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Research Article

Some Common Coupled Fixed Point Results for Generalized Contraction in Complex-Valued Metric Spaces

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We introduce and study the notion of common coupled fixed points for a pair of mappings in complex valued metric space and demonstrate the existence and uniqueness of the common coupled fixed points in a complete complex-valued metric space in view of diverse contractive conditions. In addition, our investigations are well supported by nontrivial examples.

1. Introduction

Azam et al. [1] introduced the concept of complex-valued metric spaces and obtained sufficient conditions for the existence of common fixed points of a pair of contractive type mappings involving rational expressions. Subsequently, several authors have studied the existence and uniqueness of the fixed points and common fixed points of self-mappings in view of contrasting contractive conditions. Some of these investigations are noted in [2–26].

In [27], Bhaskar and Lakshmikantham introduced the concept of coupled fixed points for a given partially ordered set X. Recently Samet et al. [28, 29] proved that most of the coupled fixed point theorems (on ordered metric spaces) are in fact immediate consequences of well-known fixed point theorems in the literature. In this paper, we deal with the corresponding definition of coupled fixed point for mappings on a complex-valued metric space along with generalized contraction involving rational expressions. Our results extend and improve several fixed point theorems in the literature.

2. Preliminaries

Let $\mathbb{C}$ be the set of complex numbers and $z_1, z_2 \in \mathbb{C}$. Define a partial order $\precsim$ on $\mathbb{C}$ as follows:

$$z_1 \precsim z_2 \quad \text{iff } \operatorname{Re}(z_1) \leq \operatorname{Re}(z_2), \ \operatorname{Im}(z_1) \leq \operatorname{Im}(z_2). \tag{1}$$

Note that $0 \precsim z_1$, z_2 and $z_1 \neq z_2$, $z_1 \precsim z_2$ implies $|z_1| < |z_2|$.

Definition 1. Let X be a nonempty set. Suppose that the self-mapping $d : X \times X \to \mathbb{C}$ satisfies the following:

(1) $0 \precsim d(x, y)$, for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$;

(2) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(3) $d(x, y) \precsim d(x, z) + d(z, y)$, for all $x, y, z \in X$.

Then d is called a complex valued metric on X, and (X, d) is known as a complex valued metric space. A point $x \in X$ is called interior point of a set $A \subseteq X$ whenever, there exists $0 \prec r \in \mathbb{C}$ such that

$$B(x, r) = \{y \in X : d(x, y) \prec r\} \subseteq A. \tag{2}$$

A point $x \in X$ is a limit point of A whenever, for every $0 \prec r \in \mathbb{C}$,

$$B(x, r) \cap (A \setminus \{x\}) \neq \emptyset. \tag{3}$$

A is called open whenever each element of A is an interior point of A. Moreover, a subset $B \subseteq X$ is called closed whenever each limit point of B belongs to B. The family

$$F = \{B(x, r) : x \in X, 0 \prec r \in \mathbb{C}\} \tag{4}$$

is a subbasis for a Hausdorff topology τ on X.

Let $\{x_n\}$ be a sequence in X and $x \in X$. If for every $c \in \mathbb{C}$ with $0 \prec c$ there is $n_0 \in \mathbb{N}$ such that, for all $n > n_0$, $d(x_n, x) \prec c$, then $\{x_n\}$ is said to be convergent, $\{x_n\}$ converges to x, and x is the limit point of $\{x_n\}$. We denote this by $\lim_{n \to +\infty} x_n = x$, or $x_n \to x$, as $n \to +\infty$. If for every $c \in \mathbb{C}$ with $0 \prec c$ there is $n_0 \in \mathbb{N}$ such that, for all $n > n_0$, $d(x_n, x_{n+m}) \prec c$, then $\{x_n\}$ is called a Cauchy sequence in (X, d). If every Cauchy sequence is convergent in (X, d), then (X, d) is called a complete complex valued metric space. We require the following lemmas.

Lemma 2 (see [1]). *Let (X, d) be a complex valued metric space, and let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ converges to x if and only if $|d(x_n, x)| \to 0$ as $n \to +\infty$.*

Lemma 3 (see [1]). *Let (X, d) be a complex valued metric space, and let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is a Cauchy sequence if and only if $|d(x_n, x_{n+m})| \to 0$ as $n \to +\infty$.*

Definition 4 (see [27]). An element $(x, y) \in X \times X$ is called a coupled fixed point of $T : X \times X \to X$ if

$$x = T(x, y), \qquad y = T(y, x). \tag{5}$$

Definition 5. An element $(x, y) \in X \times X$ is called a coupled coincidence point of $S, T : X \times X \to X$ if

$$S(x, y) = T(x, y), \qquad S(y, x) = T(y, x). \tag{6}$$

Example 6. Let $X = \mathbb{R}$ and $S, T : X \times X \to X$ defined as $S(x, y) = x^2 y^2$ and $T(x, y) = (4/3)(x + y)$ for all $x, y \in X$. Then $(0, 0)$, $(1, 2)$, and $(2, 1)$ are coupled coincidence points of S and T.

Example 7. Let $X = \mathbb{R}$ and $S, T : X \times X \to X$ defined as $S(x, y) = x + y + \sin(x + y)$ and $T(x, y) = x + y + xy + \cos(x + y)$ for all $x, y \in X$. Then $(0, \pi/4)$ and $(\pi/4, 0)$ are coupled coincidence points of S and T.

Definition 8. An element $(x, y) \in X \times X$ is called a common coupled fixed point of $S, T : X \times X \to X$ if

$$x = S(x, y) = T(x, y), \qquad y = S(y, x) = T(y, x). \tag{7}$$

Example 9. Let $X = \mathbb{R}$ and $S, T : X \times X \to X$ defined as $S(x, y) = x((x+(y-1)^2)/2)$ and $T(x, y) = x(\sqrt{x^2 + y^2 + 4} - 2)$ for all $x, y \in X$. Then $(0, 0)$, $(1, 2)$, and $(2, 1)$ are common coupled fixed points of S and T.

In the following, we provide common coupled fixed point theorem for a pair of mappings satisfying a rational inequality in complex valued metric spaces.

Theorem 10. *Let (X, d) be a complete complex-valued metric space, and let the mappings $S, T : X \times X \to X$ satisfy*

$$\begin{aligned} d(S(x, y), T(u, v)) \precsim{} & \frac{\alpha(d(x, u) + d(y, v))}{2} \\ & + (\beta d(x, S(x, y)) d(u, T(u, v)) \\ & \quad + \gamma d(u, S(x, y)) d(x, T(u, v))) \\ & \times (1 + d(x, u) + d(y, v))^{-1} \end{aligned} \tag{8}$$

for all $x, y, u, v \in X$ and α, β, and γ are nonnegative reals with $\alpha + \beta + \gamma < 1$. Then S and T have a unique common coupled fixed point.

Proof. Let x_0 and y_0 be arbitrary points in X. Define $x_{2k+1} = S(x_{2k}, y_{2k})$, $y_{2k+1} = S(y_{2k}, x_{2k})$ and $x_{2k+2} = T(x_{2k+1}, y_{2k+1})$, $y_{2k+2} = T(y_{2k+1}, x_{2k+1})$, for $k = 0, 1, \ldots$

Then,

$$\begin{aligned} & d(x_{2k+1}, x_{2k+2}) \\ & \quad = d(S(x_{2k}, y_{2k}), T(x_{2k+1}, y_{2k+1})) \\ & \quad \precsim \frac{\alpha(d(x_{2k}, x_{2k+1}) + d(y_{2k}, y_{2k+1}))}{2} \\ & \qquad + (\beta d(x_{2k}, S(x_{2k}, y_{2k})) \\ & \qquad \quad \times d(x_{2k+1}, T(x_{2k+1}, y_{2k+1}))) \\ & \qquad \times (1 + d(x_{2k}, x_{2k+1}) + d(y_{2k}, y_{2k+1}))^{-1} \\ & \qquad + (\gamma d(x_{2k+1}, S(x_{2k}, y_{2k})) \\ & \qquad \quad \times d(x_{2k}, T(x_{2k+1}, y_{2k+1}))) \\ & \qquad \times (1 + d(x_{2k}, x_{2k+1}) + d(y_{2k}, y_{2k+1}))^{-1} \\ & \quad \precsim \frac{\alpha(d(x_{2k}, x_{2k+1}) + d(y_{2k}, y_{2k+1}))}{2} \\ & \qquad + \frac{\beta d(x_{2k}, x_{2k+1}) d(x_{2k+1}, x_{2k+2})}{1 + d(x_{2k}, x_{2k+1}) + d(y_{2k}, y_{2k+1})} \\ & \qquad + \frac{\gamma d(x_{2k+1}, x_{2k+1}) d(x_{2k}, x_{2k+2})}{1 + d(x_{2k}, x_{2k+1}) + d(y_{2k}, y_{2k+1})} \end{aligned}$$

$$\preceq \frac{\alpha\left(d\left(x_{2k},x_{2k+1}\right)+d\left(y_{2k},y_{2k+1}\right)\right)}{2} + \frac{\beta d\left(x_{2k},x_{2k+1}\right)d\left(x_{2k+1},x_{2k+2}\right)}{1+d\left(x_{2k},x_{2k+1}\right)+d\left(y_{2k},y_{2k+1}\right)}, \tag{9}$$

which implies that

$$\left|d\left(x_{2k+1},x_{2k+2}\right)\right| \leq \frac{\alpha\left|d\left(x_{2k},x_{2k+1}\right)+d\left(y_{2k},y_{2k+1}\right)\right|}{2} + \frac{\beta\left|d\left(x_{2k},x_{2k+1}\right)d\left(x_{2k+1},x_{2k+2}\right)\right|}{\left|1+d\left(x_{2k},x_{2k+1}\right)+d\left(y_{2k},y_{2k+1}\right)\right|}. \tag{10}$$

Since $|1 + d(x_{2k}, x_{2k+1}) + d(y_{2k}, y_{2k+1})| > |d(x_{2k}, x_{2k+1})|$, so we get

$$\begin{aligned}\left|d\left(x_{2k+1},x_{2k+2}\right)\right| &\leq \frac{\alpha\left|d\left(x_{2k},x_{2k+1}\right)\right|+\alpha\left|d\left(y_{2k},y_{2k+1}\right)\right|}{2} \\ &\quad+\beta\left|d\left(x_{2k+1},x_{2k+2}\right)\right|,\end{aligned} \tag{11}$$

and hence

$$\begin{aligned}\left|d\left(x_{2k+1},x_{2k+2}\right)\right| &\leq \frac{1}{2}\left(\frac{\alpha}{1-\beta}\right)\left|d\left(x_{2k},x_{2k+1}\right)\right| \\ &\quad+\frac{1}{2}\left(\frac{\alpha}{1-\beta}\right)\left|d\left(y_{2k},y_{2k+1}\right)\right|.\end{aligned} \tag{12}$$

Similarly, one can show that

$$\begin{aligned}\left|d\left(y_{2k+1},y_{2k+2}\right)\right| &\leq \frac{1}{2}\left(\frac{\alpha}{1-\beta}\right)\left|d\left(y_{2k},y_{2k+1}\right)\right| \\ &\quad+\frac{1}{2}\left(\frac{\alpha}{1-\beta}\right)\left|d\left(x_{2k},x_{2k+1}\right)\right|.\end{aligned} \tag{13}$$

Also,

$$\begin{aligned} &d\left(x_{2k+2},x_{2k+3}\right) \\ &\quad= d\left(T\left(x_{2k+1},y_{2k+1}\right),S\left(x_{2k+2},y_{2k+2}\right)\right) \\ &\quad\preceq \frac{\alpha\left(d\left(x_{2k+1},x_{2k+2}\right)+d\left(y_{2k+1},y_{2k+2}\right)\right)}{2} \\ &\qquad+\left(\beta d\left(x_{2k+1},T\left(x_{2k+1},y_{2k+1}\right)\right)\right. \\ &\qquad\quad\left.\times d\left(x_{2k+2},S\left(x_{2k+2},y_{2k+2}\right)\right)\right) \\ &\qquad\times\left(1+d\left(x_{2k+1},x_{2k+2}\right)+d\left(y_{2k+1},y_{2k+2}\right)\right)^{-1} \\ &\qquad+\left(\gamma d\left(x_{2k+2},T\left(x_{2k+1},y_{2k+1}\right)\right)\right. \\ &\qquad\quad\left.\times d\left(x_{2k+1},S\left(x_{2k+2},y_{2k+2}\right)\right)\right) \\ &\qquad\times\left(1+d\left(x_{2k+1},x_{2k+2}\right)+d\left(y_{2k+1},y_{2k+2}\right)\right)^{-1} \\ &\quad\preceq \frac{\alpha\left(d\left(x_{2k+1},x_{2k+2}\right)+d\left(y_{2k+1},y_{2k+2}\right)\right)}{2} \\ &\qquad+\frac{\beta d\left(x_{2k+1},x_{2k+2}\right)d\left(x_{2k+2},x_{2k+3}\right)}{1+d\left(x_{2k+1},x_{2k+2}\right)+d\left(y_{2k+1},y_{2k+2}\right)} \\ &\qquad+\frac{\gamma d\left(x_{2k+2},x_{2k+2}\right)d\left(x_{2k+1},x_{2k+3}\right)}{1+d\left(x_{2k+1},x_{2k+2}\right)+d\left(y_{2k+1},y_{2k+2}\right)} \\ &\quad\preceq \frac{\alpha\left(d\left(x_{2k+1},x_{2k+2}\right)+d\left(y_{2k+1},y_{2k+2}\right)\right)}{2} \\ &\qquad+\frac{\beta d\left(x_{2k+1},x_{2k+2}\right)d\left(x_{2k+2},x_{2k+3}\right)}{1+d\left(x_{2k+1},x_{2k+2}\right)+d\left(y_{2k+1},y_{2k+2}\right)},\end{aligned} \tag{14}$$

so that

$$\left|d\left(x_{2k+2},x_{2k+3}\right)\right| \leq \frac{\alpha\left|d\left(x_{2k+1},x_{2k+2}\right)+d\left(y_{2k+1},y_{2k+2}\right)\right|}{2} + \frac{\beta\left|d\left(x_{2k+1},x_{2k+2}\right)\right|\left|d\left(x_{2k+2},x_{2k+3}\right)\right|}{\left|1+d\left(x_{2k+1},x_{2k+2}\right)+d\left(y_{2k+1},y_{2k+2}\right)\right|}. \tag{15}$$

As $|1 + d(x_{2k+1}, x_{2k+2}) + d(y_{2k+1}, y_{2k+2})| > |d(x_{2k+1}, x_{2k+2})|$, therefore

$$\begin{aligned}\left|d\left(x_{2k+2},x_{2k+3}\right)\right| &\leq \frac{1}{2}\left(\frac{\alpha}{1-\beta}\right)\left|d\left(x_{2k+1},x_{2k+2}\right)\right| \\ &\quad+\frac{1}{2}\left(\frac{\alpha}{1-\beta}\right)\left|d\left(y_{2k+1},y_{2k+2}\right)\right|.\end{aligned} \tag{16}$$

Similarly, one can show that

$$\begin{aligned}\left|d\left(y_{2k+2},y_{2k+3}\right)\right| &\leq \frac{\alpha}{1-\beta}\left|d\left(y_{2k+1},y_{2k+2}\right)\right| \\ &\quad+\frac{1}{2}\left(\frac{\alpha}{1-\beta}\right)\left|d\left(x_{2k+1},x_{2k+2}\right)\right|.\end{aligned} \tag{17}$$

Adding (12)–(17), we get

$$\begin{aligned}&\left|d\left(x_{2k+1}, x_{2k+2}\right)\right| + \left|d\left(y_{2k+1}, y_{2k+2}\right)\right| \\ &\quad \leq \frac{\alpha}{1-\beta}\left|d\left(x_{2k}, x_{2k+1}\right)\right| + \frac{\alpha}{1-\beta}\left|d\left(y_{2k}, y_{2k+1}\right)\right| \\ &\left|d\left(x_{2k+2}, x_{2k+3}\right)\right| + \left|d\left(y_{2k+2}, y_{2k+3}\right)\right| \\ &\quad \leq \frac{\alpha}{1-\beta}\left|d\left(x_{2k+1}, x_{2k+2}\right)\right| + \frac{\alpha}{1-\beta}\left|d\left(y_{2k+1}, y_{2k+2}\right)\right|.\end{aligned} \tag{18}$$

If $h = \alpha/(1-\beta) < 1$, then from (18), we get

$$\begin{aligned}&\left|d\left(x_n, x_{n+1}\right)\right| + \left|d\left(y_n, y_{n+1}\right)\right| \\ &\quad \leq h\left(\left|d\left(x_{n-1}, x_n\right)\right| + \left|d\left(y_{n-1}, y_n\right)\right|\right) \\ &\quad \leq \cdots \leq h^n\left(\left|d\left(x_0, x_1\right)\right| + \left|d\left(y_0, y_1\right)\right|\right).\end{aligned} \tag{19}$$

Now if $|d(x_n, x_{n+1})| + |d(y_n, y_{n+1})| = \delta_n$, then

$$\delta_n \leq h\delta_{n-1} \leq \cdots \leq h^n \delta_0. \tag{20}$$

Without loss of generality, we take $m > n$. Since $0 \leq h < 1$, so we get

$$\begin{aligned}&\left|d\left(x_n, x_m\right)\right| + \left|d\left(y_n, y_m\right)\right| \\ &\quad \leq \left|d\left(x_n, x_{n+1}\right)\right| + \left|d\left(y_n, y_{n+1}\right)\right| + \cdots \\ &\qquad + \left|d\left(x_{m-1}, x_m\right)\right| + \left|d\left(y_{m-1}, y_m\right)\right| \\ &\quad \leq \left[h^n\delta_0 + h^{n+1}\delta_0 + \cdots + h^{m-1}\delta_0\right] \\ &\quad \leq \sum_{i=n}^{m-1} h^i \delta_0 \longrightarrow 0, \quad \text{as } m, n \longrightarrow +\infty.\end{aligned} \tag{21}$$

This implies that $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences in X. Since X is complete, there exists $x, y \in X$ such that $x_n \to x$ and $y_n \to y$ as $n \to +\infty$. We now show that $x = S(x, y)$ and $y = S(y, x)$. We suppose on the contrary that $x \neq S(x, y)$ and $y \neq S(y, x)$ so that $0 \prec d(x, S(x, y)) = l_1$ and $0 \prec d(y, S(y, x)) = l_2$; we would then have

$$\begin{aligned}l_1 &= d\left(x, S\left(x, y\right)\right) \preceq d\left(x, x_{2k+2}\right) + d\left(x_{2k+2}, S\left(x, y\right)\right) \\ &\preceq d\left(x, x_{2k+2}\right) + d\left(T\left(x_{2k+1}, y_{2k+1}\right), S\left(x, y\right)\right) \\ &\preceq d\left(x, x_{2k+2}\right) + \frac{\alpha\left(d\left(x_{2k+1}, x\right) + d\left(y_{2k+1}, y\right)\right)}{2} \\ &\quad + \frac{\beta d\left(x_{2k+1}, T\left(x_{2k+1}, y_{2k+1}\right)\right) d\left(x, S\left(x, y\right)\right)}{1 + d\left(x_{2k+1}, x\right) + d\left(y_{2k+1}, y\right)} \\ &\quad + \frac{\gamma d\left(x, T\left(x_{2k+1}, y_{2k+1}\right)\right) d\left(x_{2k+1}, S\left(x, y\right)\right)}{1 + d\left(x_{2k+1}, x\right) + d\left(y_{2k+1}, y\right)} \\ &= d\left(x, x_{2k+2}\right) + \frac{\alpha\left(d\left(x_{2k+1}, x\right) + d\left(y_{2k+1}, y\right)\right)}{2} \\ &\quad + \frac{\beta d\left(x_{2k+1}, x_{2k+2}\right) d\left(x, S\left(x, y\right)\right)}{1 + d\left(x_{2k+1}, x\right) + d\left(y_{2k+1}, y\right)} \\ &\quad + \frac{\gamma d\left(x, x_{2k+2}\right) d\left(x_{2k+1}, S\left(x, y\right)\right)}{1 + d\left(x_{2k+1}, x\right) + d\left(y_{2k+1}, y\right)}\end{aligned} \tag{22}$$

so that

$$\begin{aligned}\left|l_1\right| &\leq \left|d\left(x, x_{2k+2}\right)\right| + \frac{\alpha\left|d\left(x_{2k+1}, x\right) + d\left(y_{2k+1}, y\right)\right|}{2} \\ &\quad + \frac{\beta\left|d\left(x_{2k+1}, x_{2k+2}\right)\right|\left|d\left(x, S\left(x, y\right)\right)\right|}{\left|1 + d\left(x_{2k+1}, x\right) + d\left(y_{2k+1}, y\right)\right|} \\ &\quad + \frac{\gamma\left|d\left(x, x_{2k+2}\right)\right|\left|d\left(x_{2k+1}, S\left(x, y\right)\right)\right|}{\left|1 + d\left(x_{2k+1}, x\right) + d\left(y_{2k+1}, y\right)\right|}.\end{aligned} \tag{23}$$

By taking $k \to +\infty$, we get $|d(x, S(x, y))| = 0$ which is a contradiction so that $x = S(x, y)$. Similarly, one can prove that $y = S(y, x)$. It follows similarly that $x = T(x, y)$ and $y = T(y, x)$. So we have proved that (x, y) is a common coupled fixed point of S and T. We now show that S and T have a unique common coupled fixed point. For this, assume that $(x^*, y^*) \in X$ is a second common coupled fixed point of S and T. Then

$$\begin{aligned}d\left(x, x^*\right) &= d\left(S\left(x, y\right), T\left(x^*, y^*\right)\right) \\ &\preceq \frac{\alpha\left(d\left(x, x^*\right) + d\left(y, y^*\right)\right)}{2} \\ &\quad + \frac{\beta d\left(x, S\left(x, y\right)\right) d\left(x^*, T\left(x^*, y^*\right)\right)}{1 + d\left(x, x^*\right) + d\left(y, y^*\right)} \\ &\quad + \frac{\gamma d\left(x, T\left(x^*, y^*\right)\right) d\left(x^*, S\left(x, y\right)\right)}{1 + d\left(x, x^*\right) + d\left(y, y^*\right)}\end{aligned}$$

$$\preceq \frac{\alpha(d(x,x^*)+d(y,y^*))}{2} + \frac{\beta d(x,x)\,d(x^*,x^*)}{1+d(x,x^*)+d(y,y^*)} + \frac{\gamma d(x,x^*)\,d(x^*,x)}{1+d(x,x^*)+d(y,y^*)} \tag{24}$$

so that

$$|d(x,x^*)| \le \left|\frac{\alpha(d(x,x^*)+d(y,y^*))}{2}\right| + \frac{\gamma\,|d(x,x^*)|\,|d(x^*,x)|}{|1+d(x,x^*)+d(y,y^*)|}. \tag{25}$$

Since $|1+d(x,x^*)+d(y,y^*)| > |d(x,x^*)|$, so we get

$$\begin{aligned}|d(x,x^*)| &\le \left|\frac{\alpha(d(x,x^*)+d(y,y^*))}{2}\right| + \gamma\,|d(x,x^*)| \\ &= \left(\frac{\alpha}{2-\alpha-2\gamma}\right)|d(y,y^*)|.\end{aligned} \tag{26}$$

Similarly, one can easily prove that

$$|d(y,y^*)| \le \left(\frac{\alpha}{2-\alpha-2\gamma}\right)|d(x,x^*)|. \tag{27}$$

If we add (26) and (27), we get

$$\begin{aligned}&|d(x,x^*)| + |d(y,y^*)| \\ &\quad\le \left(\frac{\alpha}{2-\alpha-2\gamma}\right)(|d(x,x^*)| + |d(y,y^*)|),\end{aligned} \tag{28}$$

which is a contradiction because $\alpha+\beta+\gamma<1$. Thus, we get $x^* = x$ and $y^* = y$, which proves the uniqueness of common coupled fixed point of S and T. □

By setting $S = T$ in Theorem 10, one deduces the following.

Corollary 11. *Let (X,d) be a complete complex-valued metric space, and let the mapping $T : X \times X \to X$ satisfy*

$$\begin{aligned}&d(T(x,y),T(u,v)) \\ &\quad\preceq \frac{\alpha(d(x,u)+d(y,v))}{2} \\ &\qquad+ (\beta d(x,T(x,y))\,d(u,T(u,v)) \\ &\qquad\quad+ \gamma d(u,T(x,y))\,d(x,T(u,v))) \\ &\qquad\times(1+d(x,u)+d(y,v))^{-1}\end{aligned} \tag{29}$$

for all $x, y, u, v \in X$, where α, β, and γ are nonnegative reals with $\alpha+\beta+\gamma<1$. Then T has a unique coupled fixed point.

Corollary 12. *Let (X,d) be a complete complex-valued metric space, and let the mapping $T : X \times X \to X$ satisfy*

$$\begin{aligned}&d(T^n(x,y),T^n(u,v)) \\ &\quad\preceq \frac{\alpha(d(x,u)+d(y,v))}{2} \\ &\qquad+ (\beta d(x,T^n(x,y))\,d(u,T^n(u,v)) \\ &\qquad\quad+ \gamma d(u,T^n(x,y))\,d(x,T^n(u,v))) \\ &\qquad\times(1+d(x,u)+d(y,v))^{-1}\end{aligned} \tag{30}$$

for all $x, y, u, v \in X$, where α, β, and γ are nonnegative reals with $\alpha+\beta+\gamma<1$. Then, T has a unique coupled fixed point.

Theorem 13. *Let (X,d) be a complete complex-valued metric space, and let the mappings $S, T : X \times X \to X$ satisfy*

$$d(S(x,y),T(u,v)) \preceq \begin{cases} \dfrac{\alpha(d(x,u)+d(y,v))}{2} + \dfrac{\beta d(x,S(x,y))\,d(u,T(u,v))}{d(x,T(u,v))+d(u,S(x,y))+d(x,u)+d(y,v)}, & \text{if } D \neq 0 \\ 0, & \text{if } D = 0 \end{cases} \tag{31}$$

for all $x, y, u, v \in X$, where $D = d(x,T(u,y)) + d(u,S(x,y)) + d(x,u) + d(y,v)$ and α, β are nonnegative reals with $\alpha+\beta<1$. Then S and T have a unique common coupled fixed point.

Proof. Let x_0 and y_0 be arbitrary points in X. Define $x_{2k+1} = S(x_{2k}, y_{2k})$, $y_{2k+1} = S(y_{2k}, x_{2k})$ and $x_{2k+2} = T(x_{2k+1}, y_{2k+1})$, $y_{2k+2} = T(y_{2k+1}, x_{2k+1})$, for $k = 0, 1, \ldots$.

Now, we assume that

$$\begin{aligned}D_S(x_{2k},y_{2k}) &= d(x_{2k},T(x_{2k+1},y_{2k+1})) \\ &\quad+ d(x_{2k+1},S(x_{2k},y_{2k})) \\ &\quad+ d(x_{2k},x_{2k+1}) + d(y_{2k},y_{2k+1}) \\ &= d(x_{2k},x_{2k+2}) + d(x_{2k},x_{2k+1}) \\ &\quad+ d(y_{2k},y_{2k+1}) \neq 0, \\ D_S(y_{2k},x_{2k}) &= d(y_{2k},T(y_{2k+1},x_{2k+1})) \\ &\quad+ d(y_{2k+1},S(y_{2k},x_{2k})) \\ &\quad+ d(x_{2k},x_{2k+1}) + d(y_{2k},y_{2k+1}) \\ &= d(y_{2k},y_{2k+2}) + d(x_{2k},x_{2k+1}) \\ &\quad+ d(y_{2k},y_{2k+1}) \neq 0.\end{aligned} \tag{32}$$

Then,

$$\begin{aligned} d(x_{2k+1}, x_{2k+2}) &= d(S(x_{2k}, y_{2k}), T(x_{2k+1}, y_{2k+1})) \\ &\preceq \frac{\alpha(d(x_{2k}, x_{2k+1}) + d(y_{2k}, y_{2k+1}))}{2} \\ &\quad + \frac{\beta d(x_{2k}, S(x_{2k}, y_{2k}))\, d(x_{2k+1}, T(x_{2k+1}, y_{2k+1}))}{D_S(x_{2k}, y_{2k})} \\ &= \frac{\alpha(d(x_{2k}, x_{2k+1}) + d(y_{2k}, y_{2k+1}))}{2} \\ &\quad + (\beta d(x_{2k}, x_{2k+1})\, d(x_{2k+1}, x_{2k+2})) \\ &\quad \times (d(x_{2k}, x_{2k+2}) + d(x_{2k}, x_{2k+1}) \\ &\quad + d(y_{2k}, y_{2k+1}))^{-1} \end{aligned} \tag{33}$$

which implies that

$$\begin{aligned} |d(x_{2k+1}, x_{2k+2})| &\leq \frac{\alpha |d(x_{2k}, x_{2k+1}) + d(y_{2k}, y_{2k+1})|}{2} \\ &\quad + (\beta |d(x_{2k}, x_{2k+1})| \, |d(x_{2k+1}, x_{2k+2})|) \\ &\quad \times (|d(x_{2k}, x_{2k+2}) + d(x_{2k}, x_{2k+1}) \\ &\quad + d(y_{2k}, y_{2k+1})|)^{-1} \\ &\leq \frac{\alpha |d(x_{2k}, x_{2k+1}) + d(y_{2k}, y_{2k+1})|}{2} \\ &\quad + \beta |d(x_{2k}, x_{2k+1})| \end{aligned} \tag{34}$$

as

$$\begin{aligned} |d(x_{2k+1}, x_{2k+2})| \leq |d(x_{2k+1}, x_{2k}) + d(x_{2k}, x_{2k+2}) + d(y_{2k}, y_{2k+1})|. \end{aligned} \tag{35}$$

Therefore,

$$\begin{aligned} |d(x_{2k+1}, x_{2k+2})| \leq \frac{(\alpha + 2\beta)}{2} |d(x_{2k}, x_{2k+1})| + \frac{\alpha}{2} |d(y_{2k}, y_{2k+1})|. \end{aligned} \tag{36}$$

Similarly, one can easily prove that

$$\begin{aligned} |d(y_{2k+1}, y_{2k+2})| \leq \frac{(\alpha + 2\beta)}{2} |d(y_{2k}, y_{2k+1})| + \frac{\alpha}{2} |d(x_{2k}, x_{2k+1})|. \end{aligned} \tag{37}$$

Now, if

$$\begin{aligned} D_T(x_{2k+1}, y_{2k+1}) &= d(x_{2k+2}, T(x_{2k+1}, y_{2k+1})) \\ &\quad + d(x_{2k+1}, S(x_{2k+2}, y_{2k+2})) \\ &\quad + d(x_{2k+2}, x_{2k+1}) + d(y_{2k+2}, y_{2k+1}) \\ &= d(x_{2k+1}, x_{2k+3}) + d(x_{2k+2}, x_{2k+1}) \\ &\quad + d(y_{2k+2}, y_{2k+1}) \neq 0, \end{aligned} \tag{38}$$

we get

$$\begin{aligned} d(x_{2k+2}, x_{2k+3}) &= d(T(x_{2k+1}, y_{2k+1}), S(x_{2k+2}, y_{2k+2})) \\ &\preceq \frac{\alpha(d(x_{2k+2}, x_{2k+1}) + d(y_{2k+2}, y_{2k+1}))}{2} \\ &\quad + (\beta d(x_{2k+2}, S(x_{2k+2}, y_{2k+2})) \\ &\quad \times d(x_{2k+1}, T(x_{2k+1}, y_{2k+1}))) \\ &\quad \times (D_T(x_{2k+1}, y_{2k+1}))^{-1} \\ &= \frac{\alpha(d(x_{2k+2}, x_{2k+1}) + d(y_{2k+2}, y_{2k+1}))}{2} \\ &\quad + (\beta d(x_{2k+2}, x_{2k+3})\, d(x_{2k+1}, x_{2k+2})) \\ &\quad \times (d(x_{2k+1}, x_{2k+3}) + d(x_{2k+2}, x_{2k+1}) \\ &\quad + d(y_{2k+2}, y_{2k+1}))^{-1}, \end{aligned} \tag{39}$$

which implies that

$$\begin{aligned} |d(x_{2k+2}, x_{2k+3})| &\leq \alpha \left| \frac{\alpha(d(x_{2k+2}, x_{2k+1}) + d(y_{2k+2}, y_{2k+1}))}{2} \right| \\ &\quad + (\beta |d(x_{2k+2}, x_{2k+3})| \, |d(x_{2k+1}, x_{2k+2})|) \\ &\quad \times (|d(x_{2k+1}, x_{2k+3}) + d(x_{2k+2}, x_{2k+1}) \\ &\quad + d(y_{2k+2}, y_{2k+1})|)^{-1} \\ &\leq \alpha \left| \frac{\alpha(d(x_{2k+2}, x_{2k+1}) + d(y_{2k+2}, y_{2k+1}))}{2} \right| \\ &\quad + \beta |d(x_{2k+1}, x_{2k+2})| \end{aligned} \tag{40}$$

as

$$\begin{aligned} |d(x_{2k+2}, x_{2k+3})| &\leq |d(x_{2k+2}, x_{2k+1}) + d(x_{2k+1}, x_{2k+3}) \\ &\quad + d(y_{2k+2}, y_{2k+1})|. \end{aligned} \tag{41}$$

Therefore,

$$\begin{aligned}\left|d\left(x_{2k+2},x_{2k+3}\right)\right| &\leq \frac{\alpha\left|d\left(x_{2k+2},x_{2k+1}\right)\right|}{2}+\frac{\alpha\left|d\left(y_{2k+2},y_{2k+1}\right)\right|}{2}\\ &\quad+\beta\left|d\left(x_{2k+1},x_{2k+2}\right)\right|\\ &=\frac{(\alpha+2\beta)}{2}\left|d\left(x_{2k+1},x_{2k+2}\right)\right|\\ &\quad+\frac{\alpha}{2}\left|d\left(y_{2k+1},y_{2k+2}\right)\right|.\end{aligned}\tag{42}$$

Similarly, if $D_T(y_{2k+1},x_{2k+1})\neq 0$, one can easily prove that

$$\begin{aligned}\left|d\left(y_{2k+2},y_{2k+3}\right)\right| \leq \frac{(\alpha+2\beta)}{2}\left|d\left(y_{2k+1},y_{2k+2}\right)\right|\\ +\frac{\alpha}{2}\left|d\left(x_{2k+1},x_{2k+2}\right)\right|.\end{aligned}\tag{43}$$

Adding the inequalities (36)–(43), we get

$$\begin{aligned}&\left|d\left(x_{2k+1},x_{2k+2}\right)\right|+\left|d\left(y_{2k+1},y_{2k+2}\right)\right|\\ &\quad\leq(\alpha+\beta)\left(\left|d\left(x_{2k},x_{2k+1}\right)\right|+\left|d\left(y_{2k},y_{2k+1}\right)\right|\right).\\ &\left|d\left(x_{2k+2},x_{2k+3}\right)\right|+\left|d\left(y_{2k+2},y_{2k+3}\right)\right|\\ &\quad\leq(\alpha+\beta)\left(\left|d\left(x_{2k+1},x_{2k+2}\right)\right|+\left|d\left(y_{2k+1},y_{2k+2}\right)\right|\right).\end{aligned}\tag{44}$$

If $h=(\alpha+\beta)<1$, then, from (44), we get

$$\begin{aligned}&\left|d\left(x_n,x_{n+1}\right)\right|+\left|d\left(y_n,y_{n+1}\right)\right|\\ &\quad\leq h\left(\left|d\left(x_{n-1},x_n\right)\right|+\left|d\left(y_{n-1},y_n\right)\right|\right)\\ &\quad\leq\cdots\leq h^n\left(\left|d\left(x_0,x_1\right)\right|\right.\\ &\qquad\left.+\left|\ d\left(y_0,y_1\right)\right|\right).\end{aligned}\tag{45}$$

Now if $|d(x_n,x_{n+1})|+|d(y_n,y_{n+1})|=\delta_n$, then

$$\delta_n\leq h\delta_{n-1}\leq\cdots\leq h^n\delta_0.\tag{46}$$

Without loss of generality, we take $m>n$. Since $0\leq h<1$, so we get

$$\begin{aligned}&\left|d\left(x_n,x_m\right)\right|+\left|d\left(y_n,y_m\right)\right|\\ &\quad\leq\left|d\left(x_n,x_{n+1}\right)\right|+\left|d\left(y_n,y_{n+1}\right)\right|+\cdots\\ &\qquad+\left|d\left(x_{m-1},x_m\right)\right|+\left|d\left(y_{m-1},y_m\right)\right|\\ &\quad\leq\left[h^n\delta_0+h^{n+1}\delta_0+\cdots+h^{m-1}\delta_0\right]\\ &\quad\leq\sum_{i=n}^{m-1}h^i\delta_0\longrightarrow 0,\quad\text{as } m,n\longrightarrow+\infty.\end{aligned}\tag{47}$$

This implies that $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences in X. Since X is complete, so there exists $x,y\in X$ such that $x_n\to x$ and $y_n\to y$ as $n\to+\infty$. We now show that $x=S(x,y)$ and $y=S(y,x)$. We suppose on the contrary that $x\neq S(x,y)$ and $y\neq S(y,x)$ so that $0\prec d(x,S(x,y))=l_1$ and $0\prec d(y,S(y,x))=l_2$; we would then have

$$\begin{aligned}l_1&=d\left(x,S\left(x,y\right)\right)\precsim d\left(x,x_{2k+2}\right)+d\left(x_{2k+2},S\left(x,y\right)\right)\\ &\precsim d\left(x,x_{2k+2}\right)+d\left(T\left(x_{2k+1},y_{2k+1}\right),S\left(x,y\right)\right)\\ &\precsim d\left(x,x_{2k+2}\right)+\frac{\alpha\left(d\left(x_{2k+1},x\right)+d\left(y_{2k+1},y\right)\right)}{2}\\ &\quad+\left(\beta d\left(x_{2k+1},T\left(x_{2k+1},y_{2k+1}\right)\right)d\left(x,S\left(x,y\right)\right)\right)\\ &\quad\times\left(d\left(x_{2k+1},S\left(x,y\right)\right)+d\left(x,T\left(x_{2k+1},y_{2k+1}\right)\right)\right.\\ &\qquad\left.+d\left(x_{2k+1},x\right)+d\left(y_{2k+1},y\right)\right)^{-1}\\ &\precsim d\left(x,x_{2k+2}\right)+\frac{\alpha\left(d\left(x_{2k+1},x\right)+d\left(y_{2k+1},y\right)\right)}{2}\\ &\quad+\left(\beta l_1 d\left(x_{2k+1},x_{2k+2}\right)\right)\\ &\quad\times\left(d\left(x_{2k+1},S\left(x,y\right)\right)+d\left(x,x_{2k+2}\right)\right.\\ &\qquad\left.+d\left(x_{2k+1},x\right)+d\left(y_{2k+1},y\right)\right)^{-1},\end{aligned}\tag{48}$$

so that

$$\begin{aligned}\left|l_1\right|&\leq\left|d\left(x,x_{2k+2}\right)\right|+\frac{\alpha}{2}\left|d\left(x_{2k+1},x\right)+d\left(y_{2k+1},y\right)\right|\\ &\quad+\left(\beta\left|l_1\right|\left|d\left(x_{2k+1},x_{2k+2}\right)\right|\right)\\ &\quad\times\left(\left|d\left(x_{2k+1},S\left(x,y\right)\right)+d\left(x,x_{2k+2}\right)\right.\right.\\ &\qquad\left.\left.+d\left(x_{2k+1},x\right)+d\left(y_{2k+1},y\right)\right|\right)^{-1}.\end{aligned}\tag{49}$$

By taking $k\to+\infty$, we get $|d(x,S(x,y))|=0$ which is a contradiction so that $x=S(x,y)$. Now

$$\begin{aligned}l_2&=d\left(y,S\left(y,x\right)\right)\precsim d\left(y,y_{2k+2}\right)+d\left(y_{2k+2},S\left(y,x\right)\right)\\ &\precsim d\left(y,y_{2k+2}\right)+d\left(T\left(y_{2k+1},x_{2k+1}\right),S\left(y,x\right)\right)\\ &\precsim d\left(y,y_{2k+2}\right)+\frac{\alpha\left(d\left(y_{2k+1},y\right)+d\left(x_{2k+1},x\right)\right)}{2}\\ &\quad+\left(\beta d\left(y_{2k+1},T\left(y_{2k+1},x_{2k+1}\right)\right)d\left(y,S\left(y,x\right)\right)\right)\\ &\quad\times\left(d\left(y_{2k+1},S\left(y,x\right)\right)+d\left(y,T\left(y_{2k+1},x_{2k+1}\right)\right)\right.\\ &\qquad\left.+d\left(y_{2k+1},y\right)+d\left(x_{2k+1},x\right)\right)^{-1}\\ &\precsim d\left(y,y_{2k+2}\right)+\frac{\alpha\left(d\left(y_{2k+1},y\right)+d\left(x_{2k+1},x\right)\right)}{2}\\ &\quad+\left(\beta l_2 d\left(y_{2k+1},y_{2k+2}\right)\right)\\ &\quad\times\left(d\left(y_{2k+1},S\left(y,x\right)\right)+d\left(y,y_{2k+2}\right)\right.\\ &\qquad\left.+d\left(y_{2k+1},y\right)+d\left(x_{2k+1},x\right)\right)^{-1},\end{aligned}\tag{50}$$

which implies that

$$\begin{aligned}|l_2| \leq & |d(y, y_{2k+2})| + \frac{\alpha}{2}|d(y_{2k+1}, y)\, d(x_{2k+1}, x)| \\ & + (\beta |l_2| |d(y_{2k+1}, y_{2k+2})|) \\ & \times (|d(y_{2k+1}, S(y, x)) + d(y, y_{2k+2}) \\ & \quad + d(y_{2k+1}, y) + d(x_{2k+1}, x)|)^{-1},\end{aligned} \tag{51}$$

Which, on making $k \to +\infty$, gives us $|d(y, S(y, x))| = 0$ which is a contradiction so that $y = S(y, x)$. It follows similarly that $x = T(x, y)$ and $y = T(y, x)$. So we have proved that (x, y) is a common coupled fixed point of S and T. As in Theorem 10, the uniqueness of common coupled fixed point remains a consequence of contraction condition (31).

We have obtained the existence and uniqueness of a unique common coupled fixed point if

$$\begin{gathered}D_S(x_{2k}, y_{2k}),\ D_S(y_{2k}, x_{2k}), \\ D_T(x_{2k+1}, y_{2k+1}),\ D_T(y_{2k+1}, x_{2k+1}) \neq 0\end{gathered} \tag{52}$$

for all $k \in \mathbb{N}$. Now, assume that $D_S(x_{2k}, y_{2k}) = 0$ for some $k \in \mathbb{N}$. From

$$d(x_{2k}, x_{2k+2}) + d(x_{2k}, x_{2k+1}) + d(y_{2k}, y_{2k+1}) = 0, \tag{53}$$

we obtain that $x_{2k} = x_{2k+1} = x_{2k+2}$ and $y_{2k} = y_{2k+1}$. If $D_S(y_{2k}, x_{2k}) \neq 0$, using (8), we deduce

$$d(y_{2k+1}, y_{2k+2}) = d(S(y_{2k}, x_{2k}), T(y_{2k+1}, x_{2k+1})) = 0. \tag{54}$$

That is, $y_{2k+1} = y_{2k+2}$ (this equality holds also if $D_S(y_{2k}, x_{2k}) = 0$). The equalities

$$x_{2k} = x_{2k+1} = x_{2k+2}, \qquad y_{2k} = y_{2k+1} = y_{2k+2}, \tag{55}$$

ensure that (x_{2k+1}, y_{2k+1}) is a unique common coupled fixed point of S and T. The same holds if either $D_S(y_{2k}, x_{2k}) = 0$, $D_T(x_{2k+1}, y_{2k+1}) = 0$, or $D_T(y_{2k+1}, x_{2k+1}) = 0$. □

From Theorem 13, if we assume $\alpha = 0$, we obtain the following corollary.

Corollary 14. *Let (X, d) be a complete complex-valued metric space, and let the self-mappings $S, T : X \times X \to X$ satisfy*

$$d(S(x, y), T(u, v)) \preceq \begin{cases} \dfrac{\beta d(x, S(x, y))\, d(u, T(u, v))}{d(x, T(u, v)) + d(u, S(x, y)) + d(x, u) + d(y, v)}, & \text{if } D \neq 0 \\ 0, & \text{if } D \neq 0 \end{cases} \tag{56}$$

for all $x, y, u, v \in X$, where $D = d(x, T(u, y)) + d(u, S(x, y)) + d(x, y) + d(y, v)$ and β is a nonnegative real such that $0 < \beta < 1$. Then S and T have a unique common coupled fixed point.

Corollary 15. *Let (X, d) be a complete complex-valued metric space, and let the mapping $T : X \times X \to X$ satisfy*

$$d(T(x, y), T(u, v)) \preceq \begin{cases} \dfrac{\alpha(d(x, u) + d(y, v))}{2} + \dfrac{\beta d(x, T(x, y))\, d(u, T(u, v))}{d(x, T(u, v)) + d(u, T(x, y)) + d(x, u) + d(y, v)}, & \text{if } D \neq 0 \\ 0, & \text{if } D = 0 \end{cases} \tag{57}$$

for all $x, y, u, v \in X$, where $D = d(x, T(u, y)) + d(u, T(x, y)) + d(x, u) + d(y, v)$ and α, β are nonnegative reals with $\alpha + \beta < 1$. Then T has a unique coupled fixed point.

Corollary 16. *Let (X, d) be a complete complex-valued metric space, and let the mapping $T : X \times X \to X$ satisfy*

$$d(T^n(x, y), T^n(u, v)) \preceq \begin{cases} \dfrac{\alpha(d(x, u) + d(y, v))}{2} + \dfrac{\beta d(x, T^n(x, y))\, d(u, T^n(u, v))}{d(x, T^n(u, v)) + d(u, T^n(x, y)) + d(x, u) + d(y, v)}, & \text{if } D \neq 0 \\ 0, & \text{if } D = 0 \end{cases} \tag{58}$$

for all $x, y, u, v \in X$, where $D = d(x, T^n(u, y)) + d(u, T^n(x, y)) + d(x, u) + d(y, v)$ and α, β are nonnegative reals with $\alpha + \beta < 1$. Then T has a unique coupled fixed point.

Now, we furnished a nontrivial example to support our main result (Theorem 10).

Example 17. Let

$$\begin{aligned}X_1 &= \{z \in \mathbb{C} : \operatorname{Re}(z) \geq 0,\ \operatorname{Im}(z) = 0\}, \\ X_2 &= \{z \in \mathbb{C} : \operatorname{Im}(z) \geq 0,\ \operatorname{Re}(z) = 0\},\end{aligned} \tag{59}$$

and let $X = X_1 \cup X_2$. Consider a complex valued metric $d : X \times X \to \mathbb{C}$ as follows:

$$d(z_1, z_2) = \begin{cases} \dfrac{2}{3}|x_1 - x_2| + \dfrac{i}{2}|x_1 - x_2|, & \text{if } z_1, z_2 \in X_1 \\ \dfrac{1}{2}|y_1 - y_2| + \dfrac{i}{3}|y_1 - y_2|, & \text{if } z_1, z_2 \in X_2 \\ \dfrac{2}{9}(x_1 + y_2) + \dfrac{i}{6}(x_1 + y_2), & \text{if } z_1 \in X_1,\ z_2 \in X_2 \\ \dfrac{i}{3}(x_2 + y_1) + \dfrac{2i}{9}(x_2 + y_1), & \text{if } z_1 \in X_2,\ z_2 \in X_1, \end{cases} \tag{60}$$

with $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$. Then (X, d) is a complex valued metric space. Define $S, T : X \times X \to X$ as follows:

$$S(z_1, z_2) = \begin{cases} 0 + \dfrac{x_1 x_2}{4} i & \text{if } z_1, z_2 \in X_1 \\ \dfrac{y_1 y_2}{5} + 0i & \text{if } z_1, z_2 \in X_2 \\ 0 + \dfrac{x_1 y_2}{8} i & \text{if } z_1 \in X_1 \text{ and } z_2 \in X_2 \\ \dfrac{y_1 x_2}{9} + 0i & \text{if } z_1 \in X_2 \text{ and } z_2 \in X_1, \end{cases}$$
$$T(z_1, z_2) = \begin{cases} 0 + \dfrac{x_1 x_2}{6} i & \text{if } z_1, z_2 \in X_1 \\ \dfrac{y_1 y_2}{7} + 0i & \text{if } z_1, z_2 \in X_2 \\ 0 + \dfrac{x_1 y_2}{10} i & \text{if } z_1 \in X_1 \text{ and } z_2 \in X_2 \\ \dfrac{y_1 x_2}{11} + 0i & \text{if } z_1 \in X_2 \text{ and } z_2 \in X_1. \end{cases} \quad (61)$$

By a routine calculation, one can easily verify that the maps S and T satisfy the contraction condition (8) with $\alpha = 3/4$, $\beta = 1/15$, and $\gamma = 2/15$. Notice that the point $(0, 0)$ remains fixed under S and T and is indeed unique common coupled fixed point.

Conflict of Interests

The authors declare that they have no competing interests.

Authors' Contribution

All authors contributed equally and significantly in writing this paper. All authors read and approved the final paper.

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