Hindawi Publishing Corporation
Advances in Decision Sciences
Volume 2011, Article ID 495392, 17 pages
doi:10.1155/2011/495392

Research Article

Rough Multisets and Information Multisystems

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Received 3 August 2011; Accepted 1 November 2011

Academic Editor: Graham Wood

Rough set theory uses the concept of upper and lower approximations to encapsulate inherent inconsistency in real-world objects. Information multisystems are represented using multisets instead of crisp sets. This paper begins with an overview of recent works on multisets and rough sets. Rough multiset is introduced in terms of lower and upper approximations and explores related properties. The paper concludes with an example of certain types of information multisystems.

1. Introduction

The advance in science and technology has given rise to a wide range of problems, where the objects under analysis are characterized by many diverse features (attributes), which may be quantitative and qualitative. Furthermore, the same objects may exist in several copies with different values of attributes, and their convolution is either impossible or mathematically incorrect. Examples of such problems are the classification of multicriteria alternatives estimated by several experts, the recognition of graphic symbols, text document processing, and so on. A convenient mathematical model for representing multiattribute objects is a multiset or a set with repeating elements. The most essential property of multisets is the multiplicity of the elements that allows us to distinguish it from a set and consider it as a qualitatively new mathematical concept.

Huge chunks of information are acquired and stored every day in computer systems for various global applications. It is hard to imagine a computer application that does not interface with a database or one that does not dump megabytes of information into a database every day. Information is constantly generated by a variety of basic day-to-day applications. The information that is being acquired is growing exponentially every minute. It gives rise to incomplete, uncertain, or vague information. Extracting useful information from such huge chunks of information is a difficult task. The convenient and effective tools to make this process easier are the theories of rough sets.

In any information system, some situations may occur, where the respective counts of objects in the universe of discourse are not single. In such situations we have to deal with collections of information in which duplicates are significant. In such cases multisets play an important role in processing the information. The information system dealing with multisets is said to be an information multisystem. Thus, information multisystems are more compact when compared to the original information system.

The authors have given a new dimension to Pawlak's rough set theory, replacing its universe by multisets. This is called a rough multiset and is a useful structure in modelling information multisystem. The process involved in the intermediate stages of reactions in chemical systems is a typical example of a situation which gives rise to multiset relations. Information multisystem is represented using rough multisets and is more convenient than ordinary rough sets. An illustrative example supporting this claim is provided in Section 4.

Rough multisets are defined in terms of lower mset approximation and upper mset approximation with the help of equivalence mset relations introduced by the authors in [1]. Grzymala-Busse introduced the concept of rough multisets using multirelations in [2]. A multirelation is a relation connecting two objects in which a pair is repeated more than once. But a multiset relation according to the authors is in entirely different concept when compared to the concept multirelation introduced by Grzymala-Busse. Multiset relation is a relation connecting two objects in which each object and the pair are repeated more than once; that is, in a multirelation "x is related to y" with the pair $(x,\ y)$ repeated more than once, but in the case of a multiset relation "x repeated m times" is related to "y repeated n times" (i.e., m/x is related to n/y) with the pair $(m/x,\ n/y)$ repeated more than once. Thus, a multirelation is a relation connecting the elements in the sets. At the same time multiset relation is a relation connecting the elements in the multisets.

This paper proposes a systematic approach to rough multisets, related properties, and information multisystems. It begins with the introduction to multisets and a brief survey of Yager's theory of bags [3]. The concept of mset relations, knowledge mset base, and classifications are given in Section 3. Section 4 includes the discussion of the underlying concepts of the rough set theory, introduction to rough multisets and related properties, and information multisystems and concludes with information and decision tables of rough multisets. The last section contains the conclusion and the scope of future work.

2. Theory of Multisets

This section begins with the origin of multisets (mset) and a brief survey of the notion of the theory of multisets introduced by Yager [3].

In mathematics, a multiset (or bag) is the generalization of a set. A member of a multiset can have more than one membership [4–6], while each member of a set has only one membership. The use of multisets in mathematics predates the name "multiset" by nearly 90 years. Richard Dedekind used the term multisets in a paper published in 1888. Knuth [7] also lists other names that were proposed for multisets, such as list, bunch, bag, and weighted set. Zermelo-Fraenkel's set theory, commonly known as ZF theory, is the classical theory based on first-order logic and is undoubtedly the foundation of mathematics. In first-order logic, a formal theory MST (multiset theory) that contains ZF as a special case has been formulated. MST was introduced by Cerf et al. [8] in 1972.

Definition 2.1. An mset M drawn from the set X is represented by a function Count M or C_M defined as $C_M : X \to N$, where N represents the set of nonnegative integers.

Here $C_M(x)$ is the number of occurrences of the element x in the mset M. We present the mset M drawn from the set $X = \{x_1, x_2, \ldots, x_n\}$ as $M = \{m_1/x_1,\ m_2/x_2, \ldots, m_n/x_n\}$, where m_i is the number of occurrences of the element x_i, $i = 1, 2, \ldots, n$, in the mset M. However, those elements which are not included in the mset M have zero count. Let M be an mset from X with x appearing n times in M. It is denoted by $x \in^n M$.

Example 2.2. Let $X = \{a,\ b,\ c,\ d,\ e\}$ be any set. Then $M = \{2/a, 4/b, 5/d, 1/e\}$ is an mset drawn from X.

To be clear, a set is a special case of an mset.

Let M and N be two msets drawn from a set X. Then, the following are defined in [3, 9, 10]:

(1) $M = N$ if $C_M(x) = C_N(x)$, for all $x \in X$,

(2) $M \subseteq N$ if $C_M(x) \leq C_N(x)$, for all $x \in X$,

(3) $P = M \cup N$ if $C_P(x) = \operatorname{Max}\{C_M(x), C_N(x)\}$, for all $x \in X$,

(4) $P = M \cap N$ if $C_P(x) = \operatorname{Min}\{C_M(x), C_N(x)\}$, for all $x \in X$,

(5) $P = M \oplus N$ if $C_P(x) = C_M(x) + C_N(x)$, for all $x \in X$,

(6) $P = M \ominus N$ if $C_P(x) = \operatorname{Max}\{C_M(x) - C_N(x), 0\}$, for all $x \in X$,

where $\oplus$ and $\ominus$ represent mset addition and mset subtraction, respectively.

Let M be an mset drawn from a set X. The support set of M denoted by M^* is a subset of X and $M^* = \{x \in X : C_M(x) > 0\}$; that is, M^* is an ordinary set. M^* is also called root set.

An mset M is said to be an empty mset if, for all $x \in X$, $C_M(x) = 0$.

The cardinality of an mset M drawn from a set X is denoted by Card (M) or $|M|$ and is given by Card $M = \sum_{x \in X} C_M(x)$.

Definition 2.3. A domain X is defined as a set of elements from which msets are constructed. The mset space $[X]^n$ is the set of all msets whose elements are in X such that no element in the mset occurs more than n times.

The set $[X]^\infty$ is the set of all msets over a domain X such that there is no limit on the number of occurrences of an element in an mset.

If $X = \{x_1, x_2, \ldots, x_k\}$, then $[X]^n = \{\{n_1/x_1, n_2/x_2, \ldots, n_k/x_k\} : \text{for } i = 1, 2, \ldots, k;\ n_i \in \{0, 1, 2, \ldots, n\}\}$.

Definition 2.4. Let X be a support set and $[X]^n$ the mset space defined over X. Then, for any mset $M \in [X]^n$, the complement M^c of M in $[X]^n$ is an element of $[X]^n$ such that

$$C_M^c(x) = n - C_M(x), \quad \forall x \in X. \tag{2.1}$$

Note 1. Using Definition 2.4, the mset sum can be modified as follows:

$$C_{M1 \oplus M2}(x) = \min\{n, C_{M1}(x) + C_{M2}(x)\}, \quad \forall x \in X. \tag{2.2}$$

3. Mset Relations and Knowledge Mset Base

This section gives the concept of relations, equivalence relations, and partitions in the context of multisets. Knowledge mset base and classifications are introduced instead of knowledge base and classifications in any information system.

A relation in mathematics is defined as an object that has its existence as such within a definite context or setting. It is literally a case wherein any change in this setting will involve a change in the relation that is being defined. The particular type of context that is needed here is formalized as a collection of elements from which are chosen the elements of the relation in question. The larger collection of elementary relations or tuples is constructed by means of the set theoretic product commonly known as the Cartesian product.

In mathematics, especially set theory and logic [11, 12], a relation is a property that assigns truth values to combinations (k-tuples) of k individuals. Typically, the property describes a possible connection between the components of a k-tuple. For a given set of k-tuples, a truth value is assigned to each k-tuple according to whether the property does or does not hold. An example of a ternary or triadic relation is "X was—connected—Y by Z," where (X, Y, Z) is a 3-tuple of cities; for example, "Kerala is connected to Delhi via Mumbai by road" is true, while "India is connected to USA via Pakistan by road" is false. The information Table 1 can represent this relation.

The data given in the table are equivalent to the following ordered triples:

$$R = \{(\text{Kerala}, \text{Mumbai}, \text{Delhi}),\ (\text{India}, \text{Pakistan}, \text{USA})\}. \tag{3.1}$$

The table for relation R is an extremely simple example of a relational database. Theoretical aspects of databases are the specializations of a branch of computer science. Their practical impacts have become all too familiar to our everyday life. Computer scientists, logicians, and mathematicians, however, tend to see things differently when they look at these concrete examples and samples of the more general concept of a relation. According to Augustus De Morgan, "when two objects, qualities, classes or attributes, viewed together by the mind, are seen under some connection, that connection is called a relation."

In any information multisystem, duplicates occur in the intermediate stages of processing information. While processing, each information can be associated within the elements of collections of information or with elements of another collection of information. This type of association is called an mset relation. For example, in chemical systems the chemical compounds are formed by interactions which take place in a systematic manner. Some molecules in the set of molecules might occur in more than a single copy. From the mathematical point of view, these sets of molecules are called msets and the rules of reactions are called mset relations.

3.1. Knowledge Mset Base and Classifications

Knowledge is deep-seated in the classificatory abilities of human beings and other species. For example, knowledge about the environment is primarily manifested as an ability to classify a variety of situations from their point of survival in the real world.

Among the given finite mset $M \neq \Phi$ (the universe) of objects any submset $N \subseteq M$ of the universe will be called an mset concept (i.e., concepts with repetition) or an mset category

Table 1: Relation: X—connected by roads—to Z—via Y.

X-cities	Y-cities	Z-cities
Kerala	Mumbai	Delhi
India	Pakistan	USA

in M and any family of mset concepts in M will be referred to as abstract knowledge with repetition. Empty mset Φ is admitted as an mset concept.

Some families of classifications with repetitions are more dealt with, like mset classifications over M. A family of mset classifications over M will be called a knowledge mset base over M denoted by K_M. Thus, a knowledge mset base represents a variety of mset classification skills of an intelligent agent or group of agents which constitute the fundamental equipments of the agent needed to define its relation to the environment or itself.

For mathematical reasons equivalence mset relations are used, instead of classifications about knowledge mset base, since these two concepts are actually interchangeable and mset relations are easier to deal with.

The authors introduced new notations [1] for the purpose of defining Cartesian product of msets, mset relations, and their domain and codomain. The entry of the form $(m/x, n/y)/k$ denotes that x is repeated m-times, y is repeated n-times, and the pair (x, y) is repeated k-times. The counts of the members of the domain and codomain vary in relation to the counts of the x coordinate and y coordinate in $(m/x, n/y)/k$. For this purpose the notations $C_1(x, y)$ and $C_2(x, y)$ are introduced. $C_1(x, y)$ denotes the count of the first coordinate in the ordered pair (x, y), and $C_2(x, y)$ denotes the count of the second coordinate in the ordered pair (x, y).

Definition 3.1. Let M_1 and M_2 be two msets drawn from a set X, then the Cartesian product of M_1 and M_2 is defined as $M_1 \times M_2 = \{(m/x, n/y)/mn : \ x \in^m M_1, \ y \in^n M_2\}$.

We can define the Cartesian product of three or more nonempty msets by generalizing the definition of the Cartesian product of two msets.

Definition 3.2. A submset R of $M \times M$ is said to be an mset relation on M if every member $(m/x, n/y)$ of R has a count of the product $C_1(x, y)$ and $C_2(x, y)$. One denotes m/x related to n/y by $m/x \ R \ n/y$.

The Domain and Range of the mset relation R on M is defined as follows:

Dom $R = \{x \in^r M : \exists y \in^s M$ such that $r/x \ R \ s/y\}$, where $C_{\text{Dom}\,R}(x) = \text{Sup}\{C_1(x, y) : x \in^r M\}$,

Ran $R = \{y \in^s M : \exists x \in^r M$ such that $r/x \ R \ s/y\}$, where $C_{\text{Ran}\,R}(y) = \text{Sup}\{C_2(x, y) : y \in^s M\}$.

Example 3.3. Let $M = \{8/x, 11/y, 15/z\}$ be an mset. Then $R = \{(2/x, 4/y)/8, (5/x, 3/x)/15, (7/x, 11/z)/77, (8/y, 6/x)/48, (11/y, 13/z)/143, (7/z, 7/z)/49, (12/z, 10/y)/120, (14/z, 5/x)/70\}$ is an mset relation defined on M. Here Dom $R = \{7/x, 11/y, 14/z\}$ and Ran $R = \{6/x, 10/y, 13/z\}$.

Definition 3.4. Let R be an mset relation defined on M and $x \in^m M$. One defines $R(m/x)$, the R-relative mset of m/x, the set of all n/y in M such that there exists some k such that $k/x \ R \ n/y$; that is, $R(m/x) = \{n/y : \ \exists \text{ some } k \text{ with } k/x \ R \ n/y\}$.

Similarly, if $M_1 \subseteq M$, then $R(M_1)$, the R-relative mset of M_1, is the mset of all n/y in M with the property that there exists some k such that $k/x\ R\ n/y$. From the preceding definitions, we see that $R(M_1)$ is the union of all msets $R(m/x)$.

Remark 3.5. If two msets have the same elements with distinct multiplicity, then their R-relative msets are the same.

Example 3.6. From Example 3.3 we see that $R(2/x) = R(5/x) = R(7/x) = \{\ 3/x, 4/y, 11/z\}$, $R(8/y) = R(11/y) = \{6/x, 13/z\}$, and $R(7/z) = R(12/z) = R(14/z) = \{5/x, 10/y, 7/z\}$. Now, if $M_1 = \{2/x, 11/y, 12/z\} \subseteq\ M$, then $R(M_1) = \{6/x, 10/y, 12/z\}$.

Definition 3.7. (i) An mset relation R on an mset M is reflexive if $m/x\ R\ m/x$ for all m/x in M.

(ii) An mset relation R on an mset M is symmetric if $m/x\ R\ n/y$ implies $n/y\ R\ m/x$.

(iii) An mset relation R on an mset M is transitive if $m/xRn/y$, $n/y\ R\ k/z$, then $m/x\ R\ k/z$.

An mset relation R on an mset M is called an equivalence mset relation if it is reflexive, symmetric, and transitive.

Example 3.8. Let $M = \{3/x, 5/y, 3/z, 7/r\}$ be an mset. Then the mset relation given by $R = \{(3/x, 3/x)/9,\ (3/x, 3/z)/9,\ (3/x, 7/r)/21,\ (7/r, 3/x)/21,\ (5/y, 5/y)/25, (3/z, 3/z)/9,\ (7/r, 7/r)/49,\ (3/z, 3/x)/9,\ (3/z, 7/r)/21,\ (7/r, 3/z)/21\}$ is an equivalence mset relation.

Definition 3.9. A partition of a nonempty mset M is a collection $\mathbf{P}$ of nonempty submsets of M such that

(1) each element of M belongs to one of the msets in $\mathbf{P}$,

(2) if M_1 and M_2 are distinct elements of $\mathbf{P}$, then $M_1 \cap\ M_2 = \Phi$.
The msets in $\mathbf{P}$ are called the blocks or cells of the partition.

Example 3.10. Let $M = \{4/x, 5/y, 7/z\}$ be an mset, and consider the partition $\mathbf{P} = \{\{4/x, 5/y\}, \{7/z\}\}$ of the mset M. Then the blocks of $\mathbf{P}$ are $\{4/x,\ 5/y\}$ and $\{7/z\}$ and each element in a block is only related to every other's element in the same block. Thus, the equivalence mset relation determined by $\mathbf{P}$ is $\{(4/x, 4/x)/16, (4/x, 5/y)/20, (5/y, 4/x)/20, (5/y, 5/y)/25, (7/z, 7/z)/49\}$.

If R is an equivalence mset relation over M, then M/R is the family of all m-equivalence classes of R (or classification of M) referred to as mset categories or mset concepts of R. $R(m/x)$ or $[m/x]_R$ denotes the mset category in R containing an element $x \in^m M$.

By a knowledge mset base, we can understand a relational multisystem $K_M = (M, R)$, where M is a nonempty finite mset called the universe and R is a family of equivalence mset relations over M.

If $P \subseteq R$ and $P \neq \Phi$, the $\cap P$ (intersection of all equivalence mset relations belonging to P) is also an equivalence mset relation and will be denoted by $\mathrm{IND}(P)$ and will be called an indiscernibility mset relation over P.

Moreover,

$$[m/x]_{\mathrm{IND}(P)} = \bigcap_R [m/x]_R. \tag{3.2}$$

Thus, $M/_{\mathrm{IND}(P)}$ (i.e., the family of all m-equivalence classes of the equivalence mset relation $\mathrm{IND}(P)$) denotes knowledge mset base associated with the family of equivalence mset relations P, called P-basic knowledge mset base about M in K_M. M-equivalence classes of $\mathrm{IND}(P)$ are called basic mset categories (mset concepts) of knowledge P.

In fact P-basic mset categories are those properties of the universe which can be expressed employing knowledge P. In other words, they are fundamental building blocks of our knowledge or basic properties of the universe which can be expressed employing knowledge P.

Let $K_M = (M, R)$ be a knowledge mset base. $\mathrm{IND}(K_M)$ denotes the family of all equivalence mset relations defined in K_M as

$$\mathrm{IND}(K_M) = \{\mathrm{IND}(P) : P \subseteq R\}. \tag{3.3}$$

Thus, $\mathrm{IND}(K_M)$ is the maximal mset of equivalence mset relations containing all elementary mset relations of K_M.

Finally the family of all mset categories in the knowledge mset base $K_M = (M, R)$ will be referred to as K_M mset categories.

4. Rough Multiset and Information Multisystems

In 1982 Pawlak [13] introduced a new mathematical tool "Rough Sets" that is devised to deal with vagueness and uncertainty. Surprisingly, this is again a contribution to humanity from one belonging to the field of computer science—during the same period, the same community that gifted several other elegant creations, like the fuzzy set theory by Lotfi Zadeh in 1965. It is also interesting to note that both of the theories address basically the same issue, namely, "vagueness" and this fact is not merely a coincidence. That "vagueness" in general is different from "probability" which is now gaining acceptance after the long, fierce debates that took place during the years immediately following the advent of the fuzzy set theory in 1965. So Pawlak did not have to fight that battle. Yet he had to utter this warning, which is an excellent distinctive criterion, namely, "vagueness is the property of sets whereas uncertainty is the property of an element" [14]. In the introduction to his short communication [15], Pawlak declares that "we compare this concept with that of the fuzzy set and we show that these two concepts are different." However, later in Pawlak a probable change in opinion is observed as reflected in the following categorical remark: "Both fuzzy and rough set theory represent two different approaches to vagueness. Fuzzy set theory addresses gradualness of knowledge, expressed by the fuzzy membership—whereas rough set theory addresses granularity of knowledge expressed by indiscernibility relation" [16].

Rough set theory is a powerful tool for dealing with the uncertainty, granularity, and incompleteness of knowledge in information systems. When there is a huge amount of data, it is very difficult to extract useful information from them, mainly because of the incompleteness of the data; that is, there might be some critical pieces of data that is missing. Vagueness and uncertainty lead to difficulty in extracting useful information from very large databases. Rough set theory differs from others by being a fundamental approach to several disciplines of artificial intelligence like expert systems, machine learning, and knowledge discovery from databases.

Definition 4.1 (information system). An information system is a pair $S = (M, A)$, where M is a nonempty finite set of objects called the universe of discourse and A is a nonempty finite set of attributes.

Information multisystems are represented using multisets instead of crisp sets. Multisets are defined as those sets in which an element may have several occurrences as against one in a crisp set. The very definition of multisets implies the omission of object identifiers. Number of occurrences of each object is denoted in an additional column named C (counts or multiplicity). Table 3 is the multiset representation of the information multisystem. Formally, an information multisystem can be defined as a triple, $S = (M, A, R)$, where M is an mset of objects, A is the set of attributes, and R is an mset relation defined on M. For example, the chemical system can be defined as $S = (M, A, R)$, where M is the mset of all possible molecules, A is an algorithm describing the reaction vessel or domain and how the rules are applied to the molecules inside the vessel, and R is the set of "collision" rules, representing the interaction among the molecules.

In the Pawlak rough set model, an equivalence relation on the universe of objects is defined based on their attribute values. After the introduction of Pawlak's rough set, many mathematicians introduced different rough set models based on various types of binary relations instead of equivalence binary relations. Based on these studies, Yao and Lin [17] introduced a new type of rough set model called nonstandard rough set model induced by various binary relations. Pawlak's standard rough set is considered and extended to the context of multisets.

Let $N \subseteq M$, and R is an equivalence mset relation. We will say that N is an R-definable mset if N is the union of some mset categories; otherwise, N is an R-undefinable mset.

The R-definable msets are those submsets of the universe which can be exactly defined in the knowledge mset base K_M, whereas the R-undefinable msets cannot be defined in this knowledge mset base.

The R-definable msets will also be called R-exact msets, and R-undefinable msets will also be called R-inexact mset or R-rough mset.

Mset $N \subseteq M$ will be called exact mset in K_M if there exists an equivalence mset relation $R \in \mathrm{IND}(K_M)$ such that N is R-exact mset and N is said to be a rough mset in K_M, if N is R-rough mset for any $R \in \mathrm{IND}(K_M)$.

A rough mset is a formal approximation of a conventional mset in terms of a pair of msets which give the lower approximation and upper approximation of the original mset.

4.1. Approximations of an Mset

Let M be an mset and R an equivalence mset relation on M. R generates a partition $M/R = \{M_1, M_2, \ldots, M_m\}$ on M, where $M_1, M_2, \ldots, M_m$ are the m-equivalence classes generated by the equivalence mset relation R.

An m-equivalence class in R containing an element $x \in^m M$ is denoted by $[m/x]$. The pair (M, R) is called an mset approximation space. For any $N \subseteq M$, we can define the lower mset approximation and upper mset approximation of N by

$$R_L(N) = \{m/x : [m/x] \subseteq M\},$$
$$R_U(N) = \{m/x : [m/x] \cap M \neq \Phi\}, \tag{4.1}$$

respectively.

The pair $(R_L(N), R_U(N))$ is referred to as the rough mset of N. The rough mset $(R_L(N), R_U(N))$ gives rise to a description of N under the present knowledge, that is, the classification of M.

Suppose that R is an arbitrary mset relation on M. With respect to R, we can define a neighborhood of an element m/x in M as follows:

$$r(m/x) = \{n/y : y \in^n M, m/xRn/y\}. \tag{4.2}$$

The mset $r(m/x)$ is also called an R-relative mset.

Definition 4.2. Let R be an arbitrary mset relation on the universal mset M, and let $r(m/x)$ be the R-relative mset (i.e., neighborhood of m/x) in M. For any $N \subseteq M$, a pair of lower and upper mset approximations $R_L(N)$ and $R_U(N)$, are defined by

$$R_L(N) = \{m/x : r(m/x) \subseteq M\},$$
$$R_U(N) = \{m/x : r(m/x) \cap M \neq \Phi\}. \tag{4.3}$$

The pair $(R_L(N), R_U(N))$ is referred to as rough mset of N.

4.2. Properties of Rough Multisets

For any equivalence mset relation R on a nonempty mset M, the following conditions hold:

(i) for every $N \subseteq M$, $R_L(N) = [R_U(N^c)]^c$;

(ii) $R_L(M) = M$;

(iii) if $N \subset M$ with $C_N(x) < C_M(x)$, then $R_L(N) = \emptyset$;

(iv) for any submsets M_1 and M_2 of M, $R_L(M_1 \cap M_2) = R_L(M_1) \cap R_L(M_2)$;

(v) if $M_1 \subseteq M_2$, then $R_L(M_1) \subseteq R_L(M_2)$;

(vi) for any submsets M_1 and M_2 of M, $R_L(M_1 \cup M_2) \supseteq R_L(M_1) \cup R_L(M_2)$;

(vii) for any submsets M_1 and M_2 of M, $R_L(M_1 \oplus M_2) \supseteq R_L(M_1) \oplus R_L(M_2)$;

(viii) for any submsets M_1 and M_2 of M, $R_L(M_1 \Theta M_2) = R_L(M_1) \Theta R_L(M_2)$;

(ix) for every $N \subseteq M, R_L(N) \subseteq R_L(R_L(N))$;

(x) for every $N \subseteq M, N \subseteq R_L(R_U(N))$;

(xi) for every $N \subseteq M, R_U(N) \subseteq R_L(R_U(N))$;

(xii) for any submsets M_1 and M_2 of M, $R_U(M_1 \oplus M_2) = R_U(M_1) \oplus R_U(M_2)$;

(xiii) for any submsets M_1 and M_2 of M, $R_U(M_1 \ominus M_2) = R_U(M_1) \ominus R_U(M_2)$.

Proof. (i) One has

$$\begin{aligned}[R_U(N^c)]^c &= \{x \in^m M : [m/x] \cap N^c \neq \emptyset\}^c \\ &= \{x \in^m M : [m/x] \cap N^c = \emptyset\} \\ &= \{x \in^m M : [m/x] \subseteq N\} \\ &= R_L(N).\end{aligned} \tag{4.4}$$

(ii) Since, for every $x \in^m M$, $[m/x] \subseteq M$, $x \in^m R_L(M)$. Thus, $M \subseteq R_L(M)$. Also since $R_L(M) \subseteq M$, hence $R_L(M) = M$.

(iii) Since R is an equivalence mset relation and $C_N(x) < C_M(x)$, there is no equivalence class in the submset of N. Hence, $R_L(N) = \emptyset$.

(iv) One has

$$\begin{aligned}R_L(M_1 \cap M_2) &= \{x \in^m M : [m/x] \subseteq M_1 \cap M_2\} \\ &= \{x \in^m M : [m/x] \subseteq M_1\} \cap \{x \in^m M : [m/x] \subseteq M_2\} \\ &= R_L(M_1) \cap R_L(M_2).\end{aligned} \tag{4.5}$$

(v) If $M_1 \subseteq M_2$ and $x \in^m R_L(M_1)$, then $[m/x] \subseteq M_1$ and so $[m/x] \subseteq M_2$; hence, $x \in^m R_L(M_2)$. Thus, $R_L(M_1) \subseteq R_L(M_2)$.

(vi) Since $M_1 \subseteq M_1 \cup M_2$ and $M_2 \subseteq M_1 \cup M_2$, so $R_L(M_1) \subseteq R_L(M_1 \cup M_2)$ and $R_L(M_2) \subseteq R_L(M_1 \cup M_2)$; hence, $R_L(M_1 \cup M_2) \supseteq R_L(M_1) \cup R_L(M_2)$.

(vii) One has

$$\begin{aligned}R_L(M_1 \oplus M_2) &= \{x \in^m M : [m/x] \subseteq M_1 \oplus M_2\} \\ &\supseteq \{x \in^m M : [m/x] \subseteq M_1\} \oplus \{x \in^m M : [m/x] \subseteq M_2\} \\ &= R_L(M_1) \oplus R_L(M_2).\end{aligned} \tag{4.6}$$

(viii) One has

$$\begin{aligned}R_L(M_1 \ominus M_2) &= \{x \in^m M : [m/x] \subseteq M_1 \ominus M_2\} \\ &= \{x \in^m M : [m/x] \subseteq M_1\} \ominus \{x \in^m M : [m/x] \subseteq M_2\} \\ &= R_L(M_1) \ominus R_L(M_2).\end{aligned} \tag{4.7}$$

(ix) Since $R_L(N) = \{x \in^m M : [m/x] \subseteq N\}$ and $R_L(R_L(N)) = \{x \in^m M : [m/x] \subseteq R_L(N)\}$, to prove that $R_L(N) \subseteq R_L(R_L(N))$, it is enough to prove that $[m/x] \subseteq R_L(N)$.

Let

$$\begin{aligned}y \in^n [m/x] &\Longrightarrow [n/y] = [m/x] \subseteq N \\ &\Longrightarrow [n/y] \subseteq N \\ &\Longrightarrow y \in^n R_L(N) \\ &\Longrightarrow [m/x] \subseteq R_L(N) \\ &\Longrightarrow R_L(N) \subseteq R_L(R_L(N)).\end{aligned} \tag{4.8}$$

(x) Let $x \in^m N$, we want to show that $x \in^m R_L(R_U(N))$, that is, $[m/x] \subseteq R_U(N)$ or $[n/y] \cap N \neq \emptyset$ for all $y \in^n [m/x]$. Since R is an equivalence mset relation, then $[m/x] = [n/y]$ or $[m/x] \cap [n/y] = \emptyset$ for all $x \in^m M$, $y \in^n M$, then, for all $x \in^m N$ and $y \in^n [m/x]$, $[m/x] \cap N \neq \emptyset$; that is, $y \in^n R_U(N)$ for all $y \in^n [m/x]$, then $[m/x] \subseteq R_U(N)$; thus, $x \in^m R_L(R_U(N))$ and so $N \subseteq R_L(R_U(N))$.

(xi) Let $x \in^m R_U(N)$; we want to show that $x \in^m R_L(R_U(N))$. Since $x \in^m R_U(N)$, then $[m/x] \cap N \neq \emptyset$; also R is an equivalence relation; hence, $[n/y] \cap N \neq \emptyset$ for all $y \in^n [m/x]$, then $y \in^n R_U(N)$ for all $y \in^n [m/x]$; that is, $[m/x] \subseteq R_U(N)$; thus, $x \in^m R_L(R_U(N))$. Hence, $R_U(N) \subseteq R_L(R_U(N))$.

(xii) One has

$$\begin{aligned} R_U(M_1 \oplus M_2) &= \{x \in^m M : [m/x] \cap M_1 \oplus M_2 \neq \emptyset\} \\ &= \{x \in^m M : ([m/x] \cap M_1) \oplus ([m/x] \cap M_2) \neq \emptyset\} \\ &= \{x \in^m M : [m/x] \cap M_1 \neq \emptyset\} \oplus \{x \in^m M : [m/x] \cap M_2 \neq \emptyset\} \\ &= R_U(M_1) \oplus R_U(M_2). \end{aligned} \tag{4.9}$$

(xiii) One has

$$\begin{aligned} R_U(M_1 \ominus M_2) &= \{x \in^m M : [m/x] \cap M_1 \ominus M_2 \neq \emptyset\} \\ &= \{x \in^m M : ([m/x] \cap M_1) \ominus ([m/x] \cap M_2) \neq \emptyset\} \\ &= \{x \in^m M : [m/x] \cap M_1 \neq \emptyset\} \ominus \{x \in^m M : [m/x] \cap M_2 \neq \emptyset\} \\ &= R_U(M_1) \ominus R_U(M_2). \end{aligned} \tag{4.10}$$

□

The following example shows that the equalities in the properties (vi), (vii), and (x) does not always hold.

Example 4.3. From Example 3.8, the equivalence classes are $[3/x] = \{3/x, 3/z, 7/r\}$, $[5/y] = \{5/y\}$, $[3/z] = \{3/x, 3/z, 7/r\}$, and $[7/r] = \{3/x, 3/z, 7/r\}$. Let $M_1\{3/x, 5/y\}$ and $M_2 = \{5/y, 3/z, 7/r\}$ be two submsets of M. Then $R_L(M_1) = \{5/y\}$ and $R_L(M_2) = \{5/y\}$. Therefore, $R_L(M_1) \oplus R_L(M_2) = \{5/y\}$. But $M_1 \oplus M_2 = \{3/x, 5/y, 3/z, 7/r\}$ and $R_L(M_1 \oplus M_2) = \{3/x, 5/y, 3/z, 7/r\}$. Thus, $R_L(M_1 \oplus M_2) \neq R_L(M_1) \oplus R_L(M_2)$.

Also $M_1 \cup M_2 = \{3/x, 5/y, 3/z, 7/r\}$ and $R_L(M_1 \cup M_2) = \{3/x, 5/y, 3/z, 7/r\}$. But $R_L(M_1) \cup R_L(M_2) = \{5/y\}$. Therefore, $R_L(M_1 \cup M_2) \neq R_L(M_1) \cup R_L(M_2) = \{5/y\}$.

Furthermore, $R_U(M_1) = \{3/x, 5/y, 3/z, 7/r\}$ and $R_L(R_U(M_1)) = R_L(M) = M$. Therefore, $R_L(R_U(M_1)) \neq M_1$.

Theorem 4.4 (see [1]). *For any submsets M_1 and M_2 of M*

(1) $(M_1 \ominus M_2)^c = M_1^c \oplus M_2$,

(2) $(M_1 \oplus M_2)^c = M_1^c \ominus M_2$.

Theorem 4.5. *For any submsets M_1 and M_2 of M*

(1) $R_L[(M_1 \ominus M_2)^c] = R_L(M_1^c) \oplus R_L(M_2)$,

(2) $R_L[(M_1 \oplus M_2)^c] = R_L(M_1^c) \ominus R_L(M_2)$.

Proof. (1) One has

$$\begin{aligned} R_L\left[(M_1 \ominus M_2)^c\right] &= \left\{x \in^m M : [m/x] \subseteq (M_1 \ominus M_2)^c\right\} \\ &= \left\{x \in^m M : [m/x] \subseteq M_1^c \oplus M_2\right\} \\ &= \left\{x \in^m M : [m/x] \subseteq M_1^c\right\} \oplus \{x \in^m M : [m/x] \subseteq M_2\} \\ &= R_L\left(M_1^c\right) \oplus R_L(M_2). \end{aligned} \tag{4.11}$$

(2) One has

$$\begin{aligned} R_L\left[(M_1 \oplus M_2)^c\right] &= \left\{x \in^m M : [m/x] \subseteq (M_1 \oplus M_2)^c\right\} \\ &= \left\{x \in^m M : [m/x] \subseteq M_1^c \ominus M_2\right\} \\ &= \left\{x \in^m M : [m/x] \subseteq M_1^c\right\} \ominus \{x \in^m M : [m/x] \subseteq M_2\} \\ &= R_L\left(M_1^c\right) \ominus R_L(M_2). \end{aligned} \tag{4.12}$$

□

The following example gives the analysis of information multisystems with a given number of objects and attributes.

Example 4.6. Let us consider mset of balls (objects) $M = \{k_1/b_1, k_2/b_2, k_3/b_3, k_4/b_4, k_5/b_5, k_6/b_6, k_7/b_7, k_8/b_8\}$ which are of different colors (attributes) $\{\text{black}, \text{blue}, \text{red}\}$, different types $\{\text{football}, \text{cricket ball}, \text{volleyball}\}$, and different prices (attributes) $\{\text{costly}, \text{cheap}\}$. Thus, the information system $S = (M, A)$ consists of $\sum_{i=1}^{8} k_i$ objects and 5 attributes.

The balls can be classified as costly black cricket ball, cheap blue football, and so forth. Thus, the mset of balls in M can be classified according to color, type, and cost. The example follows.

Balls k_1/b_1, k_3/b_3, k_7/b_7 are black, k_2/b_2, k_4/b_4 are blue, and k_5/b_5, k_6/b_6, k_8/b_8 are red.

Among the balls k_1/b_1, k_5/b_5 are foot balls, k_2/b_2, k_6/b_6 are cricket balls and k_3/b_3, $k_4/b_4, k_7/b_7, k_8/b_8$ are volleyballs which are of $k_2/b_2, k_7/b_7, k_8/b_8$ that are costly and $k_1/b_1, k_3/b_3, k_4/b_4, k_5/b_5, k_8/b_8$ that are cheap.

In other words by these classifications three equivalence mset relations R_1, R_2, and R_3 are defined with the following m-equivalence classes:

$$\begin{aligned} M/R_1 &= \{\{k_1/b_1, k_3/b_3, k_7/b_7\}, \{k_2/b_2, k_4/b_4\}, \{k_5/b_5, k_6/b_6, k_8/b_8\}\}, \\ M/R_2 &= \{\{k_1/b_1, k_5/b_5\}, \{k_2/b_2, k_6/b_6\}, \{k_3/b_3, k_4/b_4, k_7/b_7, k_8/b_8\}\}, \\ \frac{M}{R_3} &= \{\{k_2/b_2, k_7/b_7, k_8/b_8\}, \{k_1/b_1, k_3/b_3, k_4/b_4, k_5/b_5, k_8/b_8\}\}. \end{aligned} \tag{4.13}$$

These m-equivalence classes are mset concepts (mset categories) in our knowledge mset base $K_M = (M, \{R_1, R_2, R_3\})$.

Basic mset categories are mset intersections of elementary mset categories. For example, msets

$$\begin{gathered}\{k_1/b_1, k_3/b_3, k_7/b_7\} \cap \{k_3/b_3, k_4/b_4, k_7/b_7, k_8/b_8\} = \{k_3/b_3, k_7/b_7\}, \\ \{k_2/b_2, k_4/b_4\} \cap \{k_2/b_2, k_6/b_6\} = \{k_2/b_2\}, \\ \{k_5/b_5, k_6/b_6, k_8/b_8\} \cap \{k_3/b_3, k_4/b_4, k_7/b_7, k_8/b_8\} = \{k_8/b_8\}\end{gathered} \tag{4.14}$$

are $\{R_1, R_2\}$ basic mset categories of black volleyballs, blue cricket balls, and red volleyballs, respectively.

Msets

$$\begin{gathered}\{k_1/b_1, k_3/b_3, k_7/b_7\} \cap \{k_3/b_3, k_4/b_4, k_7/b_7, k_8/b_8\} \cap \{k_2/b_2, k_7/b_7, k_8/b_8\} = \{k_7/b_7\}, \\ \{k_2/b_2, k_4/b_4\} \cap \{k_2/b_2, k_6/b_6\} \cap \{k_2/b_2, k_7/b_7, k_8/b_8\} = \{k_2/b_2\}, \\ \{k_5/b_5, k_6/b_6, k_8/b_8\} \cap \{k_3/b_3, k_4/b_4, k_7/b_7, k_8/b_8\} \cap \{k_2/b_2, k_7/b_7, k_8/b_8\} = \{k_8/b_8\}\end{gathered} \tag{4.15}$$

are elementary $\{R_1, R_2, R_3\}$ basic mset categories of black volleyball of cheap cost, blue cricket ball of high cost, and red volleyball of cheap cost, respectively.

Msets

$$\begin{gathered}\{k_1/b_1, k_3/b_3, k_7/b_7\} \cup \{k_2/b_2, k_4/b_4\} = \{k_1/b_1, k_2/b_2, k_3/b_3, k_4/b_4, k_7/b_7\}, \\ \{k_2/b_2, k_4/b_4\} \cup \{k_5/b_5, k_6/b_6, k_8/b_8\} = \{k_2/b_2, k_4/b_4, k_5/b_5, k_6/b_6, k_8/b_8\}, \\ \{k_1/b_1, k_3/b_3, k_7/b_7\} \cup \{k_5/b_5, k_6/b_6, k_8/b_8\} = \{k_1/b_1, k_3/b_3, k_5/b_5, k_6/b_6, k_7/b_7, k_8/b_8\}\end{gathered} \tag{4.16}$$

are $\{R_1\}$ mset categories black or blue (not red), blue or red (not black), and black or red (not blue), respectively.

Note that some mset categories are not available in this knowledge mset base. For example, msets

$$\begin{gathered}\{k_2/b_2, k_4/b_4\} \cap \{k_1/b_1, k_5/b_5\} = \Phi, \\ \{k_1/b_1, k_3/b_3, k_7/b_7\} \cap \{k_2/b_2, k_6/b_6\} = \Phi\end{gathered} \tag{4.17}$$

are empty which means that mset categories of blue footballs and black cricket balls do not exist in our knowledge mset base (i.e., empty mset categories).

4.3. Information Tables and Decision Tables

From the information system $S = (M, A)$ as in Example 4.6, the information table representing the universe of discourse is constructed.

Table 2 can be analyzed in the following way. The first 5-tuple can be interpreted as k_1 number of black football with cheap cost, similar way for the rest of the tuples.

Consider the information Table 3 with respect to the given information system $S = (M, A)$.

Table 2

Labels	Item	Color	Price	No. of items
b_1	Football	Black	Cheap	k_1
b_2	Cricket ball	Blue	Costly	k_2
b_3	Volleyball	Black	Cheap	k_3
b_4	Volleyball	Blue	Cheap	k_4
b_5	Football	Red	Cheap	k_5
b_6	Cricket ball	Red	Cheap	k_6
b_7	Volleyball	Black	Costly	k_7
b_8	Volleyball	Red	Costly	k_8

Table 3

Blocks	Labels	Items	No. of items	Colors	Price
X_1	b_1	Football	k_1	Black	Cheap
X_1	b_3	Volleyball	k_3	Black	Cheap
X_1	b_7	Volleyball	k_7	Black	Costly
X_2	b_2	Cricket ball	k_2	Blue	Costly
X_2	b_4	Volleyball	k_4	Blue	Cheap
X_3	b_5	Football	k_5	Red	Cheap
X_3	b_6	Cricket ball	k_6	Red	Cheap
X_3	b_8	Volleyball	k_8	Red	Costly

Table 3 can be analyzed in the following way. The table contains three blocks X_1, X_2, and X_3 with the block X_1 containing k_1/b_1 black footballs and k_3/b_3 black volleyballs with cheap cost and k_7/b_7 black volleyballs with high cost. Similarly block X_2 contains k_2/b_2 blue cricket balls with high cost and k_4/b_4 blue volleyballs with cheap cost, and block X_3 contains k_5/b_5 red footballs and k_6/b_6 red cricket balls with cheap cost and k_8/b_8 red volleyballs with high cost.

With respect to the information Table 3, the decision table can be constructed in the following way. The first column of the decision table represents object identifiers with multiplicity. Identifiers are used to uniquely identify each row in the table. The decision table has six attributes $\{P_1, P_2, P_3, P_4, P_5, D, C\}$. The attribute D is the decision attribute with values 1, 2, 3, and C is the count (multiplicity) of the object. The other attributes are color and cost—the condition attributes. In Table 2 the universe of discourse M consists of $\sum_{i=1}^{8} k_i$ objects. The partition of M based on Table 3, the objects of decision attribute D, is

$$\begin{aligned} X_1 &= \{k_1/b_1, k_3/b_3, k_7/b_7\}, \\ X_2 &= \{k_2/b_2, k_4/b_4\}, \\ X_3 &= \{k_5/b_5, k_6/b_6, k_8/b_8\}. \end{aligned} \tag{4.18}$$

With respect to this decision, the decision Table 4, presented above, follows.

Table 4

Attributes	P_1	P_2	P_3	P_4	P_5	D	C
b_1	1	0	0	0	1	1	k_1
b_2	0	1	0	1	0	2	k_2
b_3	1	0	0	0	1	1	k_3
b_4	0	1	0	0	1	2	k_4
b_5	0	0	1	0	1	3	k_5
b_6	0	0	1	0	1	3	k_6
b_7	1	0	0	1	0	1	k_7
b_8	0	0	1	0	1	3	k_8

Table 5: Lower approximation decision.

Attribute	P_1	P_2	P_3	P_4	P_5	D	C
b_2	0	1	0	1	0	2	k_2
b_4	0	1	0	0	1	2	k_4

Considering the target set $X = \{k_1/b_1,\ k_2/b_2,\ k_4/b_4\}$, the lower approximations and upper approximations are given as follows:

$$\begin{aligned} R_L(X) &= \{k_2/b_2, k_4/b_4\}, \\ R_U(X) &= \{k_1/b_1, k_3/b_3, k_7/b_7\} \cup \{k_2/b_2, k_4/b_4\} \\ &= \{k_1/b_1, k_2/b_2, k_3/b_3, k_4/b_4, k_7/b_7\}. \end{aligned} \tag{4.19}$$

From the above-discussed information multisystem, decision tables (Tables 5 and 6), associated with each block, there exist two nonnegative integers representing the minimum (lower) and maximum (upper) number of copies. The minimum and maximum values may depend upon some statistical analysis of the data concerning the necessity and utility of each item and the number of items needed for the relevant programmes. The decision is taken about the number of items that are needed according to their demand and other factors involved.

5. Conclusion and Future Work

Pawlak introduced the concept of rough sets which have a wide range of applications in various fields like artificial intelligence, cognitive sciences, knowledge discovery from databases, machine learning expert systems, and so forth [18–26]. These types of situations deal with objects and attributes. Many situations may occur, where the counts of the objects in the universe of discourse are not single. In information systems we come across situations which involve the concept of a multiset. This paper begins with Yager's theory of multisets. After presenting the theoretical study, the concepts mset relations, equivalence mset relations, partitions, and knowledge mset base have been established. Finally the concept of rough multisets and related properties with the help of lower mset approximation and upper mset approximations have been introduced. It is observed that rough multisets are important frameworks for certain types of information multisystems.

Table 6: Upper approximation decision.

Attribute	P_1	P_2	P_3	P_4	P_5	D	C
b_1	1	0	0	0	1	1	k_1
b_2	0	1	0	1	0	2	k_2
b_3	1	0	0	0	1	1	k_3
b_4	0	1	0	0	1	2	k_4
b_7	1	0	0	1	0	1	k_7

Chakrabarty [27, 28] introduced two types of bags called IC bags and n^k bags, which are suitable for situations, where the counts of the objects in information system are not fixed and are represented in the form of intervals of positive integers and power set of positive integers ($P(N)$). These kinds of problems appear, for instance, during a nuclear fission, when a nucleus (consisting of protons and neutrons) is split into multiple nuclei, each of them with its own number of protons and neutrons. Thus, the information multisystem can be associated with IC bags or n^k bags by the help of lower mset approximations and upper mset approximations. This association could be used for certain types of decision analysis problems and could prove useful as mathematical tools for building decision support systems.

Acknowledgments

The authors would like to thank the anonymous referees and Editor-in-Chief, for carefully examining the paper and providing very helpful comments and suggestions. They are also grateful to Mrs. K. Sreelatha for improving the linguistic quality of the paper.

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