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Research Article

A Note on the Asymptotic Behavior of Parabolic Monge-Ampère Equations on Riemannian Manifolds

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We study the asymptotic behavior of the parabolic Monge-Ampère equation $\partial\varphi(x,t)/\partial t = \log(\det(g(x) + \mathrm{Hess}\varphi(x,t))/\det g(x)) - \lambda\varphi(x,t)$ in $\mathbb{M}\times(0,\infty)$, $\varphi(x,0) = \varphi_0(x)$ in $\mathbb{M}$, where $\mathbb{M}$ is a compact complete Riemannian manifold, λ is a positive real parameter, and $\varphi_0(x) : \mathbb{M} \to \mathbb{R}$ is a smooth function. We show a meaningful asymptotic result which is more general than those in Huisken, 1997.

1. Introduction

The main purpose of this paper is to study the asymptotic behavior of the parabolic Monge-Ampère equation:

$$\begin{aligned} \frac{\partial\varphi(x,t)}{\partial t} &= \log\left\{\frac{\det(g(x) + \mathrm{Hess}\varphi(x,t))}{\det g(x)}\right\} \\ &\quad - \lambda\varphi(x,t) \quad \text{in } \mathbb{M}\times(0,\infty), \\ \varphi(x,0) &= \varphi_0(x), \quad \text{in } \mathbb{M}, \end{aligned} \tag{1}$$

where $\mathbb{M}$ is a compact complete Riemannian manifold, λ is a positive real parameter, and $\varphi_0(x) : \mathbb{M} \to \mathbb{R}$ is a smooth function. We show a meaningful precisely asymptotic result which is more general than those in [1].

Monge-Ampère equations arise naturally from some problems in differential geometry. The existence and regularity of solutions to Monge-Ampère equations have been investigated by many mathematicians [1–8]. The long time existence and convergence of solution to (1) have been investigated in [1]. To some extent, we extend asymptotic result obtained in [1] in this paper. Hence, our main result is following analogue of Theorem 1.2 of [1].

Theorem 1. *Let φ be the solution of (1) with $\lambda > 0$. For $p > 1$, there exists $\delta > 0$ and $f > 0$ depending on φ_0 and $\|\nabla^\beta\varphi\|_{L^\infty}$ $(\beta = 0,1,2,3)$ such that*

$$\begin{aligned} &\int_{\mathbb{M}} \left(\varphi^p - \overline{\varphi^p}\right)^2 d\mu \\ &\quad \le f\exp\left(-2\left[\frac{(2p-1)\eta_1}{p} + p\lambda - \varepsilon(t)\right]t\right), \end{aligned} \tag{2}$$

where $\overline{\varphi^p}$ denotes the mean value of φ^p, η_1 is the first eigenvalue of the Laplacian, and $\varepsilon(t) \triangleq \exp(-\delta t)$.

Remark 2. If $p = 1$, Theorem 1 is in accordance with Theorem 1.2 of [1].

Lemma 3 (see [1]). *There exists positive constants $C_0(\varphi_0,\lambda)$ and C_1 depending on $(\mathbb{M}, g)$, φ_0, $\|\dot\varphi\|_{L^\infty}$, $\|\nabla^{\beta-\gamma}\varphi\|_{L^\infty}$ $(\beta = 0,1,2,3; \gamma = 1,2,3; \gamma \le \beta)$ such that*

$$\begin{aligned} \left|\dot\varphi\right| &\le C_0\exp(-\lambda t), \\ \left|\nabla^\beta\varphi\right|^2 &\le C_1\exp(-2\lambda t). \end{aligned} \tag{3}$$

Theorem 1 is proved in Section 2.

2. Asymptotic Behavior

Proof of Theorem 1. In local coordinates, we have the following evolution equation:

$$\begin{aligned}\frac{\partial \varphi}{\partial t} &= \log\left\{\frac{\det\left(g+\text{Hess}\varphi\right)}{\det g}\right\} - \lambda\varphi \\ &= \Delta\varphi + \log\det\left(g_{ij}+\nabla_{ij}\varphi\right) \\ &\quad - \log\det g_{ij} - \Delta\varphi - \lambda\varphi \\ &= \Delta\varphi + \int_0^1 \frac{d}{ds}\left\{\log\det\left(g_{ij}+s\nabla_{ij}\varphi\right)\right. \\ &\qquad \left. - (s-1)\Delta\varphi\right\} ds - \lambda\varphi.\end{aligned} \tag{4}$$

Now, setting

$$\begin{gathered}\overline{g}_s^{ij} \triangleq \left(g_{ij}+s\nabla_{ij}\right)^{-1}, \\ A \triangleq \iint_0^1 s\left|\nabla^2\varphi\right|^2_{\overline{g}_{\alpha s}} d\alpha\, ds.\end{gathered} \tag{5}$$

We rewrite (4) in more convenient notation as

$$\begin{aligned}\frac{\partial \varphi}{\partial t} &= \Delta\varphi + \int_0^1 \left(\overline{g}_s^{ij} - g^{ij}\right)\nabla_{ij}\varphi\, ds - \lambda\varphi \\ &= \Delta\varphi + \iint_0^1 \frac{d}{d\alpha}\left(g_{ij}+\alpha s\nabla_{ij}\varphi\right)^{-1}\nabla_{ij}\varphi\, d\alpha\, ds - \lambda\varphi \\ &= \Delta\varphi - \iint_0^1 \left(g_{ik}+\alpha s\nabla_{ik}\varphi\right)^{-1}\left(g_{jl}+\alpha s\nabla_{jl}\varphi\right)^{-1} \\ &\quad \times s\nabla_{kl}\varphi\nabla_{ij}\varphi\, d\alpha\, ds - \lambda\varphi \\ &= \Delta\varphi - \iint_0^1 s\left|\nabla^2\varphi\right|^2_{\overline{g}_{\alpha s}} d\alpha\, ds - \lambda\varphi \\ &= \Delta\varphi - A - \lambda\varphi.\end{aligned} \tag{6}$$

We want to apply Gronwall inequality and hence consider the following equation:

$$\begin{aligned}&\frac{\partial}{\partial t}\int_{\mathbb{M}}\left(\varphi^p - \overline{\varphi^p}\right)^2 d\mu \\ &\quad = 2p\int_{\mathbb{M}}\left(\varphi^p - \overline{\varphi^p}\right)\varphi^{p-1}\dot{\varphi}\, d\mu \\ &\quad = 2p\int_{\mathbb{M}}\left(\varphi^p - \overline{\varphi^p}\right)\varphi^{p-1}\left(\Delta\varphi - A - \lambda\varphi\right) d\mu \\ &\quad = 2p\int_{\mathbb{M}}\left(\varphi^p - \overline{\varphi^p}\right)\varphi^{p-1}\Delta\varphi\, d\mu \\ &\qquad - 2p\int_{\mathbb{M}}\left(\varphi^p - \overline{\varphi^p}\right)\varphi^{p-1}A\, d\mu \\ &\qquad - 2p\lambda\int_{\mathbb{M}}\left(\varphi^p - \overline{\varphi^p}\right)\varphi^p d\mu.\end{aligned} \tag{7}$$

Notice that

$$\int_{\mathbb{M}}\overline{\varphi^p}\left(\varphi^p - \overline{\varphi^p}\right) d\mu = 0. \tag{8}$$

We obtain

$$-2p\lambda\int_{\mathbb{M}}\left(\varphi^p - \overline{\varphi^p}\right)\varphi^p d\mu = -2p\lambda\int_{\mathbb{M}}\left(\varphi^p - \overline{\varphi^p}\right)^2 d\mu. \tag{9}$$

Furthermore we have

$$\begin{aligned}&2p\int_{\mathbb{M}}\left(\varphi^p - \overline{\varphi^p}\right)\varphi^{p-1}\Delta\varphi\, d\mu \\ &\quad = -2p\int_{\mathbb{M}}\nabla\left\{\left(\varphi^p - \overline{\varphi^p}\right)\varphi^{p-1}\right\}\nabla\varphi\, d\mu \\ &\quad = -2p\int_{\mathbb{M}}\left\{\nabla\left(\varphi^p - \overline{\varphi^p}\right)\varphi^{p-1}\right. \\ &\qquad \left. + (p-1)\varphi^{p-2}\nabla\varphi\left(\varphi^p - \overline{\varphi^p}\right)\right\}\nabla\varphi\, d\mu \\ &\quad = -2\int_{\mathbb{M}}\left|\nabla\left(\varphi^p - \overline{\varphi^p}\right)\right|^2 d\mu - \frac{2(p-1)}{p} \\ &\qquad \times \int_{\mathbb{M}}\left|\nabla\left(\varphi^p - \overline{\varphi^p}\right)\right|^2 d\mu \\ &\qquad + 2p(p-1)\overline{\varphi^p}\int_{\mathbb{M}}\varphi^{p-2}\left|\nabla\varphi\right|^2 d\mu \\ &\quad = -\frac{2(2p-1)}{p}\int_{\mathbb{M}}\left|\nabla\left(\varphi^p - \overline{\varphi^p}\right)\right|^2 d\mu \\ &\qquad + 2p(p-1)\overline{\varphi^p}\int_{\mathbb{M}}\varphi^{p-2}\left|\nabla\varphi\right|^2 d\mu.\end{aligned} \tag{10}$$

We use the Poincare inequality

$$\left\|\nabla\left(\varphi^p - \overline{\varphi^p}\right)\right\|_{L^2} \geq \eta_1\left\|\left(\varphi^p - \overline{\varphi^p}\right)\right\|_{L^2}. \tag{11}$$

It follows that

$$\begin{aligned}&2p\int_{\mathbb{M}}\left(\varphi^p - \overline{\varphi^p}\right)\varphi^{p-1}\Delta\varphi\, d\mu \\ &\quad \leq -\frac{2(2p-1)\eta_1}{p}\int_{\mathbb{M}}\left(\varphi^p - \overline{\varphi^p}\right)^2 d\mu \\ &\qquad + 2p(p-1)\overline{\varphi^p}\int_{\mathbb{M}}\varphi^{p-2}\left|\nabla\varphi\right|^2 d\mu.\end{aligned} \tag{12}$$

Moreover, we have that

$$\begin{aligned}&-2p\int_{\mathbb{M}}\left(\varphi^p - \overline{\varphi^p}\right)\varphi^{p-1}A\, d\mu \\ &\quad = 2p\int_0^1\int_{\mathbb{M}}\left(\overline{g}_s^{ij} - g^{ij}\right)\nabla_{ij}\varphi\left(\varphi^p - \overline{\varphi^p}\right)\varphi^{p-1} d\mu\end{aligned}$$

$$\begin{aligned}
&= -2p \int_0^1 \int_{\mathbb{M}} \nabla_i \left(\overline{g}_s^{ij} - g^{ij}\right) \nabla_j \varphi \left(\varphi^p - \overline{\varphi^p}\right) \varphi^{p-1} d\mu \\
&\quad - 2p \int_0^1 \int_{\mathbb{M}} \left(\overline{g}_s^{ij} - g^{ij}\right) \nabla_j \varphi \nabla_i \left(\varphi^p - \overline{\varphi^p}\right) \varphi^{p-1} d\mu \\
&\quad - 2p \int_0^1 \int_{\mathbb{M}} \left(\overline{g}_s^{ij} - g^{ij}\right) \nabla_j \varphi \left(\varphi^p - \overline{\varphi^p}\right) \nabla_i \left(\varphi^{p-1}\right) d\mu \\
&= -2 \int_0^1 \int_{\mathbb{M}} \nabla_i \left(\overline{g}_s^{ij} - g^{ij}\right) \nabla_j \left(\varphi^p - \overline{\varphi^p}\right) \left(\varphi^p - \overline{\varphi^p}\right) d\mu \\
&\quad - 2 \int_0^1 \int_{\mathbb{M}} \left(\overline{g}_s^{ij} - g^{ij}\right) \nabla_j \left(\varphi^p - \overline{\varphi^p}\right) \nabla_i \left(\varphi^p - \overline{\varphi^p}\right) d\mu \\
&\quad - 2p(p-1) \int_0^1 \int_{\mathbb{M}} \left(\overline{g}_s^{ij} - g^{ij}\right) \varphi^{2(p-1)} \nabla_i \varphi \nabla_j \varphi \, d\mu \\
&\quad + 2p(p-1) \int_0^1 \int_{\mathbb{M}} \left(\overline{g}_s^{ij} - g^{ij}\right) \varphi^{p-2} \nabla_i \varphi \nabla_j \varphi \overline{\varphi^p} \, d\mu \\
&= -2 \int_0^1 \int_{\mathbb{M}} \nabla_i \left(\overline{g}_s^{ij} - g^{ij}\right) \nabla_j \left(\varphi^p - \overline{\varphi^p}\right) \left(\varphi^p - \overline{\varphi^p}\right) d\mu \\
&\quad - \frac{2(2p-1)}{p} \int_0^1 \int_{\mathbb{M}} \left(\overline{g}_s^{ij} - g^{ij}\right) \nabla_j \left(\varphi^p - \overline{\varphi^p}\right) \nabla_i \\
&\quad \times \left(\varphi^p - \overline{\varphi^p}\right) d\mu \\
&\quad + 2p(p-1) \overline{\varphi^p} \int_0^1 \int_{\mathbb{M}} \left(\overline{g}_s^{ij} - g^{ij}\right) \varphi^{p-2} \nabla_i \varphi \nabla_j \varphi \, d\mu \\
&\leq 2 \int_0^1 \int_{\mathbb{M}} \left|\nabla \left(\overline{g}_s^{ij} - g^{ij}\right)\right| \cdot \left|\nabla \left(\varphi^p - \overline{\varphi^p}\right)\right| \cdot \left|\varphi^p - \overline{\varphi^p}\right| d\mu \\
&\quad + \frac{2(2p-1)}{p} \int_0^1 \int_{\mathbb{M}} \left|\overline{g}_s^{ij} - g^{ij}\right| \cdot \left|\nabla \left(\varphi^p - \overline{\varphi^p}\right)\right|^2 d\mu \\
&\quad + 2p(p-1) \left|\overline{\varphi^p}\right| \int_0^1 \int_{\mathbb{M}} \left|\left(\overline{g}_s^{ij} - g^{ij}\right)\right| \cdot \left|\varphi^{p-2}\right| \cdot \left|\nabla \varphi\right|^2 d\mu \\
&\leq 2C \sup \left|\nabla^3 \varphi\right| \\
&\quad \times \left(\int_{\mathbb{M}} \left|\nabla \left(\varphi^p - \overline{\varphi^p}\right)\right|^2 d\mu + \int_{\mathbb{M}} \left(\varphi^p - \overline{\varphi^p}\right)^2 d\mu \right) \\
&\quad + \frac{2C(2p-1)}{p} \sup \left|\nabla^2 \varphi\right| \int_{\mathbb{M}} \left|\nabla \left(\varphi^p - \overline{\varphi^p}\right)\right|^2 d\mu \\
&\quad + 2p(p-1) C \left|\overline{\varphi^p}\right| \sup \left|\nabla^2 \varphi\right| \int_{\mathbb{M}} \left|\varphi^{p-2}\right| \cdot \left|\nabla \varphi\right|^2 d\mu.
\end{aligned} \tag{13}$$

where C is always a constant that may change from line to line.

Substituting (9), (12), and (13) in the right-hand side of (7)

$$\begin{aligned}
&\frac{\partial}{\partial t} \int_{\mathbb{M}} \left(\varphi^p - \overline{\varphi^p}\right)^2 d\mu \\
&\quad \leq -2 \left[\frac{(2p-1)\eta_1}{p} + p\lambda \right] \int_{\mathbb{M}} \left(\varphi^p - \overline{\varphi^p}\right)^2 d\mu \\
&\qquad + 2p(p-1) \overline{\varphi^p} \int_{\mathbb{M}} \varphi^{p-2} \left|\nabla \varphi\right|^2 d\mu \\
&\qquad + 2C \sup \left|\nabla^3 \varphi\right| \left(\int_{\mathbb{M}} \left|\nabla \left(\varphi^p - \overline{\varphi^p}\right)\right|^2 d\mu \right. \\
&\qquad \left. + \int_{\mathbb{M}} \left(\varphi^p - \overline{\varphi^p}\right)^2 d\mu \right) \\
&\qquad + \frac{2C(2p-1)}{p} \sup \left|\nabla^2 \varphi\right| \int_{\mathbb{M}} \left|\nabla \left(\varphi^p - \overline{\varphi^p}\right)\right|^2 d\mu \\
&\qquad + 2p(p-1) C \left|\overline{\varphi^p}\right| \sup \left|\nabla^2 \varphi\right| \int_{\mathbb{M}} \left|\varphi^{p-2}\right| \cdot \left|\nabla \varphi\right|^2 d\mu.
\end{aligned} \tag{14}$$

By Lemma 3, that is, the exponential decay of $|\nabla^\beta \varphi|_{L^\infty}$ ($\beta = 0, 1, 2, 3$), it is easy to obtain the following.

For any $\varepsilon > 0$, there exists a $T(\varepsilon)$ such that

$$\begin{aligned}
&\frac{\partial}{\partial t} \int_{\mathbb{M}} \left(\varphi^p - \overline{\varphi^p}\right)^2 d\mu \\
&\quad \leq -2 \left[\frac{(2p-1)\eta_1}{p} + p\lambda - \varepsilon(t) \right] \\
&\qquad \times \int_{\mathbb{M}} \left(\varphi^p - \overline{\varphi^p}\right)^2 d\mu \\
&\qquad + 2\varepsilon \left(\int_{\mathbb{M}} \left|\nabla \left(\varphi^p - \overline{\varphi^p}\right)\right|^2 d\mu \right. \\
&\qquad \left. + \left|\overline{\varphi^p}\right| \int_{\mathbb{M}} \left|\varphi^{p-2}\right| \cdot \left|\nabla \varphi\right|^2 d\mu \right).
\end{aligned} \tag{15}$$

The Gronwall inequality yields

$$\begin{aligned}
&\int_{\mathbb{M}} \left(\varphi^p - \overline{\varphi^p}\right)^2 d\mu \\
&\quad \leq f \exp\left(-2 \left[\frac{(2p-1)\eta_1}{p} + p\lambda - \varepsilon(t) \right] t \right),
\end{aligned} \tag{16}$$

where the constant $f > 0$ depending on φ_0 and $\|\nabla^\beta \varphi\|_{L^\infty}$ ($\beta = 0, 1, 2, 3$).

Thus, the proof of Theorem 1 is completed. □

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