y \in K, \tag{2.8}$$

where $\mathcal{H}(\cdot,\cdot)$ is the Hausdorff metric on $C(\mathcal{X})$.

Lemma 2.4 (see [14]). *Let $\mathcal{X}$ be a real reflexive and strictly convex Banach space with its dual $\mathcal{X}^*$ and let K be a nonempty closed convex subset of $\mathcal{X}$. If $f : K \to R \cup \{+\infty\}$ is proper, convex, and* *lower semicontinuous, then Π_K^f is single valued. Moreover, if $\mathcal{X}$ has Kadec-Klee property, then Π_K^f is continuous.*

Lemma 2.5 (see [18]). *Let $\mathcal{X}$ be a real uniformly smooth Banach space. Then $\mathcal{X}$ is q-uniformly smooth if and only if there exists a constant $C_q > 0$ such that, for all $x, y \in \mathcal{X}$,*

$$\|x+y\|^q \leq \|x\|^q + q\langle y, J_q(x)\rangle + C_q\|y\|^q. \tag{2.9}$$

Lemma 2.6 (see [19]). *Let $(\mathcal{X}, d)$ be a complete metric space and let T_1, T_2 be two set-valued contractive mappings with same contractive constants $\theta \in (0,1)$. Then*

$$\mathscr{H}(F(T_1), F(T_2)) \leq \frac{1}{1-\theta} \sup_{x\in\mathcal{X}} H(T_1(x), T_2(x)), \tag{2.10}$$

where $F(T_1)$ and $F(T_2)$ are fixed-point sets of T_1 and T_2, respectively.

Lemma 2.7 (see [19]). *Let $\mathcal{X}$ be a real strictly convex, reflexive, and smooth Banach space. For any $x_1, x_2 \in \mathcal{X}$, let $\widehat{x}_1 = \Pi_K^f x_1$ and $\widehat{x}_2 = \Pi_K^f x_2$. Then*

$$\langle J(x_1) - J(x_2), \widehat{x}_1 - \widehat{x}_2\rangle \geq 2M^2\delta\left(\frac{\|\widehat{x}_1 - \widehat{x}_2\|}{2M}\right), \tag{2.11}$$

where

$$M = \sqrt{\frac{\|\widehat{x}_1\|^2 + \|\widehat{x}_2\|^2}{2}}. \tag{2.12}$$

We say that $\mathcal{X}$ is 2-uniformly convex and 2-uniformly smooth Banach space if there exist $k, c > 0$ such that

$$\begin{gathered} \delta_X(\epsilon) \geq k\epsilon^2, \\ \rho_X(t) \leq ct^2, \end{gathered} \tag{2.13}$$

where

$$\begin{gathered} \delta_X(\epsilon) = \inf\left\{1 - \left\|\frac{x+y}{2}\right\| : \|x\| = \|y\| = 1, \|x-y\| \geq \epsilon\right\}, \\ \rho_X(t) = \sup\left\{\frac{1}{2}(\|x+y\| + \|x-y\|) - 1 : \|x\| = 1, \|y\| \leq t\right\}. \end{gathered} \tag{2.14}$$

Based on Lemma 2.7, we can obtain the following lemma.

Lemma 2.8. *Let $\mathcal{X}$ be 2-uniformly convex and 2-uniformly smooth Banach space. Then*

$$\left\|\Pi_K^f x - \Pi_K^f y\right\| \leq 64\frac{c}{k}\|x-y\|, \quad \forall x, y \in \mathcal{X}. \tag{2.15}$$

Proof. According to Lemma 2.7, we have

$$\left\langle J(x) - J(y), \Pi_K^f x - \Pi_K^f y \right\rangle \geq 2M_1^2 \delta \left(\frac{\left\| \Pi_K^f x - \Pi_K^f y \right\|}{2M_1} \right), \tag{2.16}$$

where

$$M_1 = \sqrt{\frac{\|\Pi_K^f x\|^2 + \|\Pi_K^f y\|^2}{2}}. \tag{2.17}$$

Since $\delta_X(\epsilon) \geq k\epsilon^2$, (2.16) yields

$$\left\| \Pi_K^f x - \Pi_K^f y \right\| \leq \frac{2}{k} \|J(x) - J(y)\|. \tag{2.18}$$

From the property of $J(\cdot)$, we have

$$\begin{aligned} \|J(x) - J(y)\| &\leq \frac{2M_2^2 \rho_X(4\|x - y\|/M_2)}{\|x - y\|} \\ &\leq 32c\|x - y\|. \end{aligned} \tag{2.19}$$

It follows from (2.18) and (2.19) that

$$\left\| \Pi_K^f x - \Pi_K^f y \right\| \leq 64\frac{c}{k} \|x - y\|. \tag{2.20}$$

This completes the proof. □

3. Equilibrium Points Set

In this section, we prove that the equilibrium points set of the generalized set-valued dynamical system (2.3) is nonempty and closed.

Theorem 3.1. *Let $\mathcal{X}$ be 2-uniformly convex and 2-uniformly smooth Banach space. Let $N : \mathcal{X} \to C(\mathcal{X})$ be μ-Lipschitz continuous and let $g : \mathcal{X} \to \mathcal{X}$ be α-Lipschitz continuous and β-strongly accretive. If*

$$\sqrt{1 + \alpha^2 - 2\beta C_2} + 64\frac{c}{k}(\alpha + \rho\mu) < 1, \tag{3.1}$$

then the equilibrium points set of the generalized set-valued dynamical system (2.3) *is nonempty and closed.*

Proof. Let

$$T(x) = x - g(x) + \Pi_K^f(g(x) - \rho N(x)), \quad \forall x \in K. \tag{3.2}$$

Since $N : \mathcal{X} \to C(\mathcal{X})$ and Π_K^f are continuous, we know that $T : \mathcal{X} \to C(\mathcal{X})$. From Definition 2.1, it is easy to see that x^* is an equilibrium point of the generalized set-valued dynamical system (2.3) if and only if x^* is a fixed-point of T in $\mathcal{X}$, that is:

$$x^* \in T(x^*) = x^* - g(x^*) + \Pi_K^f(g(x^*) - \rho N(x^*)). \tag{3.3}$$

Thus, the equilibrium points set of (2.3) is the same as the fixed-points set of T. We first prove that $F(T)$ is nonempty. In fact, for any $x, y \in \mathcal{X}$ and $a_1 \in T(x)$, there exists $u \in N(x)$ such that

$$a_1 = x - g(x) + \Pi_K^f(g(x) - \rho u). \tag{3.4}$$

Since $u \in N(x)$, and $N : \mathcal{X} \to C(\mathcal{X})$, it follows from Nadler [16] that there exists $v \in N(y)$ such that

$$\|u - v\| \le \mathcal{H}(N(x), N(y)). \tag{3.5}$$

Let

$$a_2 = y - g(y) + \Pi_K^f(g(y) - \rho v). \tag{3.6}$$

Then $a_2 \in T(y)$. From (3.4) to (3.6), we have

$$\begin{aligned} \|a_1 - a_2\| &= \left\| x - y - (g(x) - g(y)) + \Pi_K^f(g(x) - \rho u) - \Pi_K^f(g(y) - \rho(v)) \right\| \\ &\le \|x - y - (g(x) - g(y))\| + \left\| \Pi_K^f(g(x) - \rho u) - \Pi_K^f(g(y) - \rho(v)) \right\|. \end{aligned} \tag{3.7}$$

Since g is α-Lipschitz continuous and β-strongly accretive,

$$\begin{aligned} \|x - y - (g(x) - g(y)\|^2 &\le \|x - y\|^2 - 2\langle g(x) - g(y), J(x - y)\rangle + C_2\|g(x) - g(y)\|^2 \\ &\le \left(1 + \alpha^2 - 2\beta C_2\right)\|x - y\|^2. \end{aligned} \tag{3.8}$$

From Lemma 2.8, where Π_K^f is Lipchitz continuous, we have

$$\begin{aligned} \left\| \Pi_K^f(g(x) - \rho u) - \Pi_K^f(g(y) - (\rho v)) \right\| &\le 64\frac{c}{k}(\|g(x) - g(y)\| + \rho\|u - v\|) \\ &\le 64\frac{c}{k}(\alpha\|x - y\| + \rho\|u - v\|). \end{aligned} \tag{3.9}$$

From the selection of v and the Lipschitz continuity of N,

$$\|u - v\| \leq \mathscr{H}(N(x), N(y)) \leq \mu\|x - y\|. \tag{3.10}$$

In light of (3.7)–(3.10), we have

$$\|a_1 - a_2\| \leq \left(\sqrt{1 + \alpha^2 - 2\beta C_2} + 64\frac{c}{k}(\alpha + \rho\mu)\right)\|x - y\| = L\|x - y\|, \tag{3.11}$$

where

$$L = \sqrt{1 + \alpha^2 - 2\beta C_2} + 64\frac{c}{k}(\alpha + \rho\mu). \tag{3.12}$$

Now (3.11) implies that

$$d(a_1, T(y)) = \inf_{a_2 \in T(y)} \|a_1 - a_2\| \leq L\|x - y\|. \tag{3.13}$$

Since $a_1 \in T(x)$ is arbitrary, we have

$$\sup_{a_1 \in T(x)} d(a_1, T(y)) \leq L\|x - y\|. \tag{3.14}$$

Similarly, we can prove

$$\sup_{a_2 \in T(y)} d(T(x), a_2) \leq L\|x - y\|. \tag{3.15}$$

From (3.14), (3.15), and the definition of the Hausdorff metric $\mathscr{H}$ on $C(\mathcal{X})$, we have

$$\mathscr{H}(T(x), T(y)) \leq L\|x - y\|, \quad \forall x, y \in K. \tag{3.16}$$

Now the assumption of the theorem implies that $L < 1$ and so $T(x)$ is a set-valued contractive mapping. By the fixed-point theorem of Nadler [16], there exists x^* such that $x^* \in T(x^*)$, and thus x^* is the equilibrium point of (2.3). This means that $F(T)$ is nonempty.

Now we prove that $F(T)$ is closed. Let $\{x_n\} \subset F(T)$ with $x_n \to x_0 (n \to \infty)$. Then $x_n \in T(x_n)$ and (3.16) imply that

$$\mathscr{H}(T(x_n), T(x_0)) \leq L\|x_n - x_0\|. \tag{3.17}$$

Thus,

$$\begin{aligned} d(x_0, T(x_0)) &\leq \|x_0 - x_n\| + d(x_n, T(x_n)) + \mathscr{H}(T(x_n), T(x_0)) \\ &\leq (1 + L)\|x_n - x_0\| \longrightarrow 0 \quad \text{as } n \to \infty. \end{aligned} \tag{3.18}$$

It follows that $x_0 \in F(T)$ and so $F(T)$ are closed. This completes the proof. $\square$

Remark 3.2. Theorem 3.1 is a generalization of Theorem 1 in Zou et al. [9] from $\mathbb{R}^n$ to Banach space $\mathcal{X}$.

4. Sensitivity of the Solutions Set

In this section, we study the sensitivity of the solutions set of the generalized dynamical system with set-valued perturbation for (2.3) as follows:

$$\frac{dx(t)}{dt} \in \Pi_K^f\big(g(x(t)) - \rho N(x(t))\big) - g(x(t)) + F(x(t)), \quad \text{for a.a.}[t \in 0, h], \\ x(0) = b, \tag{4.1}$$

where g and b are the same as in (2.3), $F : \mathcal{X} \to 2^{\mathcal{X}}$ is a set-valued mapping, and $N : \mathcal{X} \to \mathcal{X}$ is a single-valued mapping. Let $S(b)$ denote the set of all solutions of (4.1) on $[0, h]$ with $x(0) = b$.

Now we prove the following result.

Theorem 4.1. *Let $\mathcal{X}$ be 2-uniformly convex and 2-uniformly smooth Banach space. Let $g : \mathcal{X} \to \mathcal{X}$ be α-Lipschitz continuous, let $N : \mathcal{X} \to \mathcal{X}$ be μ-Lipschitz continuous, and let $F : \mathcal{X} \to C(\mathcal{X})$ be a ω-Lipschitz continuous set-valued mapping with compact convex values. If*

$$64\frac{c}{k}(\alpha + \rho\mu) + \alpha + \omega < 1, \qquad h\left(64\frac{c}{k}(\alpha + \rho\mu) + \alpha + \omega\right) < 1, \tag{4.2}$$

then $S(b)$ is nonempty and continuous.

Proof. Let

$$M(x) = \Pi_K^f\big(g(x) - \rho N(x)\big) - g(x) + F(x). \tag{4.3}$$

Then $M : \mathcal{X} \to C(\mathcal{X})$ is a set-valued mapping with compact convex values since $F : \mathcal{X} \to C(\mathcal{X})$ is a set-valued mapping with compact convex values. For any $x_1, x_2 \in \mathcal{X}$ and $a_1 \in M(x_1)$, there exists $u \in F(x_1)$ such that

$$a_1 = \Pi_K^f\big(g(x_1) - \rho N(x_1)\big) - g(x_1) + u. \tag{4.4}$$

Since $u \in F(x_1)$, and $F : \mathcal{X} \to C(\mathcal{X})$, it follows from Nadler [16] that there exists $v \in F(x_2)$ such that

$$\|u - v\| \le \mathcal{H}(F(x_1), F(x_2)). \tag{4.5}$$

Let

$$a_2 = \Pi_K^f\big(g(x_2) - \rho N(x_2)\big) - g(x_2) + v. \tag{4.6}$$

Then $a_2 \in M(x_2)$. From (4.4) and (4.6), we have

$$\begin{aligned}\|a_1 - a_2\| &= \left\|\Pi_K^f(g(x_1) - \rho N(x_1)) - \Pi_K^f(g(x_2) - \rho N(x_2)) - (g(x_2) - g(x_2)) + u - v\right\| \\ &\le \left\|g(x_1) - g(x_2)\right\| + \left\|\Pi_K^f(g(x) - \rho N(x_1)) - \Pi_K^f(g(y) - \rho N(x_2))\right\| + \|u - v\|.\end{aligned} \tag{4.7}$$

Since g is α-Lipschitz continuous,

$$\left\|g(x_1) - g(x_2)\right\| \le \alpha\|x_2 - x_2\|. \tag{4.8}$$

From Lemma 2.8, Π_K^f is Lipschitz continuous. It follows from the continuity of N and g that

$$\begin{aligned}\left\|\Pi_K^f(g(x_1) - \rho N(x_1)) - \Pi_K^f(g(x_2) - \rho N(x_2))\right\| &\le 64\frac{c}{k}\left(\left\|g(x_1) - g(x_2)\right\| + \rho\|N(x_1) - N(x_2)\|\right) \\ &\le 64\frac{c}{k}(\alpha + \rho\mu)\|x_1 - x_2\|.\end{aligned} \tag{4.9}$$

From the selection of v and the Lipschitz continuity of F, we know

$$\|u - v\| \le \mathscr{H}(F(x_1), F(x_2)) \le \omega\|x_1 - x_2\|. \tag{4.10}$$

In light of (4.7)–(4.10), we have

$$\|a_1 - a_2\| \le \left(\alpha + 64\frac{c}{k}(\alpha + \rho\mu) + \omega\right)\|x_1 - x_2\| = \theta\left\|x - y\right\|, \tag{4.11}$$

where

$$\theta = \alpha + 64\frac{c}{k}(\alpha + \rho\mu) + \omega. \tag{4.12}$$

Now (4.11) implies that

$$d(a_1, M(x_2)) = \inf_{a_2 \in M(x_2)} \|a_1 - a_2\| \le \theta\|x_1 - x_2\|. \tag{4.13}$$

Since $a_1 \in M(x_1)$ is arbitrary, we obtain

$$\sup_{a_1 \in M(x_1)} d(a_1, M(x_2)) \le \theta\|x_1 - x_2\|. \tag{4.14}$$

Similarly, we can prove

$$\sup_{a_2 \in M(x_2)} d(M(x_1), a_2) \le \theta\|x_1 - x_2\|. \tag{4.15}$$

From (4.13), to (4.15), and the definition of the Hausdorff metric $\mathscr{H}$ on $C(\mathcal{X})$, we have

$$\mathscr{H}(M(x_1), M(x_2)) \leq \theta\|x_1 - x_2\|, \quad \forall x_1, x_2 \in \mathcal{X}. \tag{4.16}$$

Now (4.2) implies that $0 < \theta < 1$, and so $M(x)$ is a set-valued contractive mapping. Let

$$Q(x,b) = \left\{ y \in C([0,h],\mathcal{X}) \mid y(t) = b + \int_0^t z(s)ds, z(s) \in M(x(s)) \right\}, \tag{4.17}$$

where

$$C([0,h],\mathcal{X}) = \{ f : [0,h] \longrightarrow \mathcal{X} \mid f \text{ is continuous} \}. \tag{4.18}$$

Since $M : \mathcal{X} \to C(\mathcal{X})$ is a continuous set-valued mapping with compact convex values, by the Michael's selection theorem (see, e.g., Theorem 16.1 in [20]), we know that $Q(x,b)$ is nonempty for each x and $b \in \mathcal{X}$. Moreover, it is easy to see that the set of fixed-points of $Q(x,b)$ coincides with $S(b)$. It follows from [21] or [8] that $Q(x,b)$ is compact and convex for each x and $b \in \mathcal{X}$. Suppose that b_m is the initial value of (4.1); that is, $x(0) = b_m (m = 0, 1, 2, \ldots)$ and $b_m \to b_0 (m \to \infty)$. Since

$$Q(x,b_0) = Q(x,b_m) - b_m + b_0, \tag{4.19}$$

it is obvious that $Q(x,b_m)$ converges uniformly to $Q(x,b_0)$.

Next we prove that $Q(x,b_m)$ is a set-valued contractive mapping. For any given $x_1, x_2 \in C([0,h],\mathcal{X})$, since $M : \mathcal{X} \to C(\mathcal{X})$ is a continuous set-valued mapping with compact convex values, by the Michael's selection theorem (see, e.g., Theorem 16.1 in [20]), we know that $M(x_1(s))$ has a continuous selection $r_1(s) \in M(x_1(s))$. Let

$$c_1(t) = b_m + \int_0^t r_1(s)ds. \tag{4.20}$$

Then $c_1 \in Q(x_1, b_m)$. Since $r_1(s) \in M(x_1(s))$ is measurable and $M(x_2(s))$ is a measurable mapping with compact values, we know that there exists a measurable selection $r_2(s) \in M(x_2(s))$ such that

$$\|r_1(s) - r_2(s)\| \leq \mathscr{H}(M(x_1(s)), M(x_2(s))). \tag{4.21}$$

Thus, it follows from (4.16) that

$$\|r_1(s) - r_2(s)\| \leq \mathscr{H}(M(x_1(s)), M(x_2(s))) \leq \theta\|x_1 - x_2\|. \tag{4.22}$$

Let

$$c_2(t) = b_m + \int_0^t r_2(s)ds. \tag{4.23}$$

Then $c_2 \in Q(x_2, b_m)$ and

$$\begin{aligned} \|c_1 - c_2\| &\leq \int_0^h \|r_1(s) - r_2(s)\| ds \leq h\mathscr{H}(M(x_1(s)), M(x_2(s))) \\ &\leq h\theta\|x_1 - x_2\|. \end{aligned} \tag{4.24}$$

Hence, we have

$$d(c_1, Q(x_2, b_m)) = \inf_{c_2 \in Q(x_2, b_m)} \|c_1 - c_2\| \leq h\theta\|x_1 - x_2\|. \tag{4.25}$$

Since $c_1 \in Q(x_1, b_m)$ is arbitrary, we obtain

$$\sup_{c_1 \in Q(x_1, b_m)} d(c_1, Q(x_2, b_m)) \leq h\theta\|x_1 - x_2\|. \tag{4.26}$$

Similarly, we can prove that

$$\sup_{c_2 \in Q(x_2, b_m)} d(Q(x_1, b_m), c_2) \leq h\theta\|x_1 - x_2\|. \tag{4.27}$$

From the definition of the Hausdorff metric $\mathscr{H}$ on $C(\mathcal{X})$, (4.26) and (4.27) imply that

$$\mathscr{H}(Q(x_1, b_m), Q(x_2, b_m)) \leq h\theta\|x_1 - x_2\|, \quad \forall x_1, x_2 \in \mathcal{X}, m = 0, 1, 2, \ldots. \tag{4.28}$$

Since $h\theta < 1$, it is easy to see that $Q(x, b)$ has a fixed-point for each given $b \in \mathcal{X}$, and so $S(b)$ is nonempty for each given $b \in \mathcal{X}$. Setting

$$W_m(x) = Q(x, b_m), \quad m = 0, 1, 2, \ldots, \tag{4.29}$$

we know that $W_m(x)$ are contractive mappings with the same contractive constant $h\theta$. By Lemma 2.6 and (4.28), we have

$$\mathscr{H}(F(W_m), F(W_0)) \leq \frac{1}{1 - h\theta} \sup_{x \in X} \mathscr{H}(W_m(x), W_0(x)) \longrightarrow 0. \tag{4.30}$$

Thus, $F(W_m) \to F(W_0)$, which implies that $S(b_m) \to S(b)$; that is, the solution of (4.1) is continuous with respect to the initial value of (4.1). This completes the proof. □

Remark 4.2. Theorem 4.1 is a generalization of Theorem 2 in Zou et al. [9] from $\mathbb{R}^n$ to Banach space $\mathcal{X}$.

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