ON $M^x/\binom{G_1}{G_2}/1/G(BS)/V_s$ VACATION QUEUE WITH TWO TYPES OF GENERAL HETEROGENEOUS SERVICE

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We analyze a batch arrival queue with a single server providing two kinds of general heterogeneous service. Just before his service starts, a customer may choose one of the services and as soon as a service (of any kind) gets completed, the server may take a vacation or may continue staying in the system. The vacation times are assumed to be general and the server vacations are based on Bernoulli schedules under a single vacation policy. We obtain explicit queue size distribution at a random epoch as well as at a departure epoch and also the mean busy period of the server under the steady state. In addition, some important performance measures such as the expected queue size and the expected waiting time of a customer are obtained. Further, some interesting particular cases are also discussed.

1. Introduction

The single server queues including the $M/G/1$ queue with single arrivals and $M^X/G/1$ queue with batch arrivals have been studied by numerous authors including Burke [2], Gaver [10], Chaudhry [3], Madan [18] and Medhi [23], among several others. Further, various authors studied this type of queues with server vacations under various vacation policies including Bernoulli schedules. Various aspects of Bernoulli vacation models for single server queueing systems including $M/G/1$ queue have been studied by Keilson and Servi [11, 12, 13], Scholl and Kleinrock [27], Servi [28], Ramaswamy and Servi [24], Doshi [8, 9], Takagi [29], Madan [19, 20] and Madan and Baklizi [22], among several others. More recently, most of the studies have been devoted to batch arrival vacation models under different vacation policies because of its interdisciplinary character. Numerous researchers, including Baba [1], Choudhury [4, 5], Choudhury and Borthakur [6], Lee et al. [14, 15], Lee et al. [16], Rosenberg and Yechiali [25] and Teghem [17], Madan and Abu-Dayyeh [21] and many others have studied batch arrival vacation queues under different vacation policies.

All these papers have a common assumption that the system has a single server who provides only one kind of service to the incoming customers. However, in the present

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paper, we consider an extended $M^X/G/1$ queue with batch arrivals, two types of general heterogeneous service and modified Bernoulli schedule server vacations. Just before a service starts, a customer has the option to choose one of the two kinds of services. Such a model may find applications in many day-to-day real-life queueing situations encountered at automobile stations, post offices, banks or computer centers, and so forth. where the server may offer two kinds of service one of which may be chosen by each customer. Surely, the model may have many more wider applications. Further, our model assumes that the server vacations are based on modified Bernoulli schedules under a single vacation policy which means that just after completing a service selected by a customer, the server may take a vacation of random duration or may continue staying in the system and on completion of a vacation period, the server must be back to the system even if there is no customer to serve. This type of modified server vacations under a single vacation policy was recently studied by Madan and Abu-Dayyeh [21] under the policy of restricted admissibility. For convenience, we denote the model under our study as $M^x/\binom{G_1}{G_2}/1/G(BS)/V_s$ queue where $\binom{G_1}{G_2}$ stands for two kinds of parallel general heterogeneous service, (one of which has to be chosen by each customer), $G(BS)$ denotes general service times under Bernoulli schedules and V_s denotes single vacations.

The following assumptions briefly describe the mathematical model of our problem.

2. The mathematical model

Customers (units) arrive at the system in batches of variable size in a compound Poisson process. Let $\lambda c_i dt (i = 1,2,3,\ldots)$ be the first order probability that a batch of i customers arrives at the system during the short interval of time $(t, t+dt]$, where $0 \leq c_i \leq 1$, $\sum_{i=1}^{\infty} c_i = 1$ and $\lambda > 0$ is the mean arrival rate of batches. There is a single server who provides two kinds of general heterogeneous (one by one) service to customers on a first come, first served (FCFS) basis. Before his service starts, each customer has the option to choose first service with probability θ_1 or the second service with probability θ_2 where $\theta_1 + \theta_2 = 1$. We assume that the service time random variable S_j of jth kind of service follows a general probability law with $B_j(s_j)$ as the distribution function, $b_j(s_j)$ as the probability density function and $E(S_j^k)$ as the kth moment $(k = 1, 2, \ldots)$ of the service time, $j = 1, 2$.

Let $\mu_j(x)$ be the conditional probability of completion of type j service during the interval $(x, x + dx]$, given that the elapsed time is x, so that

$$\mu_j(x) = \frac{b_j(x)}{1 - B_j(x)}, \quad j = 1, 2, \tag{2.1}$$

and therefore,

$$b_j(s_j) = \mu_j(s_j) \exp\left(- \int 0^{s_j} \mu_j(x)\, dx \right); \quad j = 1, 2. \tag{2.2}$$

As soon as the service of a customer is complete, the server may take a vacation with probability p or else with probability $1 - p$ he may continue serving the next customer, if any (or may remain idle, in case there is no customer in the system). We further assume that the vacation time random variable Y follows a general probability law with $V(y)$ as the distribution function, $v(y)$ as the probability density function and $E(Y^k)$ as its kth moment $(k = 1, 2, \dots)$.

Let $\eta(x)$ be the conditional probability of completion of a vacation period during the interval $(x, x + dx]$, given that the elapsed vacation time is x, so that

$$\eta(x) = \frac{v(x)}{1 - V(x)}, \tag{2.3}$$

and therefore,

$$v(y) = \eta(y) \exp\left(- \int_0^y \eta(x)\, dx \right). \tag{2.4}$$

We further assume that whenever the server takes a vacation, it is always a single vacation. In other words, on completion of a vacation, the server must be back to the system even if there is no customer present in the system. Such a system with modified Bernoulli schedules has been recently studied by Madan and Abu-Dayyeh [21].

Remark 2.1. Here we may remark that this assumption is unlike the assumptions of several other authors who assume that on completion of a vacation, if the server finds no customers in the queue, then he must take another vacation.

Finally, it is assumed that the inter-arrival times of the customers, the service times of each kind of service and vacation times of the server, all these stochastic processes involved in the system are independent of each other.

3. Definitions, notations, and equations governing the system

Assuming that the steady state exists, let $P_{n,j}(x)$ denote the steady state probability that there are $n(\geq 1)$ customers in the queue including one customer in type j service, $j = 1, 2$ and the elapsed service time of this customer is x. Accordingly, $P_{n,j} = \int_0^\infty P_{n,j}(x)\, dx$ denotes the corresponding steady state probability irrespective of the elapsed service time x. Next, we define $Q_n(x)$ as a steady state probability that there are $n(\geq 0)$ customers in the queue and the server is on vacation and the elapsed vacation time of the server is x. Accordingly, $Q_n = \int_0^\infty Q_n(x)\, dx$ is the corresponding steady state probability, irrespective of the elapsed vacation time x. Finally, let 'E_0' denote the steady state probability that the server is idle but available in the system and there is no customer in the system. In addition, we define the following probability generating functions (PGFs):

$$P_j(x,z) = \sum_{n=1}^{\infty} P_{n,j}(x) z^n, \quad P_j(z) = \sum_{n=1}^{\infty} P_{n,j} z^n, \quad j = 1, 2,\ (|z| \leq 1), \tag{3.1}$$

$$Q(x,z) = \sum_{n=0}^{\infty} Q_n(x) z^n, \quad Q(z) = \sum_{n=0}^{\infty} Q_n z^n, \quad |z| \leq 1, \tag{3.2}$$

$$C(z) = \sum_{i=1}^{\infty} c_i z^i, \quad |z| \leq 1. \tag{3.3}$$

Then, following the argument of Cox [7], we obtain the following steady state equations for our model ($n \geq 1$):

$$\frac{d}{dx}P_{n,1}(x)+(\lambda+\mu_1(x))P_{n,1}(x)=\lambda\sum_{i=1}^{n}c_iP_{n-i,1}(x), \tag{3.4}$$

$$\frac{d}{dx}P_{n,2}(x)+(\lambda+\mu_2(x))P_{n,2}(x)=\lambda\sum_{i=1}^{n}c_iP_{n-i,2}(x)1, \tag{3.5}$$

$$\frac{d}{dx}Q_n(x)+(\lambda+\eta(x))Q_n(x)=\lambda\sum_{i=1}^{n}c_iQ_{n-i}(x), \tag{3.6}$$

$$\frac{d}{dx}Q_0(x)+(\lambda+\eta(x))Q_0(x)=0, \tag{3.7}$$

$$E_0=(1-p)\left[\int_0^\infty P_{1,1}(x)\mu_1(x)\,dx+\int_0^\infty P_{1,2}(x)\mu_2(x)\,dx\right]+\int_0^\infty Q_0(x)\eta(x)\,dx, \tag{3.8}$$

where $P_{0,j}(x)=0$ for $j=1,2$, appearing in (3.4) and (3.5).

The above equations have to be solved subject to the boundary conditions:

$$\begin{aligned}P_{n,1}(0)=&(1-p)\theta_1\left(\int_0^\infty P_{n+1,1}(x)\mu_1(x)\,dx+\int_0^\infty P_{n+1,2}(x)\mu_2(x)dx\right)\\&+\theta_1\int_0^\infty Q_n(x)\eta(x)dx+\lambda\theta_1c_nE_0,\quad n\geq 1,\end{aligned} \tag{3.9}$$

$$\begin{aligned}P_{n,2}(0)=&(1-p)\theta_2\left(\int_0^\infty P_{n+1,1}(x)\mu_1(x)\,dx+\int_0^\infty P_{n+1,2}(x)\mu_2(x)\,dx\right)\\&+\theta_2\int_0^\infty Q_{n+1}(x)\eta(x)\,dx+\lambda\theta_2c_nE_0,\quad n\geq 1,\end{aligned} \tag{3.10}$$

$$Q_n(0)=p\left(\int_0^\infty P_{n+1,1}(x)\mu_1(x)\,dx+\int_0^\infty P_{n+1,2}(x)\mu_2(x)\,dx\right), \tag{3.11}$$

($n \geq 0$), and the normalizing condition

$$E_0+\sum_{j=1}^{2}\sum_{n=1}^{\infty}\int_0^\infty P_{n,j}(x)\,dx+\sum_{n=0}^{\infty}\int_0^\infty Q_n(x)\,dx=1. \tag{3.12}$$

Now, proceeding in the usual manner with equations (3.4)–(3.8) we obtain

$$P_1(x,z)=P_1(0,z)[1-B_1(x)]e^{-\lambda(1-C(z))},\quad x>0, \tag{3.13}$$

$$P_2(x,z)=P_2(0,z)[1-B_2(x)]e^{-\lambda(1-C(z))},\quad x>0, \tag{3.14}$$

$$Q(x,z)=Q(0,z)[1-V(x)]e^{-\lambda(1-C(z))},\quad x>0. \tag{3.15}$$

Next, we multiply equations (3.9)–(3.11) by suitable powers of z, take summation over all possible values of n and use equations (3.1)–(3.3) and (3.8) and simplify. Thus we obtain

$$\begin{aligned}&[z-(1-p)\theta_1 B_1^*(\lambda-\lambda C(z))]P_1(0,z)\\&\quad=(1-p)\theta_1 P_2(0,z)B_2^*(\lambda-\lambda C(z)+z\theta_1 Q(0,z)V^*(\lambda-\lambda C(z)))+z\lambda\theta_1(C(z)-1)E_0,\end{aligned}\tag{3.16}$$

$$\begin{aligned}&[z-(1-p)\theta_2 B_2^*(\lambda-\lambda C(z))]P_2(0,z)\\&\quad=(1-p)\theta_2 P_1(0,z)B_1^*(\lambda-\lambda C(z))+z\theta_2 Q(0,z)V^*(\lambda-\lambda C(z))+z\lambda\theta_2(C(z)-1)E_0,\end{aligned}\tag{3.17}$$

$$zQ(0,z)=p[P_1(0,z)B_1^*(\lambda-\lambda C(z))+P_2(0,z)B_2^*(\lambda-\lambda C(z))],\tag{3.18}$$

where

$$G_j^*[\lambda-\lambda e(z)]=\int_0^\infty e^{-(\lambda-\lambda e(z))x}dG_j(x),\quad j=1,2,\qquad V^*[\lambda-\lambda e(z)]=\int_0^\infty e^{-(\lambda-\lambda e(z))x}dB(x)\tag{3.19}$$

are the Laplace-Stieltjis transforms of type 1, type 2 service times and vacation time, respectively.

Then substituting for $Q(0,z)$ from (3.18) into (3.16) and (3.17), we obtain

$$\begin{aligned}&[z-\{(1-p)+pV^*(\lambda-\lambda C(z))\}\theta_1 B_1^*(\lambda-\lambda C(z))]P_1(0,z)\\&\quad=\{(1-p)+pV^*(\lambda-\lambda C(z))\}\theta_1 P_2(0,z)B_2^*(\lambda-\lambda C(z))+z\lambda\theta_1(C(z)-1)E_0,\end{aligned}\tag{3.20}$$

$$\begin{aligned}&[z-\{(1-p)+pV^*(\lambda-\lambda C(z))\}\theta_2 B_2^*(\lambda-\lambda C(z))]P_2(0,z)\\&\quad=\{(1-p)+pV^*(\lambda-\lambda C(z))\}\theta_2 P_1(0,z)B_1^*(\lambda-\lambda C(z))+z\lambda\theta_2(C(z)-1)E_0.\end{aligned}\tag{3.21}$$

Solving (3.20) and (3.21) for $P_1(0,z)$ and $P_2(0,z)$ and simplifying we have

$$P_1(0,z)=\frac{\lambda z\theta_1(1-C(z))E_0}{D(z)},\tag{3.22}$$

$$P_2(0,z)=\frac{\lambda z\theta_2(1-C(z))E_0}{D(z)},\tag{3.23}$$

where

$$D(z)=[(1-p)+pV^*(\lambda-\lambda C(z))][\theta_1 B_1^*(\lambda-\lambda C(z))+\theta_2 B_2^*(\lambda-\lambda C(z))]-z.\tag{3.24}$$

Now, utilizing (3.22)–(3.23) into (3.18), we obtain

$$Q(0,z)=\frac{p\lambda[\theta_1 B_1^*(\lambda-\lambda C(z))+\theta_2 B_2^*(\lambda-\lambda C(z))][1-C(z)]E_0}{D(z)},\tag{3.25}$$

where $D(z)$ is given by (3.24).

Next, from (3.13) and (3.22), we obtain

$$P_1(z) = \int_0^\infty P_1(x,z)\,dx = \frac{[1 - B_1^*(\lambda - \lambda C(z))]z\theta_1 E_0}{D(z)}. \tag{3.26}$$

Then, from (3.14) and (3.23), we obtain

$$P_2(z) = \int_0^\infty P_2(x,z)\,dx = \frac{[1 - B_2^*(\lambda - \lambda C(z))]z\theta_2 E_0}{D(z)}. \tag{3.27}$$

Finally, from (3.15) and (3.25), we obtain

$$\begin{aligned} Q(z) &= \int_0^\infty Q(x,z)\,dx \\ &= \frac{p[\theta_1 B_1^*(\lambda - \lambda C(z)) + \theta_2 B_2^*(\lambda - \lambda C(z))][1 - V^*(\lambda - \lambda C(z))]E_0}{D(z)}. \end{aligned} \tag{3.28}$$

The unknown probability E_0 can be determined by utilizing the normalizing condition (3.12) which is equivalent to $E_0 + P_1(1) + P_2(1) + Q(1) = 1$. Thus we have on simplifying

$$E_0 = 1 - \rho, \tag{3.29}$$

where $\rho = \lambda E(I)[\theta_1 E(S_1) + \theta_2 E(S_2) + pE(Y)] < 1$ is the utilization factor of this system and $E(I)$ is the mean size of an arriving batch.

Further, utilizing (3.29) and putting $z = 1$ into equations (3.26)–(3.28) yield

$$\Pr(\text{the server is busy providing type 1 serviceat a random epoch}) = \lambda E(I)E(S_1)\theta_1, \tag{3.30}$$

$$\Pr(\text{the server is busy providing type 2 serviceat a random epoch}) = \lambda E(I)E(S_2)\theta_2, \tag{3.31}$$

$$\Pr(\text{the server is on vacation at a random epoch}) = p\lambda E(I)E(Y). \tag{3.32}$$

Now, we let $P_Q(z) = E_0 + P_1(z) + P_2(z) + Q(z)$ denote the steady state PGF of the system size distribution at a random epoch. Then adding (3.26)–(3.29), we obtain on simplifying

$$P_Q(z) = \frac{(1 - z)(1 - \rho)[\theta_1 B_1^*(\lambda - \lambda C(z)) + \theta_2 B_2^*(\lambda - \lambda C(z))]}{[(1 - p) + pV^*(\lambda - \lambda C(z))][\theta_1 B_1^*(\lambda - \lambda C(z)) + \theta_2 B_2^*(\lambda - \lambda C(z))] - z}, \tag{3.33}$$

where $\rho = \lambda E(I)[\theta_1 E(S_1) + \theta_2 E(S_2) + pE(Y)] < 1$.

Thus (3.33) gives the PGF of $M^x/\binom{G_1}{G_2}/1/G(BS)/V_s$ queue with batch arrivals, two kinds of general heterogeneous service and general vacation times with Bernoulli schedules based on single vacations.

Note that if $\theta_2 = 0$ (i.e., $\theta_1 = 1$), which means that the server provides only the type 1 service, then (3.33) reduces to

$$P_Q(z) = \frac{(1-z)(1-\rho)\left[B_1^*(\lambda - \lambda C(z))\right]}{\left[(1-p) + pV^*(\lambda - \lambda C(z))\right]\left[B_1^*(\lambda - \lambda C(z))\right] - z}, \tag{3.34}$$

where $\rho = \lambda E(I)[E(S_1) + pE(Y)] < 1$.

Note that (3.34) gives the PGF of the queue size distribution at a random epoch of the $M^x/G/1/G(BS)/V_s$ queue with batch arrivals, general service and general vacation times with single vacations under Bernoulli schedules. Further, it may be noted that a particular case of this result is equivalent to the result obtained recently by Medhi [23] for an $M/G/1$ queue with general second optional service. In fact, this model was first studied by the present first author Madan [18] who considered first service general and the second optional service as exponential. He also gives interesting applications of such models in day to day life situations.

Further, if there are no server vacations, then with $p = 0$, (3.33) reduces to

$$P_Q(z) = \frac{(1-z)(1-\rho)\left[\theta_1 B_1^*(\lambda - \lambda C(z)) + \theta_2 B_2^*(\lambda - \lambda C(z))\right]}{\left[\theta_1 B_1^*(\lambda - \lambda C(z)) + \theta_2 B_2^*(\lambda - \lambda C(z))\right] - z}, \tag{3.35}$$

where $\rho = \lambda E(I)[\theta_1 E(S_1) + \theta_2 E(S_2)] < 1$.

Note that (3.35) gives the PGF of a $M^x/\binom{G_1}{G_2}/1$ queue with two types of general heterogeneous service without server vacations.

Further, if there is only one kind of service and also no server vacations, then with $\theta_2 = 0$, $\theta_1 = 1$, (3.35) yields

$$P_Q(z) = \frac{(1-z)(1-\rho)B_1^*(\lambda - \lambda C(z))}{B_1^*(\lambda - \lambda C(z)) - z}, \tag{3.36}$$

where $\rho = \lambda E(I)E(S_1) < 1$.

Further, note that the result in (3.36) is equivalent to the result obtained by Gaver [10] for an ordinary $M^x/G/1$ queue. Therefore, our main result in (3.33) is a generalization of the results obtained by Gaver [10], Madan [18], Medhi [23] and Choudhury [5].

Remark 3.1. Alternatively, (3.34) can also be obtained from (3.18) with $p = 0$.

4. Queue size distribution at a departure epoch

Sahbazov [26] and Chaudhry [3] obtained the PGF of queue size for the $M^x/G/1$ queue at a departure epoch through different approaches. In this section, we derive the queue size distribution at a departure epoch for our model $M^x/\binom{G_1}{G_2}/1/G(BS)/V_s$ as a generalization of the result obtained by Sahbazov [26] and Chaudhry [3]. Following the argument of PASTA (e.g., see Wolff [30]), we state that a departing customer will see $'j'$ customers in the queue just after his departure if and only if there were $j+1$ customers in the queue just before his departure. Let us denote $\{\pi_j, j \geq 0\}$ as the probability that there are $'j'$

customers in the queue at a departure epoch. Then we have

$$\pi_j = K_0(1-p)\left[\int_0^\infty P_{j+1,1}(x)\mu_1(x)\,dx + \int_0^\infty P_{j+1,2}(x)\mu_2(x)\,dx\right], \tag{4.1}$$

$(j \geq 0)$ where K_0 is the normalizing constant.

Next, we define the PGF of $\{\pi_j, j \geq 0\}$ as

$$\Pi(z) = \sum_{j=0}^{\infty} \pi_j z^j, \quad |z| \leq 1. \tag{4.2}$$

Then using (3.22)–(3.24) and (4.2), (4.1) yields

$$\Pi(z) = \frac{K_0(1-p)\lambda z(1-C(z))\left[\theta_1 B_1^*(\lambda-\lambda C(z)) + \theta_2 B_2^*(\lambda-\lambda C(z))\right]E_0}{\left[(1-p)+pV^*(\lambda-\lambda C(z))\right]\left[\theta_1 B_1^*(\lambda-\lambda C(z)) + \theta_2 B_2^*(\lambda-\lambda C(z))\right]-z}. \tag{4.3}$$

Now, using the normalizing condition $\Pi(1) = 1$, we get from (4.3)

$$K_0 = \frac{1-\rho}{\lambda(1-p)E(I)E_0}. \tag{4.4}$$

Thus using (3.33) into (4.5) we get

$$\Pi(z) = \frac{(1-\rho)z(1-C(z))\left[\theta_1 B_1^*(\lambda-\lambda C(z)) + \theta_2 B_2^*(\lambda-\lambda C(z))\right]}{E(I)\left\{\left[(1-p)+pV^*(\lambda-\lambda C(z))\right]\left[\theta_1 B_1^*(\lambda-\lambda C(z)) + \theta_2 B_2^*(\lambda-\lambda C(z))\right]-z\right\}}, \tag{4.5}$$

where $\rho = \lambda E(I)[\theta_1 E(S_1) + \theta_2 E(S_2) + pE(Y)] < 1$.

Thus from (3.33) and (4.5) we recover an interesting relationship between $\Pi(z)$ and $P_Q(z)$ as follows:

$$\Pi(z) = \left(\frac{(1-C(z))z}{E(I)(1-z)}\right)P_Q(z) = A(z)P_Q(z), \tag{4.6}$$

where $\{A(z) = (1-C(z))z\}/\{E(I)(1-z)\}$ is the PGF of the number of units placed before an arbitrary (tagged) customer in a batch in which the tagged customer arrives. This number is given as a backward recurrence time in the discrete time renewal process, where the successive renewal points are generated by the arrival size random variable. This is due to the randomness nature of arrival size random variable.

Remark 4.1. We may observe from (4.6) that the queue size distribution of $M^x/\binom{G_1}{G_2}/1/G(BS)/V_s$ queue at a departure epoch is the convolution of the following two independent random variables: (1) the PGF due to the arrival size random variable, and (2) the stationary PGF of queue size at a random epoch. This verifies that the well-known decomposition property, which is known to hold for many vacation models, also holds for our model.

Now if $\theta_2 = 0, \theta_1 = 1$ (i.e., there is no type 2 service), then the result (4.5) will reduce to

$$\Pi(z) = \frac{(1-\rho)z(1-C(z))\left[B_1^*(\lambda-\lambda C(z))\right]}{E(I)\left\{\left[(1-p)+pV^*(\lambda-\lambda C(z))\right]\left[B_1^*(\lambda-\lambda C(z))\right]-z\right\}}, \tag{4.7}$$

where $\rho = \lambda E(I)[E(S_1) + pE(Y)] < 1$.

Note that (4.7) yields the PGF of an $M^x/G/1/G(BS)/V_s$queue with batch arrivals, general service and general vacation times with Bernoulli schedule server vacations under a single vacation policy at a departure epoch. This result is equivalent to the result obtained by Choudhury [5]. Further, if we put $p = 1$ in (4.7) then we obtain

$$\Pi(z) = \frac{(1-\rho)z(1-C(z))\left[B_1^*(\lambda-\lambda C(z))\right]}{E(I)\left\{\left[V^*(\lambda-\lambda C(z))\right]\left[B_1^*(\lambda-\lambda C(z))\right]-z\right\}}, \tag{4.8}$$

where $\rho = \lambda E(I)[E(S_1) + E(Y)] < 1$.

Further, note that (4.8) is the PGF of the queue size distribution of an $M^x/G/1$queue with limited service. A particular case of this model was studied by Takagi [29], for the single Poisson arrivals.

5. The mean queue size at a random epoch and the mean waiting time

Let L_Q denote the mean queue size at a random epoch for the $M^x/\binom{G_1}{G_2}/1/G(BS)/V_s$ queue for two kinds of general heterogeneous service, Bernoulli schedule server vacations under the single vacation policy. Then we obtain from (3.33)

$$\begin{aligned} L_Q &= \frac{d}{dz}P_Q(z)|_{z=1} = \rho + \lambda^2(E(I))^2 \\ &= \frac{\left[\theta_1 E(S_1^2) + \theta_2 E(S_2^2) + pE(Y^2) + 2pE(Y)(\theta_1 E(S_1) + \theta_2 E(S_2))\right]}{2(1-\rho)} + \frac{\rho E(X_R)}{(1-\rho)}, \end{aligned} \tag{5.1}$$

where $\rho = \lambda E(I)[\theta_1 E(S_1) + \theta_2 E(S_2) + pE(Y)] < 1$ and $E(X_R) = \{E(I(I-1))\}/\{E(I)\}$ is the mean residual batch size.

It may be noted that when $\theta_2 = 0$, $\theta_1 = 1$ (i.e., there is no type 2 service), then the result (5.1) yields

$$L_Q = \rho + \frac{\lambda^2(E(I))^2\left[E(S_1^2) + 2pE(Y)E(S_1) + pE(Y^2)\right]}{2(1-\rho)} + \frac{\rho E(X_R)}{1-\rho}, \tag{5.2}$$

where $\rho = \lambda E(I)[E(S_1) + pE(Y)] < 1$.

Thus (5.2) gives the mean queue size for an $M^x/G/1/G(BS)/V_s$ vacation queue with batch arrivals, general service and general; vacation times with Bernoulli schedule server vacations based on a single vacation policy. This result is equivalent to the result obtained by Choudhury [5].

Further if $p = 0$ (i.e., no server vacation) then (3.30) reduces to

$$L_Q = \rho + \frac{\lambda^2 (E(I))^2 [E(S_1^2)]}{2(1-\rho)} + \frac{\rho E(X_R)}{1-\rho}, \tag{5.3}$$

where $\rho = \lambda E(I)E(S_1) < 1$.

Note that (5.3) can also be written as

$$L_Q = \rho + \frac{\lambda E(I)\rho E(S_R)}{1-\rho} + \frac{\rho E(X_R)}{1-\rho}, \tag{5.4}$$

where $E(S_R) = E(S_1^2)/2E(S_1)$ is the mean residual service time.

Note that (5.4) is the mean queue size at a random epoch for an $M^x/G/1$ queue and this verifies the results obtained by Chaudhry [3].

Next, let L_D denote the mean queue size at a departure epoch, then we have form (4.5)

$$L_D = \frac{d}{dz}\Pi(z)\big|_{z=1} = L_Q + E(X_R). \tag{5.5}$$

Equation (3.33) shows that $L_D > L_Q$ and equality holds if and only if $E(X_R) = 0$. This depicts an interesting phenomenon that the mean queue size as observed by a departing customer is larger than the mean queue size observed by an arbitrary (tagged) customer.

Further we can also obtain W_Q, the mean waiting time as $W_Q = L_Q/\lambda$.

6. Mean busy period

In this section, we obtain the mean busy period for our model $M^x/\binom{G_1}{G_2}/1/G(BS)/V_s$ queue with batch arrivals, two kinds of general heterogeneous service and general vacation times with Bernoulli schedule server vacations based on a single vacation policy. We define the busy period as the length of the time interval during which the server remains busy and this continues till the instant when the server becomes free again. This busy period is equivalent to the ordinary busy period generated by the units which arrive during the vacation period plus an idle period, which we may call as generalized idle period. We now define

(i) T_0 = length of the generalized idle period,

(ii) T_b = length of the busy period.

Now since T_0 and T_b generate an alternating renewal process, therefore we may write

$$\frac{E(T_b)}{E(T_0)} = \frac{\Pr[T_b]}{1-\Pr[T_b]}. \tag{6.1}$$

Now, from Section 3, we have

$$\Pr[T_b] = P_1(1) + P_2(1) = \lambda E(I)[\theta_1 E(S_1) + \theta_2 E(S_2)]. \tag{6.2}$$

Again due to the well-known property of the Poisson input queueing system, we have

$$E(T_0) = \frac{1}{\lambda} + pE(Y). \tag{6.3}$$

Next, utilizing (6.2) and (6.3) into (6.1), we get on simplifying

$$E(T_b) = \frac{E(I)[\theta_1 E(S_1) + \theta_2 E(S_2)][1 + p\lambda E(Y)]}{1 - E(I)[\theta_1 E(S_1) + \theta_2 E(S_2)]}. \tag{6.4}$$

Further, it is clear that the fraction of time the server remains in the generalized idle state T_0 (i.e., idle plus on vacation) is equivalent to

$$\frac{E(T_0)}{E(T_0) + E(T_b)}. \tag{6.5}$$

Now, using $E(T_0)$ and $E(T_b)$ from (6.3) and (6.4) in the expression (6.5) and simplifying it, one may verify that

$$\begin{aligned} \frac{E(T_0)}{E(T_0) + E(T_b)} &= (1 - \rho) + p\lambda E(I)E(Y) \\ &= \Pr[\text{the server is idle}] + \Pr[\text{server is on vacation}] = \Pr[T_0], \end{aligned} \tag{6.6}$$

as it should be.

Now, If we assume that $\theta_1 = 1, \theta_2 = 0$, which means that the server provides only one type of service, then in this case (6.4) will reduce to

$$E(T_b) = \frac{E(I)E(S_1)[1 + p\lambda E(Y)]}{1 - E(I)E(S_1)}. \tag{6.7}$$

We note that (6.7) agrees with the result obtained by Choudhury [5].

Further, if we take $p = 0$ (i.e., no server vacations), then (6.7) will reduce to

$$E(T_b) = \frac{E(I)[\theta_1 E(S_1) + \theta_2 E(S_2)]}{1 - E(I)[\theta_1 E(S_1) + \theta_2 E(S_2)]}, \tag{6.8}$$

which is the mean busy period for an ordinary $M^X/\binom{G_1}{G_2}/1$ queueing system without server vacations. In addition, if $\theta_1 = 1, \theta_2 = 0$ (there is no type 2 service), then in this case (6.5) will reduce to

$$E(T_b) = \frac{E(I)E(S_1)}{1 - E(I)E(S_1)}, \tag{6.9}$$

which agrees with the result obtained by Chaudhry [3].

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