

## Research Article

# A Simple Numerical Method for Pricing an American Put Option

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We present a simple numerical method to find the optimal exercise boundary in an American put option. We formulate an intermediate function with the fixed free boundary that has Lipschitz character near optimal exercise boundary. Employing it, we can easily determine the optimal exercise boundary by solving a quadratic equation in time-recursive way. We also present several numerical results which illustrate a comparison to other methods.

## 1. Introduction

The owner of a put (call) option has the right but no obligation to sell (buy) an underlying asset at the exercise price. European options can be exercised only on the expiry date, while American options can be exercised at any time until the expiry date. Closed-form solutions for the European options are derived in papers by Black and Scholes [1] and Merton [2]. In the case of American options, because of the early exercise possibility, the pricing problem leads to complications for analytic calculation. McKean [3] and van Moerbeke [4] show that the valuation of American options constitutes a free boundary problem looking for a boundary changing in time to maturity, mostly called an optimal exercise boundary. Hence, finance researchers have studied methods to quickly and accurately find the optimal exercise boundary. These methods are basically of two types, that is, analytical approximations such as those developed by Geske and Johnson [5], MacMillan [6], Barone-Adesi and Whaley [7], and Ju [8] and numerical methods such as those of Brennan and Schwartz [9], Hull and White [10], and Longstaff and Schwartz [11]. Zhu [12] finds an exact and explicit solution of the Black-Scholes equation for the valuation of American put options using Taylor series with infinitely many terms. His work is an excellent result for the

valuation of American put options; however, it seems difficult to perform his solution numerically. The infinite sum is likely to yield many computation errors. Zhao and Wong [13] study an extension of Zhu's work [12] to price American options under general diffusion processes.

The majority of numerical methods for pricing American options, such as the finite difference method of Brennan and Schwartz [9], the binomial method of Cox et al. [14], the Monte Carlo simulation method of Grant et al. [15], the least squares method of Longstaff and Schwartz [11], the integral-equation method of Ševčovič [16], and the Laplace transform method of Zhu [17], are time-recursive ways. Their idea is to discretize the lifetime of an option and find its optimal exercise boundary backward in time. Since time-recursive ways yield repeated calculations for every time step, they require fast computation times and small pricing errors. Also, front-fixing methods developed by Wu and Kwok [18] and Nielsen et al. [19] apply a nonlinear transformation to fix the boundary and solve the resulting nonlinear problem. A secant method developed by Zhu et al. [20] needs to solve a nonlinear problem, and a moving boundary approach developed by Muthuraman [21] converts the arising linear free boundary partial differential equation (PDE) problem into a sequence of linear fixed-boundary PDE problems. More recently, Zhu and Zhang [22] introduced

a new predictor-corrector scheme to price American put options under the Black-Scholes model, and then Zhu and Chen [23] proposed an extension of Zhu and Zhang's work [22] to solve for the valuation of American put options with stochastic volatility model.

The main contribution of this paper is the development of a simple numerical method to find optimal exercise boundary in a time-recursive way. Our result is motivated by the necessity for better understanding of the solution surface near optimal exercise boundary. We adopt the front-fixing transformation [18] to change the unknown free boundary to a known and fixed boundary. We exploit an intermediate function with the fixed free boundary that has Lipschitz character which avoids the degeneracy of the solution surface near optimal exercise boundary as in Kim et al. [24]. Indeed, our function from the Black-Scholes equation and the boundary conditions transforms the surface above the exercise region onto a new Lipschitz surface which forms a sufficiently large angle with the hyperplane corresponding to the exercise region, thereby making the borderline more easily distinguishable (see Figure 2). We use implicit scheme in the continuation region and apply extrapolation near optimal exercise boundary. Thus we can determine the optimal exercise boundary by solving a quadratic equation in a time-recursive way. Our method also provides fast and accurate results for calculating the optimal exercise boundary and pricing American put options.

The structure of the paper is as follows. Section 2 presents the model formulation. The intermediate function with the fixed free boundary to calculate the optimal exercise boundary is presented in Section 3. Numerical results and comparative studies are presented in Section 4. Section 5 summarizes the paper.

## 2. Problem Formulation

In this section, we present a mathematical formula for pricing an American put option.

Consider an American put option on an underlying asset (stock) with exercise price  $K$  and expiration  $T$ . In risk-neutral probability, an underlying asset price  $S(t)$  is governed by the following stochastic differentiable equation:

$$dS(t) = (r - \delta)S(t)dt + \sigma S(t)dW(t), \quad (1)$$

where  $r > 0$  represents the risk-free interest rate,  $\delta \geq 0$  represents the continuous dividend yield,  $\sigma > 0$  represents the volatility of the underlying asset price, and  $W(t)$  is the standard Brownian motion. The payoff function of the put option at  $T$  is defined as

$$(K - S(T))^+ = \max\{K - S(T), 0\}. \quad (2)$$

The valuation of an American put option is denoted by  $P(\tau, S)$ , where  $\tau (= T - t)$  is the time to expiration for  $\tau \in [0, T]$  and  $S$  is the underlying asset price for  $S \in [0, \infty)$ .

As seen in the previous article by McKean [3], the valuation of an American put option is considered the solution to a free boundary problem with a parabolic PDE. We suppose that the optimal exercise boundary  $\beta(\tau)$  is continuously

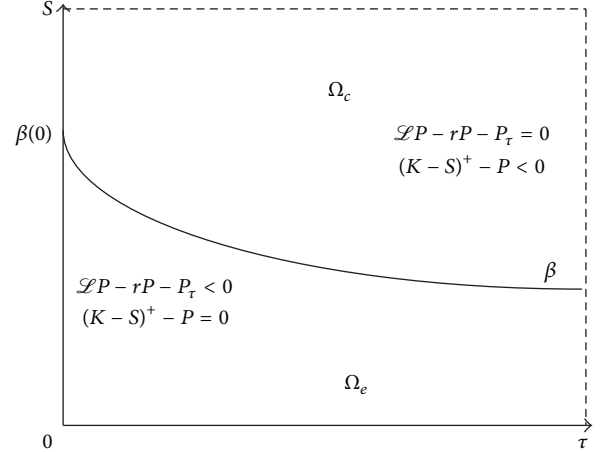


FIGURE 1: Boundary conditions.

nonincreasing with  $\beta(0) = \min\{K, Kr/\delta\}$ . The region where it is optimal to hold, generally called the continuation region, is defined as  $\Omega_c = [0, T] \times (\beta(\tau), \infty)$ , and the region where it is optimal to exercise, generally called the exercise region, is defined as  $\Omega_e = [0, T] \times [0, \beta(\tau)]$ . Then,  $P(\cdot, \cdot)$  and  $\beta(\cdot)$  uniquely solve

$$\begin{aligned} \mathcal{L}P - rP - P_\tau &= 0 \quad \text{in } \Omega_c, \\ P(0, S) &= (K - S)^+, \\ P(\tau, \beta(\tau)) &= K - \beta(\tau), \\ \lim_{S \uparrow \infty} \max_{0 \leq \tau \leq T} |P(\tau, S)| &= 0, \\ \lim_{S \downarrow \beta(\tau)} P_s(\tau, S) &= -1, \\ P(\tau, S) &= K - S \quad \text{in } \Omega_e, \\ P(\tau, S) &\geq (K - S)^+, \end{aligned} \quad (3)$$

where  $\mathcal{L}P = (1/2)\sigma^2 S^2 P_{ss} + (r - \delta)SP_s$  and  $P_s, P_{ss}$ , and  $P_\tau$  are defined by the infinitesimal generator and partial derivatives, respectively. Here, we assume  $r > \delta$  for using the Black-Scholes equation at  $\tau = 0$ . See, for example, Karatzas and Shreve [25] for general reference.

Figure 1 shows an illustration of an optimal exercise boundary with  $P$ . The two regions are separated by the optimal exercise boundary. From Figure 1 it is necessary that  $P$  must satisfy  $\max_{\tau, S} \{\mathcal{L}P - rP - P_\tau, (K - S)^+ - P\} = 0$ . This condition is known as the Hamilton-Jacobi Bellman (HJB) equation. As Chockalingam and Muthuraman [26] point out, the continuation and exercise regions are determined by which term in the HJB equation is tight. Their method requires iterations till convergence of the boundaries. However, we emphasize that there is no iteration in our method using a square root transformation. Refer to Chockalingam and Muthuraman [26], and Pham [27].

A front-fixing method, proposed in Wu and Kwok [18], uses a change in variables to transform the free boundary

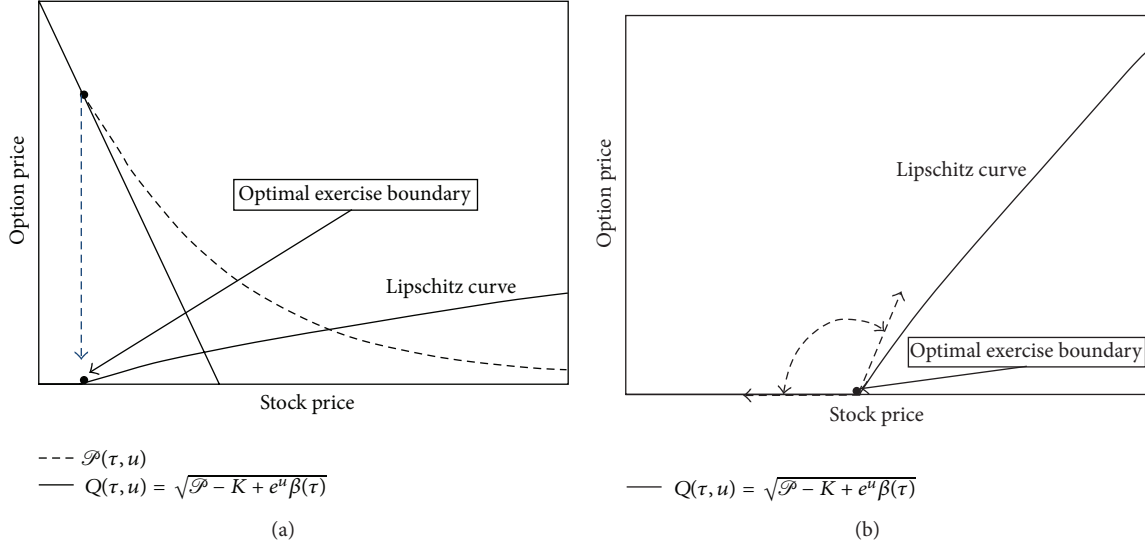


FIGURE 2: Transformation from  $\mathcal{P}(\tau, u)$  to  $Q(\tau, u)$ .

problem into a nonlinear problem on a fixed domain. The following transformation of state variable serves for such a purpose:

$$u = \ln S - \ln \beta(\tau). \quad (4)$$

They derive the equation and the boundary conditions with respect to  $u$  as follows:

$$\begin{aligned} \tilde{\mathcal{L}}\mathcal{P} - r\mathcal{P} - \mathcal{P}_\tau &= 0, \quad u \in (0, \infty), \\ \mathcal{P}(0, u) &= 0, \quad u \in (0, \infty), \\ \mathcal{P}(\tau, 0) &= K - \beta(\tau), \\ \mathcal{P}_u(\tau, 0) &= -\beta(\tau), \\ \mathcal{P}(\tau, \infty) &= 0, \end{aligned} \quad (5)$$

where  $\tilde{\mathcal{L}}\mathcal{P} = (1/2)\sigma^2\mathcal{P}_{uu} + (r - \delta - (\sigma^2/2) + (\beta'(\tau)/\beta(\tau)))\mathcal{P}_u$  and  $\mathcal{P}_u$  and  $\mathcal{P}_{uu}$  are defined by infinitesimal generator and partial derivatives, respectively. Note that  $\beta(\tau)$  is a monotonically decreasing function of  $\tau$  with a nontrivial asymptotic limit as follows:

$$\beta(\infty) = \frac{\theta}{\theta - 1}K, \quad (6)$$

where

$$\theta = \frac{-(r - \delta - (1/2)\sigma^2) - \sqrt{(r - \delta - (1/2)\sigma^2)^2 + 2\sigma^2r}}{\sigma^2}. \quad (7)$$

Namely, the optimal exercise boundary does not change with time. Especially, plugging  $\delta = 0$  into (6), we have the asymptotically optimal exercise boundary as follows:

$$\beta(\infty) = \frac{\gamma}{\gamma + 1}K, \quad (8)$$

where  $\gamma = 2r/\sigma^2$ . Note that transformation (4) is valid because  $\beta(\tau) > 0$  holds for all  $\tau \geq 0$ . Refer to Kim [28].

### 3. Intermediate Function with the Fixed Free Boundary

In this section, we present an intermediate function with the fixed free boundary and can determine the optimal exercise boundary by solving a quadratic equation in a time-recursive way.

Under the assumption of the Black-Scholes model, the time for optimal exercise can be shown to be the first hitting time of a boundary, the optimal exercise boundary, in the plane consisting of pairs of the underlying asset price and the time to expiration. Namely, the price curve of an American put option touches the line representing the intrinsic value tangentially. With a careful examination of the solution surface near optimal exercise boundary, we find a Lipschitz surface which avoids the degeneracy of the solution surface near optimal exercise boundary. To find the optimal exercise boundary, we present an intermediate function with the fixed free boundary that has Lipschitz character near optimal exercise boundary as follows:

$$Q(\tau, u) := \sqrt{\mathcal{P}(\tau, u) - K + e^u \beta(\tau)}. \quad (9)$$

The transformed function  $Q(\tau, u)$  provides that the solution surface in  $\Omega_e$  is a horizontal plane, and it is an inclined plain in  $\Omega_c$ . Namely, this function forms a sufficiently large angle with the hyperplane corresponding to the exercise region, thereby making the borderline more easily distinguishable.  $Q(\tau, u)$  also has a Lipschitz character with nonsingularity in  $(0, \infty)$  and a nondegeneracy property near optimal exercise boundary. Hence, we have

$$\begin{cases} Q(\tau, u) = 0 & \text{if } u \in [\ln \beta(\infty) - \ln \beta(\tau), 0], \\ Q(\tau, u) > 0 & \text{if } u \in (0, \infty). \end{cases} \quad (10)$$

Figure 2(a) shows that  $\mathcal{P}(\tau, u)$  is transformed to  $Q(\tau, u)$ , and Figure 2(b) is a magnified view of the optimal exercise boundary.

We find the intermediate function with the fixed free boundary to decide the optimal exercise boundary by the Taylor series. From  $\mathcal{P}(\tau, u) = Q^2(\tau, u) + K - e^u \beta(\tau)$ , we obtain the following relations near optimal exercise boundary ( $u = 0$ ):

$$\begin{aligned} \mathcal{P}_{uu} &= 2Q_u^2 - \beta, & \mathcal{P}_u &= -\beta, \\ \mathcal{P} &= K - \beta, & \mathcal{P}_\tau &= -\beta'. \end{aligned} \quad (11)$$

Plugging (11) into (5), we obtain

$$\begin{aligned} \frac{1}{2}\sigma^2(2Q_u^2 - \beta) + \left(r - \delta - \frac{1}{2}\sigma^2 + \frac{\beta'}{\beta}\right)(-\beta) \\ - r(K - \beta) + \beta' = 0, \end{aligned} \quad (12)$$

and then we get  $Q_u^2 = (rK - \delta\beta)/\sigma^2$ . More precisely, we have

$$Q_u(\tau, 0) = \frac{\sqrt{rK}}{\sigma}, \quad (13)$$

$$\frac{\sqrt{r}}{\sigma\sqrt{K}} < Q_u(\tau, 0) < \frac{\sqrt{rK}}{\sigma e^{u^*}}$$

with  $\delta = 0$ , where  $u^* = \ln \beta(\infty) - \ln \beta(\tau) < 0$ . Then, we have  $|\mathcal{P} - K + e^u \beta(\tau)| \leq c|u - 0|^2$  for some constant  $c$  because  $Q(\tau, u)$  is Lipschitz and a natural candidate for computation in  $(0, \infty)$ . We obtain an angle between exercise surface ( $Q = 0$ ) and  $Q$  surface ( $Q > 0$ ) such that  $0 < \lambda_0 < \partial Q/\partial u < \lambda_1$  for some constants  $\lambda_0$  and  $\lambda_1$ . We also calculate the partial derivative with respect to  $u$  in (5) as follows:

$$\frac{1}{2}\sigma^2 \mathcal{P}_{uuu} + \left(r - \delta - \frac{1}{2}\sigma^2 + \frac{\beta'}{\beta}\right) \mathcal{P}_{uu} - r\mathcal{P}_u - \mathcal{P}_{\tau u} = 0. \quad (14)$$

From  $\mathcal{P}(\tau, u) = Q^2(\tau, u) + K - e^u \beta(\tau)$ , we obtain the following relations near optimal exercise boundary ( $u = 0$ ):

$$\mathcal{P}_{uuu} = 6Q_u Q_{uu} - \beta, \quad \mathcal{P}_{\tau u} = -\beta'. \quad (15)$$

Hence, plugging (15) into (14), we get

$$Q_{uu} = -\frac{2\xi\eta}{3\sigma^3} - \frac{\delta\beta}{3\sigma\eta}, \quad (16)$$

where  $\xi = r - \delta - \sigma^2/2 + \beta'/\beta$  and  $\eta = \sqrt{rK - \delta\beta}$ . From (6) we easily show that  $\xi$  and  $\eta$  are bounded parameters such that  $c_1 \leq \xi \leq c_2$  for some negative constants  $c_1$  and  $c_2$  and  $\sqrt{(r - \delta)K} \leq \eta \leq \sqrt{rK}$ , respectively.

Using the similar arguments, we can obtain the following equations near optimal exercise boundary ( $u = 0$ ) as follows:

$$\frac{1}{2}\sigma^2 \mathcal{P}_{uuuu} + \left(r - \delta - \frac{1}{2}\sigma^2 + \frac{\beta'}{\beta}\right) \mathcal{P}_{uuu} - r\mathcal{P}_{uu} - \mathcal{P}_{\tau uu} = 0, \quad (17)$$

$$\mathcal{P}_{uuuu} = 6Q_{uu}^2 + 8Q_u Q_{uuu} - \beta, \quad \mathcal{P}_{\tau uu} = -\left(\frac{2\delta}{\sigma^2} + 1\right)\beta'. \quad (18)$$

Hence, plugging (18) into (17), we get

$$Q_{uuu} = \frac{2\xi^2\eta}{3\sigma^5} + \frac{\xi\delta\beta}{6\sigma^3\eta} - \frac{\delta^2\beta^2}{12\sigma\eta^3} - \frac{\delta\beta}{4\sigma\eta} + \frac{r\eta}{2\sigma^3} - \frac{\delta\beta'}{2\sigma^3\eta}. \quad (19)$$

Furthermore, we recognize that  $\mathcal{P}$  is analytic up to the optimal exercise boundary and  $u$  is locally analytic. Hence, the approximation for  $Q(\tau, u)$  at  $\beta(\tau)$  can be written as

$$\begin{aligned} Q(\tau, u) &= Q(\tau, 0) + Q_u(\tau, 0)u + \frac{1}{2!}Q_{uu}(\tau, 0)u^2 \\ &+ \frac{1}{3!}Q_{uuu}(\tau, 0)u^3 + \mathcal{O}(u^4). \end{aligned} \quad (20)$$

We introduce the equilibrium parameter  $\tilde{u} > 0$  which enables us to adjust the location of optimal exercise boundary in a mesh size. So, plugging  $Q_u(\tau, 0)$ ,  $Q_{uu}(\tau, 0)$ , and  $Q_{uuu}(\tau, 0)$  into (20), we obtain  $Q(\tau, \tilde{u})$  as follows:

$$\begin{aligned} Q(\tau, \tilde{u}) &= \frac{\eta}{\sigma}\tilde{u} - \frac{1}{2}\left(\frac{2\xi\eta}{3\sigma^3} + \frac{\delta\beta}{3\sigma\eta}\right)\tilde{u}^2 \\ &+ \frac{1}{6}\left(\frac{2\xi^2\eta}{3\sigma^5} + \frac{\xi\delta\beta}{6\sigma^3\eta} - \frac{\delta^2\beta^2}{12\sigma\eta^3}\right. \\ &\quad \left. - \frac{\delta\beta}{4\sigma\eta} + \frac{r\eta}{2\sigma^3} - \frac{\delta\beta'}{2\sigma^3\eta}\right)\tilde{u}^3. \end{aligned} \quad (21)$$

We rewrite (21) with respect to  $\xi$  as follows:

$$\tilde{a}(\beta')^2 + \tilde{b}(\beta') + \tilde{c} = 0, \quad (22)$$

where

$$\tilde{a} = \frac{\eta\tilde{u}^3}{9\sigma^5\beta^2},$$

$$\tilde{b} = -\frac{\eta\tilde{u}^2}{3\sigma^3\beta} + \frac{2\eta\tilde{u}^3}{9\sigma^5\beta} + \frac{\delta\tilde{u}^3}{36\sigma^3\eta} - \frac{\delta\tilde{u}^3}{12\sigma^3\eta},$$

$$\begin{aligned} \tilde{c} &= -Q(\tau, \tilde{u}) + \frac{\eta\tilde{u}}{\sigma} - \frac{1}{3}\left(\frac{\eta\nu}{\sigma^3} + \frac{\delta\beta}{2\sigma\eta}\right)\tilde{u}^2 \\ &+ \frac{1}{6}\left(\frac{2\eta\nu^2}{3\sigma^5} + \frac{\delta\nu\beta}{6\sigma^3\eta} - \frac{\delta^2\beta^2}{12\sigma\eta^3} - \frac{\delta\beta}{4\sigma\eta} + \frac{r\eta}{2\sigma^3}\right)\tilde{u}^3, \end{aligned}$$

$$\xi = \nu + \frac{\beta'}{\beta},$$

$$\nu = r - \delta - \frac{\sigma^2}{2}. \quad (23)$$

Combining (22) with  $\beta' < 0$ , we have

$$\beta' = \frac{-\tilde{b} - \sqrt{\tilde{b}^2 - 4\tilde{a}\tilde{c}}}{2\tilde{a}}. \quad (24)$$

For discretization  $(\Delta\tau, \Delta u)$ , we introduce a two-dimensional mesh in the first quadrant of the  $\tau - u$  plane. From (24) we have

$$\frac{\beta_{n+1} - \beta_n}{\Delta\tau} = \frac{-\hat{b} - \sqrt{\hat{b}^2 - 4\hat{a}\hat{c}}}{2\hat{a}}, \quad (25)$$

where

$$n = 0, 1, 2, \dots, N,$$

$$m = 0, 1, 2, \dots, M,$$

$$\hat{u} = \rho\Delta u \quad (\rho > 0),$$

$$Q_z^n \text{ is the numerical approximation to } Q(n\Delta\tau, z\Delta u) \\ (z = \rho, m),$$

$\beta_n$  is the numerical approximation to  $\beta(n\Delta\tau)$ ,

$$\hat{a} = \frac{\eta\hat{u}^3}{9\sigma^5\beta^2},$$

$$\hat{b} = -\frac{\eta\hat{u}^2}{3\sigma^3\beta} + \frac{2\eta\nu\hat{u}^3}{9\sigma^5\beta} + \frac{\delta\hat{u}^3}{36\sigma^3\eta} - \frac{\delta\hat{u}^3}{12\sigma^3\eta},$$

$$\hat{c} = -Q(\tau, \hat{u}) + \frac{\eta\hat{u}}{\sigma} - \frac{1}{3} \left( \frac{\eta\nu}{\sigma^3} + \frac{\delta\beta}{2\sigma\eta} \right) \hat{u}^2 \\ + \frac{1}{6} \left( \frac{2\eta\nu^2}{3\sigma^5} + \frac{\delta\nu\beta}{6\sigma^3\eta} - \frac{\delta^2\beta^2}{12\sigma\eta^3} - \frac{\delta\beta}{4\sigma\eta} + \frac{r\eta}{2\sigma^3} \right) \hat{u}^3, \quad (26)$$

respectively. Hence, we rewrite (25) with respect to  $\beta_n$  as follows:

$$\beta_{n+1} = \beta_n + \left( \frac{-\hat{b} - \sqrt{\hat{b}^2 - 4\hat{a}\hat{c}}}{2\hat{a}} \right) \Delta\tau. \quad (27)$$

When the initial values are given by  $Q_\rho^0$  (transformed price of the American put option) and  $\beta_0$  (optimal exercise boundary) at  $\tau = 0$ , we can determine  $\beta_1$  (optimal exercise boundary) at  $\Delta\tau$  using (27). More importantly, for updating the optimal exercise boundary our method dose not include iteration until sufficient accuracy is obtained. So, we repeat the previously mentioned process until  $N\Delta\tau$  and obtain the optimal exercise boundary in a time-recursive way.

#### 4. Numerical Examples

In this section, we provide numerical examples to illustrate our method. We also make runtimes and computation errors compared with the results obtained by other numerical methods such as the binomial method (Binomial) developed by Cox et al. [14], the front-fixing method (Front-fixing) developed by Wu and Kwok [18], and the finite difference implementation of the moving boundary method (MBM-FDM) developed by Muthuraman [21].

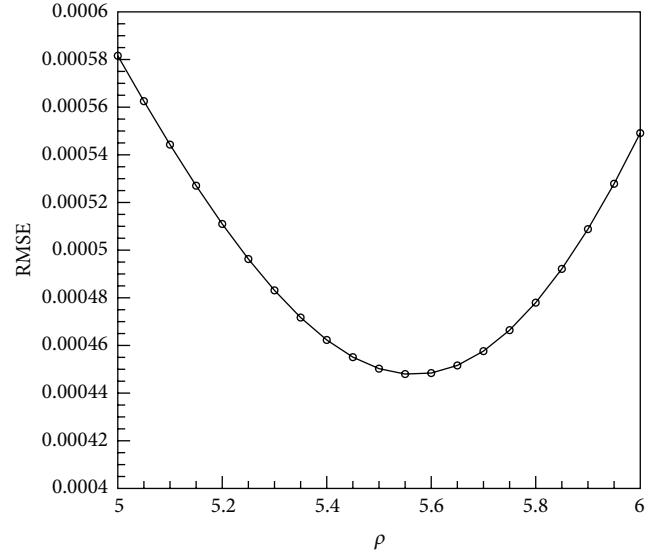


FIGURE 3: RMSE of the safety parameter  $\rho$ .

TABLE 1: Valuation of the American put options.

Safety parameter $\rho$	RMSE	$\beta(T)$	$P(T, 90)$	$P(T, 100)$	$P(T, 110)$
5.00	0.0005816	80.8760	11.4926	6.0911	2.9869
5.10	0.0005443	80.8761	11.4925	6.0911	2.9868
5.20	0.0005110	80.8762	11.4925	6.0910	2.9868
5.30	0.0004831	80.8764	11.4924	6.0909	2.9867
5.40	0.0004623	80.8765	11.4924	6.0908	2.9866
5.50	0.0004503	80.8767	11.4923	6.0907	2.9865
5.60	0.0004484	80.8769	11.8760	6.0906	2.9864
5.70	0.0004576	80.8771	11.8771	6.0906	2.9863
5.80	0.0004780	80.8773	11.4922	6.0905	2.9863
5.90	0.0005088	80.8776	11.4921	6.0904	2.9862
6.00	0.0005491	80.8779	11.4920	6.0903	2.9861
Maximum difference	0.0001336	0.0019	0.0006	0.0008	0.0008

All implementations are carried out using a C++ implementation with the a 2.66 Ghz Intel 4 Core CPU with 3 GB RAM. A finite difference method with Crank-Nicolson scheme is proposed for our method. The benchmark results are obtained using the Binomial with 10,000 time steps, and we consider these results to be the exact values of the American put options. Here, root mean squared error (RMSE) is calculated by the values of the Binomial.

The parameter values used to calculate the optimal exercise boundary and values of the American put options are  $r = 0.05$ ,  $\sigma = 0.20$ ,  $\delta = 0$ ,  $K = 100$ ,  $T = 1$ ,  $u \in [0, 2]$ , and a discrete mesh of  $2000 \times 300$  nodes.

In Figure 3, we find a numerical optimization  $\rho = 5.5702$ . Table 1 also shows the results of the optimal exercise boundary and the values of American put options with

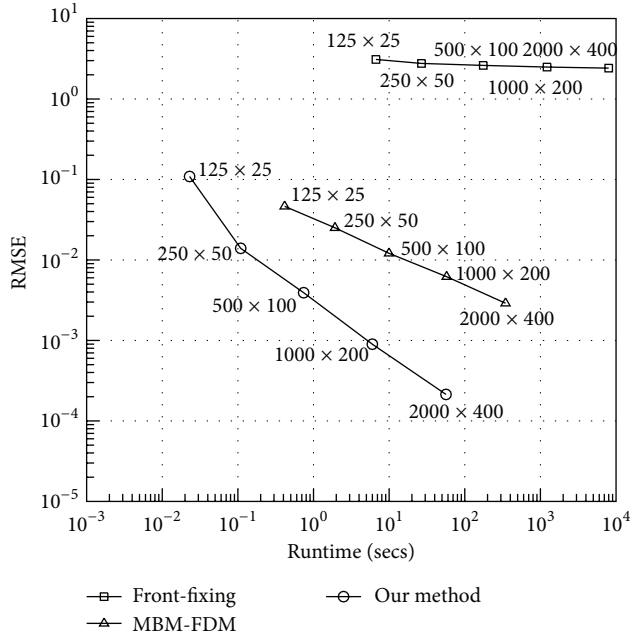


FIGURE 4: Comparison of RMSE and runtime.

safety parameters. One can see from Table 1 that the optimal exercise boundary monotonically increases as the value of  $\rho$  increases, but the value of American put options monotonically decreases when  $\rho$  increases. They are so gradual that they are not very susceptible to change in the value of  $\rho$ .

Table 2 reports the values of the American put options for the specific parameter set associated with the table. In Figure 4 and Table 3, we take the parameter values used in Figure 3 except for the discrete mesh and plot runtimes and computational errors compared with various methods. Note that the discrete meshes of  $125 \times 25$ ,  $250 \times 50$ ,  $500 \times 100$ ,  $1000 \times 200$ , and  $2000 \times 400$  nodes are plotted in Figure 4 and Table 3.

As is shown in Table 2, Figure 4, and Table 3, although four different methods have similar values of the American put option, our method is computationally faster and more accurate than other methods. Especially, Figure 4 and Table 3 show the numerical convergence of our method. So our method is superior to the others in accuracy and computational efficiency.

## 5. Final Remarks

The front-fixing method suggested by Wu and Kwok [18] shows a degeneracy near optimal exercise boundary, while our method adopts a square root function to avoid the quadratic behavior of solution surface that causes degeneracy. Our method employing an intermediate function with the fixed free boundary solves a nonlinear problem on a fixed domain derived from a free boundary problem. Since the computation process depends on Lipschitz surface, we need to focus on the motion of the solution surface which would be simple to see the minute behavior of solution surface.

TABLE 2: Comparison of the values of the American put options.

$(S, T, r, \sigma, \delta)$	Binomial	Front fixing	MBM-FDM	Our method
(80, 0.5, 0.05, 0.20, 0.00)	20.0000	20.0000	20.0000	20.0000
(90, 0.5, 0.05, 0.20, 0.00)	10.6661	10.6643	10.6680	10.6661
(100, 0.5, 0.05, 0.20, 0.00)	4.6556	4.6501	4.6504	4.6549
(110, 0.5, 0.05, 0.20, 0.00)	1.6681	1.6629	1.6631	1.6686
(120, 0.5, 0.05, 0.20, 0.00)	0.4976	0.4961	0.4993	0.4985
(80, 0.5, 0.05, 0.20, 0.03)	20.0000	20.0000	20.0000	20.0000
(90, 0.5, 0.05, 0.20, 0.03)	11.1551	11.1513	11.1526	11.1544
(100, 0.5, 0.05, 0.20, 0.03)	5.1496	5.1435	5.1444	5.1496
(110, 0.5, 0.05, 0.20, 0.03)	1.9491	1.9461	1.9455	1.9509
(120, 0.5, 0.05, 0.20, 0.03)	0.6132	0.6113	0.6155	0.6153
(80, 1.0, 0.05, 0.20, 0.00)	20.0000	20.0000	20.0000	20.0000
(90, 1.0, 0.05, 0.20, 0.00)	11.4928	11.4924	11.4857	11.4929
(100, 1.0, 0.05, 0.20, 0.00)	6.0903	6.0893	6.0829	6.0905
(110, 1.0, 0.05, 0.20, 0.00)	2.9866	2.9856	2.9854	2.9868
(120, 1.0, 0.05, 0.20, 0.00)	1.3672	1.3654	1.3643	1.3674
(80, 1.0, 0.07, 0.40, 0.03)	24.0068	24.0054	23.9987	24.0057
(90, 1.0, 0.07, 0.40, 0.03)	18.2760	18.2741	18.2697	18.2746
(100, 1.0, 0.07, 0.40, 0.03)	13.7886	13.7879	13.7852	13.7873
(110, 1.0, 0.07, 0.40, 0.03)	10.3317	10.3312	10.3235	10.3307
(120, 1.0, 0.07, 0.40, 0.03)	7.7027	7.7014	7.7016	7.7018
(80, 3.0, 0.08, 0.20, 0.00)	20.0000	20.0000	20.0000	20.0000
(90, 3.0, 0.08, 0.20, 0.00)	11.6974	11.9029	11.6892	11.6977
(100, 3.0, 0.08, 0.20, 0.00)	6.9320	7.2527	6.9221	6.9321
(110, 3.0, 0.08, 0.20, 0.00)	4.1550	4.4841	4.1443	4.1548
(120, 3.0, 0.08, 0.20, 0.00)	2.5102	2.7760	2.4997	2.5102
(80, 3.0, 0.08, 0.20, 0.03)	20.1345	20.2396	20.1282	20.1349
(90, 3.0, 0.08, 0.20, 0.03)	12.9694	13.1798	12.9611	12.9697
(100, 3.0, 0.08, 0.20, 0.03)	8.3791	8.5901	8.3690	8.3792
(110, 3.0, 0.08, 0.20, 0.03)	5.4152	5.5223	5.4041	5.4151
(120, 3.0, 0.08, 0.20, 0.03)	3.4981	3.5983	3.4879	3.4979

TABLE 3: Comparison of RMSE and runtime.

Mesh size ( $N \times M$ )	Front fixing (runtime, RMSE)	MBM-FDM (runtime, RMSE)	Our Method (runtime, RMSE)
$125 \times 25$	(6.690, 3.10070)	(0.415, 0.04640)	(0.023, 0.10901)
$250 \times 50$	(26.767, 2.76880)	(1.922, 0.02510)	(0.109, 0.01395)
$500 \times 100$	(175.508, 3.61630)	(9.953, 0.01210)	(0.741, 0.00392)
$1000 \times 200$	(1223.615, 2.49460)	(57.531, 0.00620)	(5.997, 0.00089)
$2000 \times 400$	(7998.379, 2.41980)	(345.759, 0.00290)	(56.444, 0.00021)

The moving boundary approach developed by Mutheraman [21] requires iterations till the convergence of the boundaries. However, we emphasize that there is no iteration in our method. In such a rapidly changing environment, our straightforward method is a very powerful tool to understand financial market. Numerical study also shows that overall speed and accuracy comparisons have demonstrated the superiority of our method over other methods. Our method can be easily extended to other models under stochastic volatility and jump diffusion processes. These remain as topics for future research.

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