

Research Article

Fixed-Term Homotopy

**Hector Vazquez-Leal,¹ Yasir Khan,² Uriel Filobello-Nino,¹ Arturo Sarmiento-Reyes,³
Alejandro Diaz-Sanchez,³ and Luis-F. Cisneros-Sinencio¹**

¹ *Electronic Instrumentation and Atmospheric Sciences School, University of Veracruz,
Circuito Gonzalo Aguirre Beltrán s/n, 91000 Xalapa, VER, Mexico*

² *Department of Mathematics, Zhejiang University, Hangzhou 310027, China*

³ *Department of Electronics, National Institute for Astrophysics, Optics, and Electronics,
Luis Enrique Erro No. 1, 72840 Sta. María Tonantzintla, PUE, Mexico*

Correspondence should be addressed to Hector Vazquez-Leal; hvazquez@uv.mx

Received 6 September 2012; Revised 6 December 2012; Accepted 20 December 2012

Academic Editor: Chein-Shan Liu

Copyright © 2013 Hector Vazquez-Leal et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

A new tool for the solution of nonlinear differential equations is presented. The Fixed-Term Homotopy (FTH) delivers a high precision representation of the nonlinear differential equation using only a few linear algebraic terms. In addition to this tool, a procedure based on Laplace-Padé to deal with the truncate power series resulting from the FTH method is also proposed. In order to assess the benefits of this proposal, two nonlinear problems are solved and compared against other semianalytic methods. The obtained results show that FTH is a power tool capable of generating highly accurate solutions compared with other methods of literature.

1. Introduction

Many physical phenomena are commonly modelled using nonlinear differential equations, which is a straightforward way to describe the behaviour of their dynamics. Among these methods, the most commonly used is the Homotopy Perturbation Method (HPM) [1–49]. This method is based in the use of a power series of the homotopy parameter, which transforms the original nonlinear differential equation into a series of linear differential equations. In this paper, a generalization of this concept using a product of two power series of the homotopy parameter called Fixed Term Homotopy (FTH) method is proposed. FTH method transforms the nonlinear differential equation into a series of linear differential equations, generating high precision expressions with fewer algebraic terms, reducing the computing cost. Furthermore, in order to deal with the truncate power series obtained with FTH method, the use of Laplace-Padé after-treatment is also proposed. To assess the potential of the proposed methodology, two nonlinear problems, Van Der Pol Oscillator [6, 50] and Troesch's equation [51–57], will be solved and compared using similar methodologies.

This paper is organized as follows. In Section 2, the fundamental idea of the FTH method is described. Section 3 presents a study of convergence for the proposed method. Section 4 introduces the Laplace-Padé after-treatment. In Sections 5 and 6, the solution procedure of two nonlinear problems is presented. Additionally, a discussion of the obtained results and the finds of this work are summarized in Section 7. Finally, the conclusions are presented in Section 8.

2. Basic Concept of FTH Method

The FTH and HPM methods share common foundations. Both methods consider that a nonlinear differential equation can be expressed as

$$A(u) - f(r) = 0, \quad \text{where } r \in \Omega \quad (1)$$

which has as boundary condition

$$B\left(u, \frac{\partial u}{\partial \eta}\right) = 0, \quad \text{where } r \in \Gamma, \quad (2)$$

where A is a general differential operator, $f(r)$ is a known analytic function, B is a boundary operator, Γ is the boundary of domain Ω , and $\partial u/\partial \eta$ denotes differentiation along the normal drawn outwards from Ω [58]. In general, the A operator can be divided into two operators L and N , which are the corresponding linear and nonlinear operators, respectively. Hence, (1) can be rewritten as

$$L(u) + N(u) - f(r) = 0. \tag{3}$$

Now, a possible homotopy formulation is given by the expression

$$H(v, p) = (1 - p) [L(v) - L(u_0)] + p(L(v) + N(v) - f(r)) = 0, \quad p \in [0, 1], \tag{4}$$

where u_0 is the trial function (initial approximation) for (3) which satisfies the boundary conditions, and p is known as the perturbation homotopy parameter. From analyzing (4), it can be concluded that

$$H(v, 0) = L(v) - L(u_0) = 0, \tag{5}$$

$$H(v, 1) = L(v) + N(v) - f(r) = 0. \tag{6}$$

For $p \rightarrow 0$, the homotopy map (4) is reduced to the problem (5) that possesses a trivial solution u_0 . Moreover, for $p \rightarrow 1$, the homotopy map (4) is transformed into the original nonlinear problem (6) that possesses the sought solution.

For the HPM method [9–12], we assume that the solution for (4) can be written as a power series of p , such that

$$v = p^0 v_0 + p^1 v_1 + p^2 v_2 + \dots \tag{7}$$

Considering that $p \rightarrow 1$, it results that the approximate solution for (1) is

$$u = \lim_{p \rightarrow 1} v = v_0 + v_1 + v_2 + \dots \tag{8}$$

The series (8) is convergent on most cases [9, 12].

In [59], a homotopy which uses the auxiliary term $\alpha p(1 - p)$ was reported. Then, modifying that version, it results in the following proposed homotopy:

$$H(v, p) = (1 - p) [L(v) - L(u_0)] + p(L(v) + N(v) - f(r)) + p(1 - p)G(r) = 0, \quad p \in [0, 1], \tag{9}$$

where $G(r)$ is an arbitrary function.

When $p \rightarrow 0$ or $p \rightarrow 1$, the auxiliary term $p(1 - p)G(r)$ is set to zero. Then, $G(r)$ does not affect or change the initial solution when $p \rightarrow 0$ or the sought solution at $p \rightarrow 1$. Moreover, a properly selection of $G(r)$ can be useful

to improve convergence of the homotopy. Now, for the FTH method, (7) can be rewritten as the product of two power series, such that

$$v = (p^0 v_0 + p^1 v_1 + p^2 v_2 + \dots)(p^0 w_0 + p^1 w_1 + p^2 w_2 + \dots), \tag{10}$$

where v_0, v_1, \dots are unknown functions to be determined by the FTH method, and w_0, w_1, \dots are arbitrarily chosen fixed terms.

After substituting (10) into (9), and equating terms with the same order of p , we obtain a set of linear equations that lead us to calculate v_0, v_1, \dots . The limit of (10), when $p \rightarrow 1$, provides an approximate solution for (3) in the form of

$$u = \lim_{p \rightarrow 1} v = (v_0 + v_1 + v_2 + \dots)(w_0 + w_1 + w_2 + \dots). \tag{11}$$

The upper limit exists in the event that both limits exists so

$$\lim_{p \rightarrow 1} \left(\sum_{i=0}^{\infty} v_i \right), \tag{12}$$

and fixed term

$$\lim_{q \rightarrow 1} \left(\sum_{i=0}^{\infty} w_i \right), \quad \text{where } \sum_{i=0}^{\infty} w_i \neq 0, \text{ (to avoid trivial solution),} \tag{13}$$

exist.

3. Convergence of FTH Method

To analyze the convergence of FTH, (9) is rewritten as

$$L(v) = L(u_0) + p[f(r) - N(v) - L(u_0)] + p(1 - p)G(r) = 0. \tag{14}$$

Applying the inverse operator L^{-1} to both sides of (14), we obtain

$$v = u_0 + p[L^{-1}f(r) - L^{-1}N(v) - u_0] + p(1 - p)L^{-1}G(r). \tag{15}$$

Assuming that (see (10))

$$v = \left(\sum_{i=0}^{\infty} p^i v_i \right) \left(\sum_{i=0}^{\infty} p^i w_i \right), \tag{16}$$

and substituting (16) into the right-hand side of (15), we

obtain

$$v = u_0 + p \left[L^{-1} f(r) - (L^{-1}N) \left[\left(\sum_{i=0}^{\infty} p^i v_i \right) \left(\sum_{i=0}^{\infty} p^i w_i \right) \right] - u_0 \right] + p(1-p)L^{-1}G(r). \tag{17}$$

The exact solution of (3) is obtained when $p \rightarrow 1$ of (17), resulting in

$$u = \lim_{p \rightarrow 1} \left(pL^{-1} f(r) - p(L^{-1}N) \left[\left(\sum_{i=0}^{\infty} p^i v_i \right) \left(\sum_{i=0}^{\infty} p^i w_i \right) \right] + u_0 - pu_0 + p(1-p)L^{-1}G(r) \right) = L^{-1} f(r) - \left[\sum_{i=0}^{\infty} (L^{-1}N)(v_i\beta) \right], \quad \beta = \sum_{i=0}^{\infty} w_i. \tag{18}$$

For the convergence analysis of the FTH method, we used the Banach Theorem as reported in [1, 2, 5, 60]. Such theorem relates the solution of (3) and the fixed point problem of the nonlinear operator N .

Theorem 1 (Sufficient Condition of Convergence). *Suppose that X and Y are Banach spaces and $N : X \rightarrow Y$ is a contractive nonlinear mapping, then*

$$\forall w, w^* \in X; \quad \|N(w) - N(w^*)\| \leq \gamma \|w - w^*\|; \quad 0 < \gamma < 1. \tag{19}$$

According to Banach Fixed Point Theorem, N has a unique fixed point u such that $N(u) = u$. Assume that the sequence generated by the FTH method can be written as

$$W_n = N(W_{n-1}), \quad W_{n-1} = \sum_{i=0}^{n-1} (v_i\beta), \quad n = 1, 2, 3, \dots, \tag{20}$$

If one assumes that $W_0 = v_0\beta \in B_r(u)$, where $B_r(u) = \{w^* \in X \mid \|w^* - u\| < r\}$. Then, under these conditions:

- (i) $W_n \in B_r(u)$,
- (ii) $\lim_{n \rightarrow \infty} W_n = u$.

Proof. (i) By inductive approach, we have for $n = 1$

$$\|W_1 - u\| = \|N(W_0) - N(u)\| \leq \gamma \|w_0 - u\|. \tag{21}$$

Assuming as induction hypothesis that $\|W_{n-1} - u\| \leq \gamma^{n-1} \|w_0 - u\|$, then

$$\|W_n - u\| = \|N(W_{n-1}) - N(u)\| \leq \gamma \|W_{n-1} - u\| \leq \gamma^n \|w_0 - u\|. \tag{22}$$

Using (i), we have

$$\|W_n - u\| \leq \gamma^n \|w_0 - u\| \leq \gamma^n r < r \implies W_n \in B_r(u). \tag{23}$$

(ii) Because of $\|W_n - u\| \leq \gamma^n \|w_0 - u\|$ and $\lim_{n \rightarrow \infty} \gamma^n = 0$, $\lim_{n \rightarrow \infty} \|W_n - u\| = 0$, that is,

$$\lim_{n \rightarrow \infty} W_n = u. \tag{24}$$

□

4. Laplace-Padé after Treatment for FTH Series

The coupling of Laplace transform and Padé approximant [61] is used in order to recover part of the lost information due to the truncated power series [60, 62-70]. The process can be recast as follows.

- (1) First, Laplace transformation is applied to power series obtained by FTH method.
- (2) Next, s is substituted by $1/t$ in the resulting equation.
- (3) After that, we convert the transformed series into a meromorphic function by forming its Padé approximant of order $[N/M]$. N and M are arbitrarily chosen, but they should be of smaller value than the order of the power series. In this step, the Padé approximant extends the domain of the truncated series solution to obtain better accuracy and convergence.
- (4) Then, t is substituted by $1/s$.
- (5) Finally, by using the inverse Laplace s transformation, we obtain the modified approximate solution.

We will denominate to this process as the Laplace-Padé fixed term homotopy (LPFTH) method.

5. Van Der Pol Oscillator

Consider the Van der Pol Oscillator problem [6, 50]

$$u'' + u' + u + u^2 u' - 2 \cos(t) + \cos^3(t) = 0, \tag{25}$$

$$u(0) = 0, \quad u'(0) = 1,$$

which have the exact solution

$$u(t) = \sin(t). \tag{26}$$

In order to find an approximate solution for (25) by means of LPFTH, we obtain the Taylor series of the trigonometric terms, resulting is

$$-2 \cos(t) + \cos^3(t) = -1 - \frac{1}{2}t^2 + \frac{19}{24}t^4 - \frac{181}{720}t^6, \tag{27}$$

where the expansion order is 7.

From (9), we establish the following homotopy equation

$$(1 - p)(v'') + p\left(v'' + v' + v + v^2v' - 1 - \frac{1}{2}t^2 + \frac{19}{24}t^4 - \frac{181}{720}t^6\right) + bp(1 - p)t^3 = 0, \tag{28}$$

where b is an adjustment parameter due to the auxiliary term $bp(1 - p)t^3$.

From (10), we assume that solution for (28) has the following form:

$$v = (v_0 + v_1p + v_2p^2)(1 + at^2p), \tag{29}$$

where a is an adjustment parameter of the fixed term of the homotopy map.

Substituting (29) into (28), and rearranging the terms of the same order of p ,

$$\begin{aligned} p^0 : v_0'' = 0, \quad v_0(0) = 0, \quad v_0'(0) = 1 \\ p^1 : v_1'' - 1 + av_0''t^2 + 4atv_0' + 2av_0 + bt^3 \\ - \frac{1}{2}t^2 + \frac{19}{24}t^4 + v_0 + v_0' - \frac{181}{720}t^6 + v_0^2v_0' = 0, \\ v_1(0) = 0, \quad v_1'(0) = 0, \\ p^2 : v_2'' + av_1''t^2 + 3av_0^2v_0't^2 + 2av_0^3t + 2v_0v_1v_0' + 4atv_1' \\ - bt^3 + v_0^2v_1' + 2atv_0 + v_0at^2 + v_1 + av_0^2t^2 + v_1' + 2av_1 = 0, \\ v_2(0) = 0, \quad v_2'(0) = 0. \\ \vdots \end{aligned} \tag{30}$$

By solving (30), we obtain

$$\begin{aligned} v_0 &= t, \\ v_1 &= \frac{181}{40320}t^8 - \frac{1}{20}bt^5 - \frac{19}{720}t^6 - \frac{1}{24}t^4 + \left(-a - \frac{1}{6}\right)t^3, \\ v_2 &= \frac{181}{443520}t^{11} + \left(-\frac{181}{40320}a - \frac{181}{3628800}\right)t^{10} + \frac{883}{362880}t^9 \\ &+ \left(\frac{19}{720}a + \frac{1}{160}b + \frac{19}{40320}\right)t^8 \\ &+ \left(\frac{1}{20}ab + \frac{1}{840}b + \frac{7}{720}\right)t^7 + \left(\frac{1}{24}a + \frac{1}{120}b + \frac{7}{240}\right)t^6 \\ &+ \left(a^2 + \frac{1}{6}a + \frac{1}{20}b + \frac{1}{60}\right)t^5 + \frac{1}{24}t^4. \\ &\vdots \end{aligned} \tag{31}$$

Substituting (31) into (29), and calculating the limit when $p \rightarrow 1$, we obtain the second order approximated solution

$$u(t) = \lim_{p \rightarrow 1} v = (v_0 + v_1 + v_2)(1 + at^2). \tag{32}$$

Then, we select the adjustment parameters as: $a = 0.4431998183E-3$, $b = -2.1761220576$; where the parameters are calculated using the NonlinearFit command from Maple Release 13 [5, 32–34]. Moreover, NonlinearFit command finds values of the approximate model parameters such that the sum of the squared k -residuals is minimized.

In order to guarantee the validity of the approximate solution (32) for large t , the series solution is transformed by the Laplace-Padé after-treatment. First, Laplace transformation is applied to (32) and then $1/t$ is written in place of s in the equation. Then, the Padé approximant [2/2] is applied and $1/s$ is written in place of t . Finally, by using the inverse Laplace s transformation, we obtain the modified approximate solution

$$u(t) = 0.9986730464 \sin(1.0013287167t). \tag{33}$$

6. Troesch's Problem

The Troesch's equation is a boundary value problem (BVP) that arises in the investigation of confinement of a plasma column by a radiation pressure [71] and also in the theory of gas porous electrodes [72, 73]. The problem is expressed as

$$y'' = n \sinh(ny), \quad y(0) = 0, \quad y(1) = 1, \tag{34}$$

where prime denotes differentiation with respect to x and n is known as Troesch's parameter.

Straightforward application of FTH to solve (34) is not possible due to the hyperbolic sin term of dependent variable. Nevertheless, the polynomial type nonlinearities are easier to handle by the FTH method. Therefore, in order to apply FTH successfully, we convert the hyperbolic-type nonlinearity in Troesch's problem into a polynomial type nonlinearity, using the variable transformation reported in [51, 52]

$$u(x) = \tanh\left(\frac{n}{4}y(x)\right). \tag{35}$$

After using (35), we obtain the following transformed problem:

$$(1 - u^2)u'' + 2u(u')^2 - n^2u(1 + u^2) = 0, \tag{36}$$

where conditions are obtained by using variable transformation (35).

Substituting original boundary conditions $y(0) = 0$ and $y(1) = 1$ into (35) results in

$$u(0) = 0, \quad u(1) = \tanh\left(\frac{n}{4}\right). \tag{37}$$

TABLE 1: Comparison between exact solution (26) for (25) and the results of approximations: LPFTH (33), HPM [6], and VIM [6] (A.E. means absolute error).

t	Exact (26)	LPFTH (33)	HPM [6]	VIM [6]	A.E. of LPFTH	A.E. of HPM	A.E. of VIM
1.0	0.84147098	0.84107061	0.83819112	0.82124433	0.00040038	0.00327987	0.02022665
2.0	0.90929743	0.90698321	0.78090770	0.23121704	0.00231422	0.12838972	0.67808039
3.0	0.14112001	0.13699062	-0.19760326	-3.14507873	0.00412939	0.33872327	3.28619874
4.0	-0.75680250	-0.75925698	-0.94834234	-8.80326033	0.00245449	0.19153985	8.04645783
5.0	-0.95892427	-0.95574867	-1.02405737	-16.02894628	0.00317560	0.06513310	15.07002201
6.0	-0.27941550	-0.27139133	-0.11322005	-24.21046104	0.00802417	0.16619545	23.93104554
7.0	0.65698660	0.66308908	0.88967763	-34.18610854	0.00610248	0.23269103	34.84309514
8.0	0.98935825	0.98644505	1.23972759	-46.77116918	0.00291320	0.25036935	47.76052743
9.0	0.41211849	0.40066121	0.43840592	-62.53129090	0.01145728	0.02628744	62.94340939
10.0	-0.54402111	-0.55438502	-0.70915755	-80.67754120	0.01036391	0.16513644	80.13352009

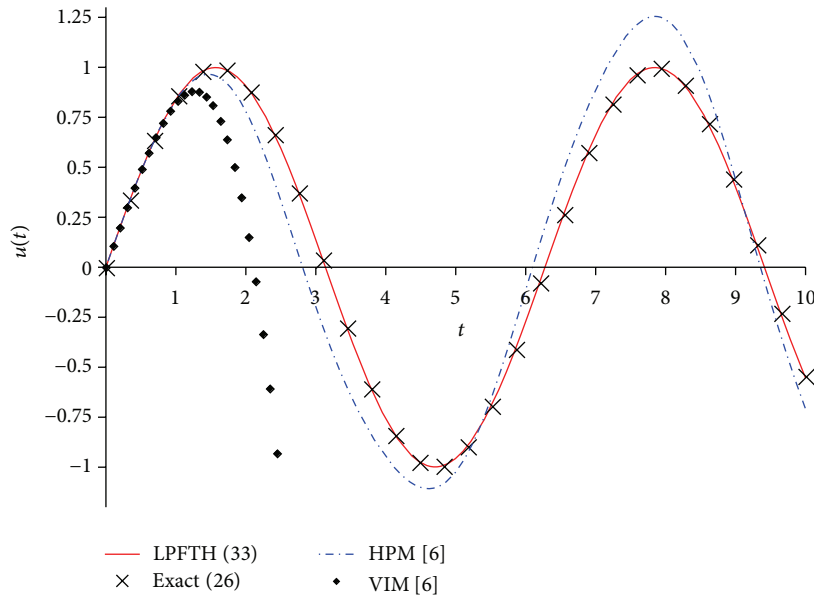


FIGURE 1: Exact solution (26) (diagonal cross) for (25) and its approximate solutions (33) (solid line), HPM [6] (dash-dot), and VIM [6] (solid diamond).

From (9) and (36), we can formulate the following homotopy [7–9]:

$$\begin{aligned}
 H(v, p) &= (1 - p)v'' \\
 &+ p\left(\left(1 - v^2\right)v'' + 2v\left(v'\right)^2 - n^2v\left(1 + v^2\right)\right) \\
 &+ pc(1 - p)x^6 = 0,
 \end{aligned}
 \tag{38}$$

where p is the homotopy parameter and c is an adjustment parameter due to the auxiliary term $pc(1 - p)x^6$.

From (10), we assume that solution for (38) has the following form:

$$v = \left(v_0 + v_1p + v_2p^2\right)\left(1 + pax + p^2bx^2\right), \tag{39}$$

where a, b are the adjustment parameters due to the fixed term $(1 + pax + p^2bx^2)$ of the homotopy map.

Substituting (39) into (38) and equating identical powers of p terms, we obtain

$$\begin{aligned}
 p^0 : v_0'' &= 0, \quad v_0(0) = 0, \quad v_0(1) = \gamma, \\
 p^1 : v_1'' - v_0^2v_0'' - n^2v_0^3 + 2v_0v_0'^2 + v_0''ax + 2v_0'a - n^2v_0 & \\
 &+ cx^6 = 0, \quad v_1(0) = 0, \quad v_1(1) = 0, \\
 p^2 : v_2'' + n^2v_1 + 2v_1'a + 6v_0v_0'^2ax - v_0^2v_1'' + 2v_0^2v_0'a & \tag{40} \\
 &- 3n^2v_0^3ax - n^2v_0ax + 2bv_0 + 4v_0'bx + 4v_0v_1'v_0' \\
 &- cx^6 + v_0''bx^2 - 3n^2v_1v_0^2 - 2v_0v_1v_0'' + 2v_1v_0'^2 \\
 &+ v_1''ax - 3v_0^2axv_0'' = 0, \quad v_2(0) = 0, \quad v_2(1) = 0,
 \end{aligned}$$

where $\gamma = \tanh(n/4)$.

TABLE 2: Comparison between (43), exact solution [74, 75], and other reported approximate solutions, using $n = 0.5$.

x	Exact [74, 75]	This work (43)	HPM [52]	DMA [53]	HPM [54]	HPM [55]	HAM [56]	LDTM [57]
0.1	0.0959443493	0.0959443679	0.0959443155	0.0959383534	0.0959395656	0.095948026	0.0959446190	0.0959443520
0.2	0.1921287477	0.1921287510	0.1921286848	0.1921180592	0.1921193244	0.192135797	0.1921292845	0.1921287539
0.3	0.2887944009	0.2887943893	0.2887943176	0.2887803297	0.2887806940	0.288804238	0.2887952148	0.2887944107
0.4	0.3861848464	0.3861848355	0.3861847539	0.3861687095	0.3861675428	0.386196642	0.3861859313	0.3861848612
0.5	0.4845471647	0.4845471662	0.4845470753	0.4845302901	0.4845274183	0.4845599	0.4845485110	0.4845471832
0.6	0.5841332484	0.5841332589	0.5841331729	0.5841169798	0.5841127822	0.584145785	0.5841348222	0.5841332650
0.7	0.6852011483	0.6852011536	0.6852010943	0.6851868451	0.6851822495	0.685212297	0.6852028604	0.6852011675
0.8	0.7880165227	0.7880165163	0.7880164925	0.7880055691	0.7880018367	0.788025104	0.7880181729	0.7880165463
0.9	0.8928542161	0.8928542136	0.8928542059	0.8928480234	0.8928462193	0.892859085	0.8928553997	0.8928542363
Order		2	2	6	2	2	6	3
A.A.R.E.		3.5462E(-08)	1.83327E(-07)	3.47802E(-05)	3.57932E(-05)	2.44418E(-05)	2.51374E(-06)	3.10957E(-08)

We solve (40) by using Maple software, resulting in

$$\begin{aligned}
 v_0 &= \gamma x, \\
 v_1 &= \left(\frac{1}{56}c - \frac{1}{20}\gamma^3 n^2 + \frac{1}{3}\gamma^3 - \frac{1}{6}\gamma n^2 + \gamma a \right) x - \gamma a x^2 \\
 &\quad + \left(-\frac{1}{3}\gamma^3 + \frac{1}{6}\gamma n^2 \right) x^3 + \frac{1}{20}\gamma^3 n^2 x^5 - \frac{1}{56}c x^8, \\
 v_2 &= \left(-\frac{1}{56}c + b\gamma + \frac{1}{45}\gamma^2 c - \frac{31}{420}\gamma^5 n^2 + \frac{17}{840}n^4 \gamma^3 + \frac{7}{360}n^4 \gamma \right. \\
 &\quad \left. - \frac{5}{36}\gamma^3 n^2 - \frac{1}{360}c n^2 - \frac{3}{20}a\gamma^3 n^2 - \frac{1}{6}a\gamma n^2 - \frac{1}{440}n^2 \gamma^2 c \right. \\
 &\quad \left. + \frac{13}{2400}n^4 \gamma^5 + a\gamma^3 + \frac{1}{5}\gamma^5 \right) x \\
 &\quad + \left(\frac{1}{20}a\gamma^3 n^2 + \frac{1}{6}a\gamma n^2 - \frac{1}{56}ac - \frac{1}{3}a\gamma^3 - \gamma a^2 \right) x^2 \\
 &\quad + \left(\frac{1}{20}\gamma^5 n^2 - a\gamma^3 - \frac{1}{3}\gamma^5 + \frac{2}{9}\gamma^3 n^2 - b\gamma - \frac{1}{36}n^4 \gamma \right. \\
 &\quad \left. + \frac{1}{6}a\gamma n^2 - \frac{1}{56}\gamma^2 c + \gamma a^2 + \frac{1}{336}c n^2 - \frac{1}{120}n^4 \gamma^3 \right) x^3 \\
 &\quad + \left(\frac{1}{3}a\gamma^3 - \frac{1}{6}a\gamma n^2 \right) x^4 + \left(\frac{1}{120}n^4 \gamma + \frac{2}{15}\gamma^5 - \frac{1}{12}\gamma^3 n^2 \right. \\
 &\quad \left. + \frac{3}{1120}n^2 \gamma^2 c + \frac{3}{20}a\gamma^3 n^2 \right. \\
 &\quad \left. + \frac{1}{20}\gamma^5 n^2 - \frac{1}{40}n^4 \gamma^3 \right. \\
 &\quad \left. - \frac{3}{400}n^4 \gamma^5 \right) x^5 - \frac{1}{20}a\gamma^3 n^2 x^6 \\
 &\quad + \left(\frac{11}{840}n^4 \gamma^3 - \frac{11}{420}\gamma^5 n^2 \right) x^7 + \frac{1}{56}c x^8 \\
 &\quad + \left(\frac{1}{56}ac + \frac{1}{480}n^4 \gamma^5 \right) x^9 + \left(-\frac{11}{2520}\gamma^2 c - \frac{1}{5040}c n^2 \right) x^{10} \\
 &\quad - \frac{1}{2464}n^2 \gamma^2 c x^{12}.
 \end{aligned} \tag{41}$$

Substituting (41) into (39), and calculating the limit when $p \rightarrow 1$, we obtain the second order approximated solution of (36)

$$u_2(x) = \lim_{p \rightarrow 1} v = (v_0 + v_1 + v_2)(1 + ax + bx^2). \tag{42}$$

Finally, from (35) and (42), the proposed solution of Troesch's problem is

$$y(x) = \frac{4}{n} \tanh^{-1}(u_2(x)), \quad 0 \leq x \leq 1. \tag{43}$$

If we consider $n = 0.5$, then we choose the adjustment parameter as: $a = 0.4144179560E-5$, $b = 1.7343105421E-8$,

and $c = 0.2046343553E-2$, by using the procedure explained for the Van der Pol Oscillator problem.

7. Numerical Simulation and Discussion

A comparison of the exact solution (26) of the Van der Pol Oscillator against the approximated solutions obtained by LPFTH (33), HPM [6], and VIM [6] is shown in Table 1 and Figure 1. In the comparison can be seen that the LPFTH method exhibited the higher accuracy for a large period of time reproducing successfully the oscillatory behaviour of the exact solution (26), therefore, being a suitable alternative to solve nonlinear oscillators.

The Troesch's BVP problem is a benchmark equation for numerical [74, 75] and semi-analytical methods [51–57] which has been solved by FTH to obtain the approximated solution (43). Table 2 shows a comparison of the results obtained with other semi-analytical methods as: homotopy perturbation method (HPM) [52, 54, 55], decomposition method approximation (DMA) [53, 54], homotopy analysis method (HAM) [56], and Laplace transform decomposition method (LTDm) [57]. The comparison shows that the average absolute relative error (A.A.R.E.) of (43) is lower than most of the reported results and similar to LTDm [57].

For both cases of study, polynomial functions were used for fixed and auxiliary terms as they generate the best results. Nonetheless, functions as exponential, trigonometric or hyperbolic may enlarge the domain of convergence. Thereby, a methodology that allows the selection of fixed and auxiliary terms like these to obtain more accurate solutions is a window of opportunity to be exploited.

Since FTH does not rely in the linearization of the input equations, a perturbation parameter nor assumption of weak nonlinearity, the solution generated may be general and more realistic than the method of simplifying the physical problems.

8. Conclusions

In this work, the Fixed Term Homotopy and the Laplace-Padé Fixed Term Homotopy methods are presented as novel tools to solve nonlinear ordinary differential equations. The proposed methods were tested using two nonlinear problems: a second order nonlinear oscillator and a high nonlinear boundary value problem. From a comparison against several semi-analytical methods from the literature, FTH and LPFTH proved to be power tools that generate highly accurate easy manageable expressions. Furthermore, the homotopy given in (9) may be replaced or modified by a formulation inspired by some other homotopy method proposed in literature [76–96], which can lead to improvement of convergence of proposed methods. Additionally, being FTH a modified version of HPM can be assumed that the differential equations solved by HPM should be also solvable by FTH. In that fashion, further research can be focused on applying FTH method in the solution of nonlinear partial differential

equations, nonlinear fractional differential equations, among others.

Acknowledgments

The authors gratefully acknowledge the financial support of the National Council for Science and Technology of Mexico (CONACyT) through Grant CB-2010-01 no. 157024. The authors would like to thank Roberto Castaneda-Sheissa, Rogelio-Alejandro Callejas-Molina, and Roberto Ruiz-Gomez for their contribution to this project.

References

- [1] J. Biazar and H. Aminikhah, "Study of convergence of homotopy perturbation method for systems of partial differential equations," *Computers & Mathematics with Applications*, vol. 58, no. 11-12, pp. 2221-2230, 2009.
- [2] J. Biazar and H. Ghazvini, "Convergence of the homotopy perturbation method for partial differential equations," *Nonlinear Analysis. Real World Applications*, vol. 10, no. 5, pp. 2633-2640, 2009.
- [3] D. Ağırseven and T. Öziş, "An analytical study for Fisher type equations by using homotopy perturbation method," *Computers & Mathematics with Applications*, vol. 60, no. 3, pp. 602-609, 2010.
- [4] T. Öziş and D. Ağırseven, "He's homotopy perturbation method for solving heat-like and wave-like equations with variable coefficients," *Physics Letters A*, vol. 372, no. 38, pp. 5944-5950, 2008.
- [5] H. Vazquez-Leal, "Rational homotopy perturbation method," *Journal of Applied Mathematics*, vol. 2012, Article ID 490342, 14 pages, 2012.
- [6] A. Barari, M. Omidvar, A. R. Ghotbi, and D. D. Ganji, "Application of homotopy perturbation method and variational iteration method to nonlinear oscillator differential equations," *Acta Applicandae Mathematicae*, vol. 104, no. 2, pp. 161-171, 2008.
- [7] J.-H. He, "Comparison of homotopy perturbation method and homotopy analysis method," *Applied Mathematics and Computation*, vol. 156, no. 2, pp. 527-539, 2004.
- [8] J.-H. He, "An elementary introduction to the homotopy perturbation method," *Computers & Mathematics with Applications*, vol. 57, no. 3, pp. 410-412, 2009.
- [9] J.-H. He, "Homotopy perturbation technique," *Computer Methods in Applied Mechanics and Engineering*, vol. 178, no. 3-4, pp. 257-262, 1999.
- [10] J.-H. He, "The homotopy perturbation method nonlinear oscillators with discontinuities," *Applied Mathematics and Computation*, vol. 151, no. 1, pp. 287-292, 2004.
- [11] J.-H. He, "Homotopy perturbation method: a new nonlinear analytical technique," *Applied Mathematics and Computation*, vol. 135, no. 1, pp. 73-79, 2003.
- [12] J.-H. He, "A coupling method of a homotopy technique and a perturbation technique for non-linear problems," *International Journal of Non-Linear Mechanics*, vol. 35, no. 1, pp. 37-43, 2000.
- [13] J. H. He, "Application of homotopy perturbation method to nonlinear wave equations," *Chaos, Solitons and Fractals*, vol. 26, no. 3, pp. 695-700, 2005.
- [14] J.-H. He, "Homotopy perturbation method for solving boundary value problems," *Physics Letters A*, vol. 350, no. 1-2, pp. 87-88, 2006.
- [15] Y. Khan, H. Vazquez-Leal, and Q. Wu, "An efficient iterated method for mathematical biology model," *Neural Computing and Applications*. In press.
- [16] J. Biazar and M. Eslami, "A new homotopy perturbation method for solving systems of partial differential equations," *Computers & Mathematics with Applications*, vol. 62, no. 1, pp. 225-234, 2011.
- [17] J. Biazar and B. Ghanbari, "The homotopy perturbation method for solving neutral functional-differential equations with proportional delays," *Journal of King Saud University*, vol. 24, no. 1, pp. 33-37, 2012.
- [18] D. D. Ganji, H. Tari, and M. B. Jooybari, "Variational iteration method and homotopy perturbation method for nonlinear evolution equations," *Computers & Mathematics with Applications*, vol. 54, no. 7-8, pp. 1018-1027, 2007.
- [19] M. Sheikholeslami, H. R. Ashorynejad, D. D. Ganji, and A. Yildirim, "Homotopy perturbation method for three-dimensional problem of condensation film on inclined rotating disk," *Scientia Iranica*, vol. 19, no. 3, pp. 437-442, 2012.
- [20] D. D. Ganji and A. Sadighi, "Application of homotopy-perturbation and variational iteration methods to nonlinear heat transfer and porous media equations," *Journal of Computational and Applied Mathematics*, vol. 207, no. 1, pp. 24-34, 2007.
- [21] D. D. Ganji, "The application of He's homotopy perturbation method to nonlinear equations arising in heat transfer," *Physics Letters A*, vol. 355, no. 4-5, pp. 337-341, 2006.
- [22] M. Rafei, D. D. Ganji, and H. Daliani, "Solution of the epidemic model by homotopy perturbation method," *Applied Mathematics and Computation*, vol. 187, no. 2, pp. 1056-1062, 2007.
- [23] D. D. Ganji and A. Rajabi, "Assessment of homotopy-perturbation and perturbation methods in heat radiation equations," *International Communications in Heat and Mass Transfer*, vol. 33, no. 3, pp. 391-400, 2006.
- [24] Y. Khan, Q. Wu, N. Faraz, A. Yildirim, and M. Madani, "A new fractional analytical approach via a modified riemannliouville derivative," *Applied Mathematics Letters*, vol. 25, no. 10, pp. 1340-1346, 2012.
- [25] N. Faraz and Y. Khan, "Analytical solution of electrically conducted rotating flow of a second grade fluid over a shrinking surface," *Ain Shams Engineering Journal*, vol. 2, no. 34, pp. 221-226, 2011.
- [26] Y. Khan, Q. Wu, N. Faraz, and A. Yildirim, "The effects of variable viscosity and thermal conductivity on a thin film flow over a shrinking/stretching sheet," *Computers & Mathematics with Applications*, vol. 61, no. 11, pp. 3391-3399, 2011.
- [27] M. Fathizadeh, M. Madani, Y. Khan, N. Faraz, A. Yildirim, and S. Tutkun, "An effective modification of the homotopy perturbation method for mhd viscous flow over a stretching sheet," *Journal of King Saud University*. In press.
- [28] M. Madani, M. Fathizadeh, Y. Khan, and A. Yildirim, "On the coupling of the homotopy perturbation method and Laplace transformation," *Mathematical and Computer Modelling*, vol. 53, no. 9-10, pp. 1937-1945, 2011.
- [29] H. Aminikhah, "The combined Laplace transform and new homotopy perturbation methods for stiff systems of ODEs," *Applied Mathematical Modelling*, vol. 36, no. 8, pp. 3638-3644, 2012.
- [30] F. I. Compean, D. Olvera, F. J. Campa, L. N. Lopez de Lacalle, A. Elias-Zuniga, and C. A. Rodriguez, "Characterization and stability analysis of a multivariable milling tool by the enhanced multistage homotopy perturbation method," *International Journal of Machine Tools and Manufacture*, vol. 57, pp. 27-33, 2012.

- [31] A. M. A. El-Sayed, A. Elsaid, I. L. El-Kalla, and D. Hammad, "A homotopy perturbation technique for solving partial differential equations of fractional order in finite domains," *Applied Mathematics and Computation*, vol. 218, no. 17, pp. 8329–8340, 2012.
- [32] H. Vázquez-Leal, R. Castaneda-Sheissa, U. Filobello-Nino, A. Sarmiento-Reyes, and J. Sánchez-Orea, "High accurate simple approximation of normal distribution related integrals," *Mathematical Problems in Engineering*, vol. 2012, Article ID 124029, 22 pages, 2012.
- [33] H. Vazquez-Leal, U. Filobello-Nino, R. Castaneda-Sheissa, L. Hernandez-Martinez, and A. Sarmiento-Reyes, "Modified hpms inspired by homotopy continuation methods," *Mathematical Problems in Engineering*, vol. 2012, Article ID 309123, 19 pages, 2012.
- [34] U. Filobello-Nino, H. Vazquez-Leal, R. Castaneda-Sheissa et al., "An approximate solution of blasius equation by using hpm method," *Asian Journal of Mathematics and Statistics*, vol. 5, pp. 50–59, 2012.
- [35] U. Filobello-Nino, H. Vazquez-Leal, Y. Khan et al., "HPM applied to solve nonlinear circuits: a study case," *Applied Mathematical Sciences*, vol. 6, no. 85-88, pp. 4331–4344, 2012.
- [36] J. Biazar, F. Badpeima, and F. Azimi, "Application of the homotopy perturbation method to Zakharov-Kuznetsov equations," *Computers & Mathematics with Applications*, vol. 58, no. 11-12, pp. 2391–2394, 2009.
- [37] J. Biazar and H. Ghazvini, "Homotopy perturbation method for solving hyperbolic partial differential equations," *Computers & Mathematics with Applications*, vol. 56, no. 2, pp. 453–458, 2008.
- [38] J. Biazar and H. Ghazvini, "Exact solutions for non-linear Schrödinger equations by He's homotopy perturbation method," *Physics Letters A*, vol. 366, no. 1-2, pp. 79–84, 2007.
- [39] J. Biazar, H. Ghazvini, and M. Eslami, "He's homotopy perturbation method for systems of integro-differential equations," *Chaos, Solitons & Fractals*, vol. 39, no. 3, pp. 1253–1258, 2009.
- [40] J. Biazar, M. Eslami, and H. Aminikhah, "Application of homotopy perturbation method for systems of volterra integral equations of the first kind," *Chaos, Solitons and Fractals*, vol. 42, no. 5, pp. 3020–3026, 2009.
- [41] J. Biazar and H. Ghazvini, "He's homotopy perturbation method for solving systems of Volterra integral equations of the second kind," *Chaos, Solitons & Fractals*, vol. 39, no. 2, pp. 770–777, 2009.
- [42] J. Biazar and H. Ghazvini, "Numerical solution for special nonlinear Fredholm integral equation by HPM," *Applied Mathematics and Computation*, vol. 195, no. 2, pp. 681–687, 2008.
- [43] M. Jalaal, D. D. Ganji, and G. Ahmadi, "Analytical investigation on acceleration motion of a vertically falling spherical particle in incompressible Newtonian media," *Advanced Powder Technology*, vol. 21, no. 3, pp. 298–304, 2010.
- [44] M. Jalaal and D. D. Ganji, "On unsteady rolling motion of spheres in inclined tubes filled with incompressible Newtonian fluids," *Advanced Powder Technology*, vol. 22, no. 1, pp. 58–67, 2011.
- [45] Z. Ziabakhsh and G. Domairry, "Solution of the laminar viscous flow in a semi-porous channel in the presence of a uniform magnetic field by using the homotopy analysis method," *Communications in Nonlinear Science and Numerical Simulation*, vol. 14, no. 4, pp. 1284–1294, 2009.
- [46] D. D. Ganji, G. A. Afrouzi, and R. A. Talarposhti, "Application of variational iteration method and homotopy-perturbation method for nonlinear heat diffusion and heat transfer equations," *Physics Letters A*, vol. 368, no. 6, pp. 450–457, 2007.
- [47] M. Rafei, H. Daniali, and D. D. Ganji, "Variational iteration method for solving the epidemic model and the prey and predator problem," *Applied Mathematics and Computation*, vol. 186, no. 2, pp. 1701–1709, 2007.
- [48] G. Domairry, A. Mohsenzadeh, and M. Famouri, "The application of homotopy analysis method to solve nonlinear differential equation governing Jeffery-Hamel flow," *Communications in Nonlinear Science and Numerical Simulation*, vol. 14, no. 1, pp. 85–95, 2009.
- [49] A. Sadighi and D. D. Ganji, "Analytic treatment of linear and nonlinear Schrödinger equations: a study with homotopy-perturbation and Adomian decomposition methods," *Physics Letters A*, vol. 372, no. 4, pp. 465–469, 2008.
- [50] S. H. Behiry, H. Hashish, I. L. El-Kalla, and A. Elsaid, "A new algorithm for the decomposition solution of nonlinear differential equations," *Computers & Mathematics with Applications*, vol. 54, no. 4, pp. 459–466, 2007.
- [51] S.-H. Chang, "A variational iteration method for solving Troesch's problem," *Journal of Computational and Applied Mathematics*, vol. 234, no. 10, pp. 3043–3047, 2010.
- [52] H. Vazquez-Leal, Y. Khan, G. Fernandez-Anaya et al., "A general solution for troesch's problem," *Mathematical Problems in Engineering*, vol. 2012, Article ID 208375, 14 pages, 2012.
- [53] E. Deeba, S. A. Khuri, and S. Xie, "An algorithm for solving boundary value problems," *Journal of Computational Physics*, vol. 159, no. 2, pp. 125–138, 2000.
- [54] X. Feng, L. Mei, and G. He, "An efficient algorithm for solving Troesch's problem," *Applied Mathematics and Computation*, vol. 189, no. 1, pp. 500–507, 2007.
- [55] S. H. Mirmoradi, I. Hosseinpour, S. Ghanbarpour, and A. Barari, "Application of an approximate analytical method to nonlinear Troesch's problem," *Applied Mathematical Sciences*, vol. 3, no. 29-32, pp. 1579–1585, 2009.
- [56] H. N. Hassan and M. A. El-Tawil, "An efficient analytic approach for solving two-point nonlinear boundary value problems by homotopy analysis method," *Mathematical Methods in the Applied Sciences*, vol. 34, no. 8, pp. 977–989, 2011.
- [57] S. A. Khuri, "A numerical algorithm for solving Troesch's problem," *International Journal of Computer Mathematics*, vol. 80, no. 4, pp. 493–498, 2003.
- [58] Y.-G. Wang, W.-H. Lin, and N. Liu, "A homotopy perturbation-based method for large deflection of a cantilever beam under a terminal follower force," *International Journal for Computational Methods in Engineering Science and Mechanics*, vol. 13, no. 3, pp. 197–201, 2012.
- [59] J.-H. He, "Homotopy perturbation method with an auxiliary term," *Abstract and Applied Analysis*, vol. 2012, Article ID 857612, 7 pages, 2012.
- [60] H. Vazquez-Leal, A. Sarmiento-Reyes, Y. Khan, U. Filobello-Nino, and A. Diaz-Sanchez, "Rational biparameter homotopy perturbation method and laplace-padé coupled version," *Journal of Applied Mathematics*, vol. 2012, Article ID 923975, 21 pages, 2012.
- [61] Y. C. Jiao, Y. Yamamoto, C. Dang, and Y. Hao, "An aftertreatment technique for improving the accuracy of Adomian's decomposition method," *Computers & Mathematics with Applications*, vol. 43, no. 6-7, pp. 783–798, 2002.
- [62] N. H. Sweilam and M. M. Khader, "Exact solutions of some coupled nonlinear partial differential equations using the

- homotopy perturbation method," *Computers & Mathematics with Applications*, vol. 58, no. 11-12, pp. 2134–2141, 2009.
- [63] S. Momani, G. H. Erjaee, and M. H. Alnasr, "The modified homotopy perturbation method for solving strongly nonlinear oscillators," *Computers & Mathematics with Applications*, vol. 58, no. 11-12, pp. 2209–2220, 2009.
- [64] Y. Khan and N. Faraz, "Application of modified Laplace decomposition method for solving boundary layer equation," *Journal of King Saud University*, vol. 23, no. 1, pp. 115–119, 2011.
- [65] D. Bahuguna, A. Ujlayan, and D. N. Pandey, "A comparative study of numerical methods for solving an integro-differential equation," *Computers & Mathematics with Applications*, vol. 57, no. 9, pp. 1485–1493, 2009.
- [66] S. Momani and V. S. Ertürk, "Solutions of non-linear oscillators by the modified differential transform method," *Computers & Mathematics with Applications*, vol. 55, no. 4, pp. 833–842, 2008.
- [67] A. Gökdoğan, M. Merdan, and A. Yildirim, "The modified algorithm for the differential transform method to solution of Genesis systems," *Communications in Nonlinear Science and Numerical Simulation*, vol. 17, no. 1, pp. 45–51, 2012.
- [68] P.-Y. Tsai and C.-K. Chen, "An approximate analytic solution of the nonlinear Riccati differential equation," *Journal of the Franklin Institute. Engineering and Applied Mathematics*, vol. 347, no. 10, pp. 1850–1862, 2010.
- [69] A. E. Ebaid, "A reliable aftertreatment for improving the differential transformation method and its application to nonlinear oscillators with fractional nonlinearities," *Communications in Nonlinear Science and Numerical Simulation*, vol. 16, no. 1, pp. 528–536, 2011.
- [70] M. Merdan, A. Gökdoğan, and A. Yildirim, "On the numerical solution of the model for HIV infection of CD4⁺ T cells," *Computers & Mathematics with Applications*, vol. 62, no. 1, pp. 118–123, 2011.
- [71] E. S. Weibel, "On the confinement of a plasma by magnetostatic fields," *Physics of Fluids*, vol. 2, no. 1, pp. 52–56, 1959.
- [72] D. Gidaspow and B. S. Baker, "A model for discharge of storage batteries," *Journal of the Electrochemical Society*, vol. 120, no. 8, pp. 1005–1010, 1973.
- [73] V. S. Markin, A. A. Chernenko, Y. A. Chizmadehev, and Y. G. Chirkov, "Aspects of the theory of gas porous electrodes," in *Fuel Cells: Their Electrochemical Kinetics*, pp. 22–33, Consultants Bureau, New York, NY, USA, 1966.
- [74] U. Erdogan and T. Ozis, "A smart nonstandard finite difference scheme for second order nonlinear boundary value problems," *Journal of Computational Physics*, vol. 230, no. 17, pp. 6464–6474, 2011.
- [75] Y. Lin, J. A. Enszer, and M. A. Stadtherr, "Enclosing all solutions of two-point boundary value problems for ODEs," *Computers and Chemical Engineering*, vol. 32, no. 8, pp. 1714–1725, 2008.
- [76] H. Vazquez-Leal, R. Castaneda-Sheissa, A. Yildirim et al., "Biparameter homotopy-based direct current simulation of multistable circuits," *British Journal of Mathematics & Computer Science*, vol. 2, no. 3, pp. 137–150, 2012.
- [77] K. Reif, K. Weinzierl, A. Zell, and R. Unbehauen, "A homotopy approach for nonlinear control synthesis," *IEEE Transactions on Automatic Control*, vol. 43, no. 9, pp. 1311–1318, 1998.
- [78] N. Aslam and A. K. Sunol, "Homotopy continuation based prediction of azeotropy in binary and multicomponent mixtures through equations of state," *Physical Chemistry Chemical Physics*, vol. 6, no. 9, pp. 2320–2326, 2004.
- [79] F. Kubler and K. Schmedders, "Computing equilibria in stochastic finance economies," *Computational Economics*, vol. 15, no. 1-2, pp. 145–172, 2000.
- [80] R. C. Melville and L. Trajković, "Artificial parameter homotopy methods for the dc operating point problem," *IEEE Transactions on Computer-Aided Design of Integrated Circuits and Systems*, vol. 12, no. 6, pp. 861–877, 1997.
- [81] K. Yamamura, T. Sekiguchi, and Y. Inoue, "A fixed-point homotopy method for solving modified nodal equations," *IEEE Transactions on Circuits and Systems*, vol. 46, no. 6, pp. 654–665, 1999.
- [82] R. Geoghegan, J. C. Lagarias, and R. C. Melville, "Threading homotopies and dc operating points of nonlinear circuits," *SIAM Journal on Optimization*, vol. 9, no. 1, pp. 159–178, 1999.
- [83] J. Lee and H.-D. Chiang, "Convergent regions of the Newton homotopy method for nonlinear systems: theory and computational applications," *IEEE Transactions on Circuits and Systems I*, vol. 48, no. 1, pp. 51–66, 2001.
- [84] A. Ushida, Y. Yamagami, Y. Nishio, I. Kinouchi, and Y. Inoue, "An efficient algorithm for finding multiple DC solutions based on the SPICE-oriented Newton homotopy method," *IEEE Transactions on Computer-Aided Design of Integrated Circuits and Systems*, vol. 21, no. 3, pp. 337–348, 2002.
- [85] H. Vazquez-Leal, L. Hernaandez-Martinez, and A. Sarmiento-Reyes, "Double-bounded homotopy for analysing nonlinear resistive circuits," *International Symposium on Circuits and Systems*, vol. 4, pp. 3203–3206, 2005.
- [86] M. Van Barel, Kh. D. Ikramov, and A. A. Chesnokov, "A continuation method for solving symmetric toeplitz systems," *Computational Mathematics and Mathematical Physics*, vol. 48, no. 12, pp. 2126–2139, 2008.
- [87] H. Vázquez-Leal, L. Hernández-Martínez, A. Sarmiento-Reyes, and R. Castañeda-Sheissa, "Numerical continuation scheme for tracing the double bounded homotopy for analysing nonlinear circuits," in *Proceedings of the International Conference on Communications, Circuits and Systems*, pp. 1122–1126, May 2005.
- [88] M. M. Green, "An efficient continuation method for use in globally convergent dc circuit simulation," in *Proceedings of URSI International Symposium on Signals, Systems, and Electronics (ISSSE '95)*, pp. 497–500, October 1995.
- [89] L. B. Goldgeisser and M. M. Green, "A method for automatically finding multiple operating points in nonlinear circuits," *IEEE Transactions on Circuits and Systems*, vol. 52, no. 4, pp. 776–784, 2005.
- [90] D. M. Wolf and S. R. Sanders, "Multiparameter homotopy methods for finding dc operating points of nonlinear circuits," *IEEE Transactions on Circuits and Systems*, vol. 43, no. 10, pp. 824–838, 1996.
- [91] Y. Yamamoto, "A variable dimension fixed point algorithm and the orientation of simplices," *Mathematical Programming*, vol. 30, no. 3, pp. 301–312, 1984.
- [92] H. Vázquez-Leal, L. Hernández-Martínez, A. Sarmiento-Reyes, and R. S. Murphy-Arteaga, "Improving multi-parameter homotopy via symbolic analysis techniques for circuit simulation," in *Proceedings of the European Conference on Circuit Theory and Design*, vol. 2, pp. 402–405, 2003.
- [93] J. S. Roychowdhury and R. C. Melville, "Homotopy techniques for obtaining a DC solution of large-scale MOS circuits," in *Proceedings of the 33rd Annual Design Automation Conference*, pp. 286–291, June 1996.
- [94] J. Roychowdhury and R. Melville, "Delivering global DC convergence for large mixed-signal circuits via homotopy/

continuation methods," *IEEE Transactions on Computer-Aided Design of Integrated Circuits and Systems*, vol. 25, no. 1, pp. 66–78, 2006.

- [95] Y. Inoue, S. Kusanobu, K. Yamamura, and T. Takahashi, "Newton-fixed-point homotopy method for finding dc operatingpoints of nonlinear circuits," in *Proceedings of the International Technical Conference on Circuits/Systems, Computer and Communications (ITC-CSCC '01)*, vol. 1, pp. 370–373, Tokushima, Japan, July 2001.
- [96] H. Vazquez-leal, L. Hernandez-Martinez, A. Sarmiento-Reyes, R. Castaneda-Sheissa, and A. Gallardo-Del-Angel, "Homotopy method with a formal stop criterion applied to circuit simulation," *IEICE Electronics Express*, vol. 8, no. 21, pp. 1808–1815, 2011.



Hindawi

Submit your manuscripts at
<http://www.hindawi.com>

