

Research Article

Invariants for Weighted Digraphs under One-Sided State Splittings

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Using Matrix-Forest theorem and Matrix-Tree theorem, we present some invariants for weighted digraphs under state in-splittings or out-splittings.

1. Introduction

State in-splittings and out-splittings are very important operations in the theory of one-sided, or two-sided Markov shifts ([1, 2]). Lind and Tuncel introduced a spanning tree invariant for Markov shifts in [3]. Spanning tree invariants are further studied in [4–6]. Motivated by these works, we consider some other graph structures like cycles and forests and present some invariants for weighted digraphs under state in-splittings or out-splittings.

Firstly we give some basic definitions in graph theory and a brief introduction of Matrix-Forest theorem for digraphs. Readers can refer to [7, 8] for more details.

In this paper, a digraph is an ordered pair $D = (V, E)$ of finite sets, where V is called the vertex set and $E \subseteq V \times V$ is called the edge set. For an edge $(u, v) \in E$, u and v are called the initial and terminal ends of the edge, respectively. The number of edges having u as the initial end is defined to be the outdegree of u and denoted by $d(u)$. The number of edges having v as the terminal end is defined to be the indegree of v . A walk of length n is a sequence of edges $\{(u_i, u_{i+1})\}$ ($i = 1, \dots, n$) and can be denoted by $(u_1, u_2, \dots, u_{n+1})$; moreover, if u_{n+1} is the same as u_1 , we call the walk a closed one. A directed forest is a digraph without closed walks such that the indegree of each vertex is no more than one. The vertices with indegree zero of a forest are called roots. We say that $D_0 = (V_0, E_0)$ is a spanning subgraph of D if $V_0 = V$ and $E_0 \subseteq E$.

Suppose that D is a digraph with vertex set $V(D) = \{1, \dots, n\}$. Let $w : E(D) \rightarrow \mathbb{R}^+$ be a weight function on the edge

set. We then say that $\mathcal{D} = (D, w)$ is a weighted digraph and $M = (w(i, j))_{n \times n}$ is the weight matrix of \mathcal{D} . The Kirchhoff matrix of \mathcal{D} is defined as $L = R - M$, where $R = (r_{i,j})$ is a diagonal matrix and $r_{i,i} = \sum_{j=1}^n w(i, j)$. The product of the weights of all edges that belong to a subgraph \mathcal{H} of \mathcal{D} is defined to be the weight of \mathcal{H} and denoted by $w(\mathcal{H})$.

Let $\mathcal{F}(\mathcal{D}) = \mathcal{F}$ be the set of all spanning rooted forests of \mathcal{D} and $\mathcal{F}^{i \rightarrow j}(\mathcal{D}) = \mathcal{F}^{i \rightarrow j}$ the set of those spanning rooted forests of \mathcal{D} such that i and j belong to the same tree rooted at i . For a matrix A , $A^{i,j}$ denotes the cofactor of the (i, j) -entry of A . The Matrix-Forest theorem then states as follows.

Lemma 1 (cf. [8]). *Let $\mathcal{D} = (D, w)$ be a weighted digraph. Let L be the Kirchhoff matrix of \mathcal{D} . Then one has*

- (1) $\sum_{F \in \mathcal{F}} w(F) = \det(I + L)$;
- (2) for any $i, j \in V(D)$, $\sum_{F \in \mathcal{F}^{i \rightarrow j}} w(F) = (I + L)^{i,j}$.

2. Invariants for Weighted Digraphs under State In-Splitting

Before giving the main result, we recall the definition of state in-splitting.

Definition 2. Let $\mathcal{D} = (D, w)$ be a weighted digraph. For a vertex u of D , E^u denotes the set of edges of D with terminal end u . The state in-splitting of \mathcal{D} at u induces a new weighted digraph $\tilde{\mathcal{D}} = (\tilde{D}, \tilde{w})$ in the following way: let $\mathcal{S} =$

$\{S_1, S_2, \dots, S_r\}$ be a partition of E^u . The vertex set of the new digraph is $V(\tilde{D}) = (V(D) \setminus \{u\}) \cup \{u_1, u_2, \dots, u_r\}$. The edge set $E(\tilde{D})$ and weight \tilde{w} of \tilde{D} are defined as follows.

- (i) For $x, y \in V(D) \setminus \{u\}$, $(x, y) \in E(\tilde{D})$ if and only if $(x, y) \in E(D)$ and in this case $\tilde{w}(x, y) = w(x, y)$.
- (ii) For $x \in V(D) \setminus \{u\}$, $(x, u_i) \in E(\tilde{D})$ if and only if $(x, u) \in S_i$ and in this case $\tilde{w}(x, u_i) = w(x, u)$.
- (iii) For $x \in V(D) \setminus \{u\}$, $(u_j, x) \in E(\tilde{D})$ if and only if $(u, x) \in E(D)$ and in this case $\tilde{w}(u_j, x) = w(u, x)$.
- (iv) If $(u, u) \in S_i$, then $(u_j, u_i) \in E(\tilde{D})$, for $j = 1, 2, \dots, r$, and in this case $\tilde{w}(u_j, u_i) = w(u, u)$.

For more details about state splittings, readers can refer to [2, 3, 9]. Now we give the definition of our new invariant.

Definition 3. Let $\mathcal{D} = (D, w)$ be a weighted digraph. We define $W_k(\mathcal{D})$ ($k \geq 1$) as

$$W_k(\mathcal{D}) = \sum_v d(v) \sum_{C \in C_v^k} w(C), \quad (1)$$

where v runs over $V(D)$ and C_v^k denotes the set of closed walks of \mathcal{D} with length k at vertex v . Furthermore, we define the generating function $W_{\mathcal{D}}(t)$ as

$$W_{\mathcal{D}}(t) = \sum_{k \geq 1} W_k(\mathcal{D}) t^k. \quad (2)$$

Let A be a square matrix. The trace of A is defined to be the sum of the elements on the main diagonal and denoted by $\text{tr}(A)$. For a digraph D , the diagonal matrix $O(D) = (o_{i,i})$ denotes the outdegree matrix of D that is, $o_{i,i} = d(v_i)$. Then we have the following result.

Theorem 4. Let \mathcal{D} be a weighted digraph with weight matrix M . Then $W_{\mathcal{D}}(t)$ is an invariant under state in-splitting and can be computed in the following way:

$$W_{\mathcal{D}}(t) = \frac{\text{tr}(O \cdot \text{adj}(I - tM))}{\det(I - tM)} - \text{tr}(O). \quad (3)$$

Proof. We firstly prove the invariance of $W_k(\mathcal{D})$ for $k \geq 1$. Without loss of generality, if there is a loop at vertex u , we assume that it belongs to S_1 , where $\mathcal{S} = \{S_1, S_2, \dots, S_r\}$ denotes the partition of E^u as in the definition of state in-splitting.

We define the mapping

$$\varphi: \bigcup_{v \in V(D)} C_v^k(\mathcal{D}) \longrightarrow \bigcup_{v \in V(\tilde{D})} C_v^k(\tilde{\mathcal{D}}) \quad (4)$$

in the following way: for a closed walk C of \mathcal{D} with length k , if $C = (u, u, \dots, u)$, then $\varphi(C) = (u_1, u_1, \dots, u_1)$; otherwise, we replace each maximum path of C of the form (v, u, u, \dots, u) ($v \neq u$) with $(v, u_i, u_1, \dots, u_1)$ if $(v, u) \in S_i$. it is not difficult to see that

$$\varphi: C_v^k(\mathcal{D}) \longrightarrow C_v^k(\tilde{\mathcal{D}}), \quad (5)$$

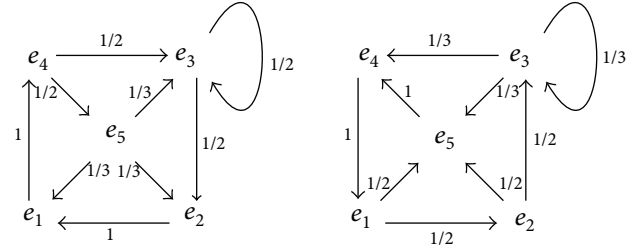


FIGURE 1

where $v \neq u$, and

$$\varphi: C_u^k(\mathcal{D}) \longrightarrow \bigcup_{i=1}^r C_{u_i}^k(\tilde{\mathcal{D}}) \quad (6)$$

are both weight-preserving bijections.

Since $d(v)$ ($v \neq u$) is the same for \mathcal{D} and $\tilde{\mathcal{D}}$ and $d(u) = d(u_1) = d(u_2) = \dots = d(u_r)$, we know that $W_k(\mathcal{D}) = W_k(\tilde{\mathcal{D}})$ for $k \geq 1$, and the invariance of $W_{\mathcal{D}}(t)$ follows.

Finally, we notice that $W_k(\mathcal{D}) = \text{tr}(OM^k)$. Thus

$$\begin{aligned} W_{\mathcal{D}}(t) &= \sum_{k \geq 1} \text{tr}\{O \cdot (tM)^k\} \\ &= \text{tr}\left\{O \cdot \sum_{k \geq 1} (tM)^k\right\} \\ &= \frac{\text{tr}(O \cdot \text{adj}(I - tM))}{\det(I - tM)} - \text{tr}(O). \end{aligned} \quad (7) \quad \square$$

Example 5. Let $\mathcal{D} = (D, w)$ be a weighted digraph as in the left of Figure 1. \tilde{D} is the opposite of D (see the right of Figure 1), that is, the digraph obtained from D by reversing the direction of all its edges. It is easy to see that D and \tilde{D} have the same outdegree sequence $\{1, 1, 2, 2, 3\}$. The weight of any edge $(u, v) \in E(D)$ or $(u, v) \in E(\tilde{D})$ is defined to be $1/d(u)$. Since $W_3(\mathcal{D}) = 5/4$ and $W_3(\tilde{\mathcal{D}}) = 19/9$, we know that $\tilde{\mathcal{D}}$ cannot be obtained from \mathcal{D} by a sequence of in-splittings or reverse operations.

Let P be a nonnegative matrix. P is called row stochastic if the summation of each row equals 1 and column stochastic if the summation of each column equals 1. P is called double stochastic if it is row and column stochastic.

Definition 6. Let P be a row-stochastic matrix and t a real positive number. Let \mathcal{D} be the weighted digraph with weight matrix $M = tP$. We define $K(\mathcal{D}, t)$ as

$$K(\mathcal{D}, t) = (1 + t) \frac{\sum_v d(v) \sum_{F_v} w(F_v)}{\sum_F w(F)}, \quad (8)$$

where v runs over all vertices of $V(D)$, F runs over all spanning directed forests of D , and F_v runs over all spanning directed forests including v as a root.

In general, $K(\mathcal{D}, t)$ is not an invariant under state in-splitting, but the following result shows that it indeed reflects some invariance.

Corollary 7. *Let P be a row-stochastic matrix and t a real positive number. Let \mathcal{D} be a weighted digraph with weight matrix $M = tP$. Then $K(\mathcal{D}, t) - K(\widetilde{\mathcal{D}}, t)$ is an integer independent of t .*

Proof. Let $O = (o_{i,j})$ be the outdegree matrix of D . Then we get by Lemma 1 that

$$\begin{aligned} K(\mathcal{D}, t) &= (1+t) \frac{\sum_v d(v) \sum_{F_v} w(F_v)}{\sum_F w(F)} \\ &= (1+t) \frac{\text{tr}(O \cdot \text{adj}[I + (tI - M)])}{\det[I + (tI - M)]} \quad (9) \\ &= \text{tr}\left(O \cdot \left[I - \frac{t}{1+t}P\right]^{-1}\right). \end{aligned}$$

Since P is stochastic and $1/(1+t) \in (0, 1)$, we have

$$\left[I - \frac{t}{1+t}P\right]^{-1} = [I - r(tP)]^{-1} = \sum_{i \geq 0} (M)^i r^i, \quad (10)$$

where $r = 1/(1+t)$.

Therefore

$$\begin{aligned} K(\mathcal{D}, t) &= \text{tr}\left(O \cdot \left[I - \frac{t}{1+t}P\right]^{-1}\right) \\ &= \sum_{i \geq 0} \text{tr}\{O(M)^i\} r^i \quad (11) \\ &= W_{\mathcal{D}}(r) + \text{tr}(O). \end{aligned}$$

By Theorem 4, we know that $W_{\mathcal{D}}(r)$ is an invariant under in-splitting; thus

$$K(\mathcal{D}, t) - K(\widetilde{\mathcal{D}}, t) = \text{tr}(O) - \text{tr}(\widetilde{O}) \in \mathbb{Z}. \quad (12)$$

The result follows. \square

Lind and Tuncel defined a spanning tree invariant $\tau(\mathcal{D})$ for Markov shifts in [3] as follows:

$$\tau(\mathcal{D}) = \sum_T w(T). \quad (13)$$

Here the weight matrix P of \mathcal{D} is an irreducible row-stochastic matrix, and T runs over all spanning trees of \mathcal{D} .

By considering the outdegree matrix as in Definitions 3 and 6, we can define a new spanning tree invariant as

$$\tau_d(\mathcal{D}) = \sum_T d(T) w(T), \quad (14)$$

where T is as above, and $d(T)$ denotes the outdegree of the root of T .

Corollary 8. $\tau_d(\mathcal{D})$ is an invariant under in-splitting.

Proof. Let P be the weight matrix of \mathcal{D} and thus row stochastic as in [3]. By the Matrix-Tree theorem (Theorem 2 in [8]), we have

$$\begin{aligned} \tau_d(\mathcal{D}) &= \text{tr}(O \cdot \text{adj}[I - P]) \\ &= \lim_{t \rightarrow 1^-} \left\{ \det[I - tP] \cdot \text{tr}(O \cdot [I - tP]^{-1}) \right\} \\ &= \lim_{t \rightarrow 1^-} \left\{ \det[I - tP] \cdot \sum_{i \geq 0} \text{tr}(OP^i) t^i \right\} \quad (15) \\ &= \lim_{t \rightarrow 1^-} \left\{ \det[I - tP] \cdot (W_{\mathcal{D}}(t) + \text{tr}(O)) \right\}. \end{aligned}$$

By Theorem 4, we know that $W_{\mathcal{D}}(t)$ is an invariant under state in-splitting. It is also well known that $\det[I - tP]$ is an invariant under state splitting. Therefore

$$\tau_d(\mathcal{D}) - \tau_d(\widetilde{\mathcal{D}}) = \lim_{t \rightarrow 1^-} \left\{ \det[I - tP] \cdot (\text{tr}(O) - \text{tr}(\widetilde{O})) \right\}. \quad (16)$$

Since $\text{tr}(O) - \text{tr}(\widetilde{O})$ is a constant and $\lim_{t \rightarrow 1^-} \det[I - tP] = 0$, we have

$$\tau_d(\mathcal{D}) - \tau_d(\widetilde{\mathcal{D}}) = 0. \quad (17)$$

The result follows. \square

Let $\mathcal{D} = (D, w)$ be a weighted digraph. The out-weighted line digraph $L^+(\mathcal{D}) = (L(D), w^+)$ of \mathcal{D} is a weighted digraph defined in the following way: the vertex set of $L(D)$ is $E(D)$; $((u, v), (x, y)) \in E(L(D))$ if and only if $v = x$, and in this case, $w^+(((u, v), (x, y)))) = w(x, y)$. Similarly, if we let $w^-(((u, v), (x, y)))) = w(u, v)$ in the above definition, then we get the in-weighted line digraph $L^-(\mathcal{D}) = (L(D), w^-)$. Galeana-Sánchez and Gómez show that $L^+(\mathcal{D})$ can be obtained by sequences of state in-splittings from \mathcal{D} (see Proposition 2.2 in [9], which has a small typo there by stating $L^-(\mathcal{D})$ can be obtained by sequences of state in-splittings). Now the following conclusion is an immediate result of Corollary 8.

Corollary 9. $\tau_d(\mathcal{D})$ is an invariant under out-weighted line digraph operation.

3. The State Out-Splitting Case

Let P be a row-stochastic matrix. Let $\mathcal{D} = (D, W)$ be the weighted digraph with weight matrix $W = P$. We first give the definition of state out-splitting, which is a little more complicated than the case of state in-splitting. Readers can refer to [3] for more details.

Definition 10. For a vertex u of D , let E^{*u} denote the set of edges of D with initial end u . The state out-splitting of \mathcal{D} at u induces a new weighted digraph $\widetilde{\mathcal{D}}^* = (\widetilde{D}^*, \widetilde{w})$ in the following way: let $\mathcal{S}^* = \{S_1^*, S_2^*, \dots, S_r^*\}$ be a partition of E^{*u} . Let q_i denote the sum of the weights of edges in S_i^* . The vertex set of the new digraph is $V(\widetilde{D}^*) = (V(D) \setminus \{u\}) \cup \{u_1, u_2, \dots, u_r\}$. The edge set and weight of $\widetilde{\mathcal{D}}^*$ are defined as follows.

- (i) For $x, y \in V(D) \setminus \{u\}$, $(x, y) \in E(\widetilde{D}^*)$ if and only if $(x, y) \in E(D)$ and in this case $\widetilde{w}(x, y) = w(x, y)$.
- (ii) For $y \in V(D) \setminus \{u\}$, $(u_i, y) \in E(\widetilde{D}^*)$ if and only if $(u, y) \in S_i^*$ and in this case $\widetilde{w}(u_i, y) = w(u, y)/q_i$.
- (iii) For $x \in V(D) \setminus \{u\}$, $(x, u_i) \in E(\widetilde{D}^*)$ if and only if $(x, u) \in E(D)$ and in this case $\widetilde{w}(u_i, x) = q_i w(x, u)$.
- (iv) If $(u, u) \in S_i^*$, then $(u_i, u_j) \in E(\widetilde{D}^*)$, for $j = 1, 2, \dots, r$, and in this case $\widetilde{w}(u_i, u_j) = w(u, u)q_j/q_i$.

In the definition of $W_k(\mathcal{D})$ and $W_D(t)$, by replacing outdegrees with indegrees, we get $W_k^*(\mathcal{D})$ and $W_{\mathcal{D}}^*(t)$; that is,

$$W_k^*(\mathcal{D}) = \sum_v d^*(v) \sum_{C \in \mathcal{C}_v^k} w(C),$$

$$W_{\mathcal{D}}^*(t) = \sum_{k \geq 1} W_k^*(\mathcal{D}) t^k, \quad (18)$$

where $d^*(v)$ is the indegree of v .

Theorem 11. *Let P be a row-stochastic matrix. Let \mathcal{D} be the weighted digraph with weight matrix P . Then $W_{\mathcal{D}}^*(t)$ is an invariant under state out-splitting, and can be computed as*

$$W_{\mathcal{D}}^*(t) = \frac{\text{tr}(O^* \cdot \text{adj}(I - tP))}{\det(I - tP)} - \text{tr}(O^*), \quad (19)$$

where O^* is the indegree matrix of D .

Proof. We just need to prove the invariance of $W_k^*(\mathcal{D})$ for $k \geq 1$. Without loss of generality, if there is a loop at vertex u , we assume that it belongs to S_1^* , where $\mathcal{S}^* = \{S_1^*, S_2^*, \dots, S_r^*\}$ denotes the partition of E^{*u} as in the definition of state out-splitting.

We define the mapping

$$\varphi: \bigcup_{v \in V(D)} C_v^k(\mathcal{D}) \longrightarrow \bigcup_{v \in V(\widetilde{D}^*)} C_v^k(\widetilde{\mathcal{D}}^*) \quad (20)$$

in the following way: for a closed walk C of \mathcal{D} with length k , if $C = (u, u, \dots, u)$, then $\varphi(C) = (u_1, u_1, \dots, u_1)$; otherwise, we replace each maximum path of C of the form (u, u, \dots, u, u, v) ($v \neq u$) with $(u_1, u_1, \dots, u_1, u_i, v)$ if $(u, v) \in S_i^*$. By the definition of state out-splitting, it is not difficult to prove that

$$\varphi: C_v^k(\mathcal{D}) \longrightarrow C_v^k(\widetilde{\mathcal{D}}^*), \quad (21)$$

where $v \neq u$, and

$$\varphi: C_u^k(\mathcal{D}) \longrightarrow \bigcup_{i=1}^r C_{u_i}^k(\widetilde{\mathcal{D}}^*) \quad (22)$$

are both bijections.

We now prove that they are also weight-preserving. In fact, if $C = (u, u, \dots, u)$, then $w(C) = w(\varphi(C))$, since $\widetilde{w}(u_i, u_i) = w(u, u)$. On the other hand, for any walk of C of the form $S = (r, u, \dots, u, u, v)$ ($v \neq u$), we have $w(S) = w(r, u)w(u, u)^k w(u, v)$.

- (1) If $(u, v) \in S_1^*$, we have

$$w(\varphi(S)) = w(r, u_1)w(u_1, u_1)^k w(u_1, v)$$

$$= \frac{q_1 w(r, u)w(u, u)^k w(u, v)}{q_1} \quad (23)$$

$$= w(S).$$

- (2) If $(u, v) \in S_i^*$ ($i \neq 1$) and $k \geq 1$, we have

$$w(\varphi(S)) = w(r, u_1)w(u_1, u_1)^{k-1} w(u_1, u_i)w(u_i, v)$$

$$= \frac{q_1 w(r, u)w(u, u)^{k-1} (w(u, u)q_i/q_1) w(u, v)}{q_i}$$

$$= w(S). \quad (24)$$

- (3) If $(u, v) \in S_i^*$ ($i \neq 1$) and $k = 0$, we have

$$w(\varphi(S)) = w(r, u_i)w(u_i, v)$$

$$= \frac{q_i w(r, u)w(u, v)}{q_i} \quad (25)$$

$$= w(S).$$

Thus the maps above are weight preserving. Since $d^*(v)$ ($v \neq u$) is the same for \mathcal{D} and $\widetilde{\mathcal{D}}^*$, and $d^*(u) = d^*(u_1) = d^*(u_2) = \dots = d^*(u_r)$, we know that $W_k^*(\mathcal{D}) = W_k^*(\widetilde{\mathcal{D}}^*)$, for $k \geq 1$, and the invariance of $W_{\mathcal{D}}^*(t)$ follows.

The proof of the equality is similar to that of Theorem 4. \square

Similarly, we can define $\tau_d^*(\mathcal{D})$ and prove that it is also an invariant under state out-splitting on the basis of the above result.

Now, we consider some weighted digraphs from [10] in the following two examples.

Example 12. The weight matrices of two weighted digraphs are as follows:

$$A = \begin{bmatrix} \frac{3}{8} & \frac{1}{2} & \frac{1}{8} \\ 0 & \frac{4}{5} & \frac{1}{5} \\ \frac{2}{7} & \frac{4}{7} & \frac{1}{7} \end{bmatrix}, \quad B = \begin{bmatrix} \frac{1}{7} & 0 & \frac{6}{7} \\ \frac{5}{56} & \frac{3}{8} & \frac{15}{28} \\ \frac{2}{15} & \frac{1}{15} & \frac{4}{5} \end{bmatrix}. \quad (26)$$

By some computation, we get that $W_{\mathcal{A}}(1/2) = 1316/471$, $W_{\mathcal{B}}(1/2) = 1615/471$, and $W_{\mathcal{A}}^*(1/2) = W_{\mathcal{B}}^*(1/2) = 1559/471$. Thus B cannot be archived by a sequence of in-splittings or reverse operations, but may be archived by a sequence of out-splittings or reverse operations.

Example 13. The weight matrices of three weighted digraphs are as follows:

$$A = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & \frac{2}{3} \end{bmatrix}, \quad B = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} \end{bmatrix}, \quad C = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} \end{bmatrix}. \quad (27)$$

By some computation, we get that $\tau_d^*(A) = 2 = \tau_d(A)$, $\tau_d^*(B) = 8/3 = \tau_d(B)$, $\tau_d^*(C) = 4/3 = \tau_d(C)$. Thus for any pair of them, we cannot get one from the other and by a sequence of in-splittings or reverse operations either nor by a sequence of out-splittings or reverse operations.

4. Invariants for Weighted Digraphs with Double-Stochastic Matrices

Let $\mathcal{D} = (D, P)$ be a weighted digraph. If the weight matrix P is column stochastic, the weight distribution after state out-splitting can be defined in an easier way, that is, without multiplying by the coefficients about q_i in Definition 10. Under this definition, we can get that $\tau_d^*(\mathcal{D})$ is still an invariant under state out-splitting, the proof of which is similar to that of Corollary 8. We also know from [9] that the in-weighted line digraph can be obtained by a sequence of such state out-splittings, so the following result is immediate.

Corollary 14. *Let $\mathcal{D} = (D, P)$ be a weighted digraph. If the weight matrix P is column stochastic, then $\tau_d^*(\mathcal{D})$ is an invariant under in-weighted line digraph operation.*

Especially, if the weight matrix is doubly stochastic, we have the following result.

Corollary 15. *Let $\mathcal{D} = (D, P)$ be a weighted digraph. If the weight matrix P is doubly stochastic, then $\tau_d(L^+(\mathcal{D})) = \tau_d(L^-(\mathcal{D}))$.*

Proof. Since P is doubly stochastic, we have by Corollary 8 that

$$\tau_d(L^+(\mathcal{D})) = \tau_d(\mathcal{D}) = \text{tr}(O \cdot \text{adj}[I - P]) \quad (28)$$

and by Corollary 14 that

$$\tau_d^*(L^-(\mathcal{D})) = \tau_d^*(\mathcal{D}) = \text{tr}(O^* \cdot \text{adj}[I - P]). \quad (29)$$

By Matrix-Tree theorem (Theorem 2 in [8]), we know that both $\text{adj}[I - P]$ and $\text{adj}[I - P'] = (\text{adj}[I - P])'$ are row-constant matrices, where P' is P transposed. Thus $\text{adj}[I - P]$ is a constant matrix. Since the sum of indegrees is equal to that of outdegrees, the result follows. \square

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References

- [1] B. P. Kitchens, *Symbolic Dynamics: One-Sided, Two-Sided and Countable State Markov Shifts*, Springer, Berlin, Germany, 1998.
- [2] D. Lind and B. Marcus, *An Introduction to Symbolic Dynamics and Coding*, Cambridge University Press, Cambridge, UK, 1995.
- [3] D. Lind and S. Tuncel, "A spanning tree invariant for Markov shifts," in *Codes, Systems, and Graphical Models*, vol. 123 of *The IMA Volumes in Mathematics and Its Applications*, pp. 487–497, Springer, New York, NY, USA, 2001.
- [4] R. B. Bapat, "On a conjecture concerning spanning tree invariants and loop systems," *Linear Algebra and Its Applications*, vol. 433, no. 8–10, pp. 1642–1645, 2010.
- [5] R. Gómez, "On mean recurrence times of Markov chains and spanning tree invariants," *Linear Algebra and Its Applications*, vol. 433, no. 11–12, pp. 1714–1718, 2010.
- [6] R. Gómez and J. M. Salazar, "Spanning tree invariants, loop systems and doubly stochastic matrices," *Linear Algebra and Its Applications*, vol. 432, no. 2–3, pp. 556–565, 2010.
- [7] B. Bollobás, *Modern Graph Theory*, vol. 184, Springer, New York, NY, USA, 1998.
- [8] P. Yu. Chebotarev and E. V. Shamis, "The Matrix-Forest theorem and measuring relations in small social groups," *Automation and Remote Control*, vol. 58, no. 9, pp. 1505–1514, 1997.
- [9] H. Galeana-Sánchez and R. Gómez, "Monochromatic paths on edge colored digraphs and state splittings," *Advances and Applications in Discrete Mathematics*, vol. 4, no. 1, pp. 33–51, 2009.
- [10] R. Cowen and E. M. Lungu, "When are two Markov chains the same?" *Quaestiones Mathematicae*, vol. 23, no. 4, pp. 507–513, 2000.



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