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Research Article

A Simulation Approach to Statistical Estimation of Multiperiod Optimal Portfolios

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This paper discusses a simulation-based method for solving discrete-time multiperiod portfolio choice problems under AR(1) process. The method is applicable even if the distributions of return processes are unknown. We first generate simulation sample paths of the random returns by using AR bootstrap. Then, for each sample path and each investment time, we obtain an optimal portfolio estimator, which optimizes a constant relative risk aversion (CRRA) utility function. When an investor considers an optimal investment strategy with portfolio rebalancing, it is convenient to introduce a value function. The most important difference between single-period portfolio choice problems and multiperiod ones is that the value function is time dependent. Our method takes care of the time dependency by using bootstrapped sample paths. Numerical studies are provided to examine the validity of our method. The result shows the necessity to take care of the time dependency of the value function.

1. Introduction

Portfolio optimization is said to be “myopic” when the investor does not know what will happen beyond the immediate next period. In this framework, basic results about single period portfolio optimization (such as mean-variance analysis) are justified for short-term investments without portfolio rebalancing. Multiperiod problems are much more realistic than single-period ones. In this framework, we assume that an investor makes a sequence of decisions to maximize a utility function at each time. The fundamental method to solve this problem is the dynamic programming. In this method, a value function which expresses the expected terminal wealth is introduced. The recursive equation with respect to the value function is so-called Bellman equation. The first order conditions (FOCs) to satisfy the Bellman equation are key tool in order to solve the dynamic problem.

The original literature on dynamic portfolio choice, pioneered by Merton [1] in continuous time and by Samuelson [2] and Fama [3] in discrete time, produced many important insights into the properties of optimal portfolio policies. Unfortunately, since it is known that the closed-form solutions are obtained only for a few special cases, the recent literature uses a variety of numerical and approximate solution methods to incorporate realistic features into the dynamic portfolio problem such as Ait-Sahalia and Brandet [4] and Brandt et al. [5].

We introduce an procedure to construct the dynamic portfolio weights based on AR bootstrap.

The simulation algorithm is as follows; first, we generate simulation sample paths of the vector random returns by using AR bootstrap. Based on the bootstrapping samples, an optimal portfolio estimator, which is applied from time $T-1$ to the end of trading time T, is obtained under a constant relative risk aversion (CRRA) utility function. Note that this optimal portfolio corresponds "myopic" (single period) optimal portfolio. Next we approximate the value function by linear functions of the past observation. This idea is similar to that of [4, 5]. Then, optimal portfolio weight estimators at each trading time are obtained based on the value function. Finally, we construct an optimal investment strategy as a sequence of the optimal portfolio weight estimators.

This paper is organized as follows. We describe the basic idea to solve multiperiod optimal portfolio weights under a CRRA utility function in Section 2. In Section 3, we discuss an algorithm to construct the estimator involving the method of AR bootstrap. The applications of our method are in Section 4.

2. Multiperiod Optimal Portfolio

Suppose the existence of a finite number of risky assets indexed by i, $(i = 1, \ldots, m)$. Let $\mathbf{X}_t = (X_1(t), \ldots, X_m(t))'$ denote the random excess returns on m assets from time t to $t+1$ (suppose that $S_i(t)$ is a value of asset i at time t. Then, the return is described as $1+X_i(t) = S_i(t)/S_i(t-1)$). Suppose too that there exists a risk-free asset with the excess return X_f (Suppose that $B(t)$ is a value of risk-free asset at time t. Then, the return is described as $1 + X_f = B(t)/B(t-1)$). Based on the process $\{\mathbf{X}_t\}_{t=1}^{T}$ and X_f, we consider an investment strategy from time 0 to time T where $T(\in \mathbb{N})$ denotes the end of the investment time. Let $\mathbf{w}_t = (w_1(t), \ldots, w_m(t))'$ be vectors of portfolio weight for the risky assets at the beginning of time $t+1$. Here we assume that the portfolio weights $\mathbf{w}_t$ can be rebalanced at the beginning of time $t+1$ and measurable (predictable) with respect to the past information $\mathcal{F}_t = \sigma(\mathbf{X}_t, \mathbf{X}_{t-1}, \ldots)$. Here we make the following assumption.

Assumption 2.1. There exists an optimal portfolio weight $\tilde{\mathbf{w}}_t \in \mathbb{R}^m$ satisfied with $|\tilde{\mathbf{w}}_t'\mathbf{X}_{t+1} + (1 - \tilde{\mathbf{w}}_t\mathbf{e})X_f| \ll 1$ (we assume that the risky assets exclude ultra high-risk and high-return ones, for instance, the asset value $S_i(t+1)$ may be larger than $2S_i(t)$), almost surely for each time $t = 0, 1, \ldots, T-1$ where $\mathbf{e} = (1, \ldots, 1)'$.

Then the return of the portfolio from time t to $t+1$ is written as $1 + X_f + \mathbf{w}_t'(\mathbf{X}_{t+1} - X_f\mathbf{e})$ (assuming that $\mathbf{S}_t := (S_1(t), \ldots, S_m(t))' = B(t)\mathbf{e}$, the portfolio return is written as $(\mathbf{w}_t'\mathbf{S}_{t+1} + (1 - \mathbf{w}_t\mathbf{e})B(t+1))/(\mathbf{w}_t'\mathbf{S}_t + (1 - \mathbf{w}_t\mathbf{e})B(t)) = 1 + X_f + \mathbf{w}_t'(\mathbf{X}_{t+1} - X_f\mathbf{e})$) and the return from time 0 to time T (called terminal wealth) is written as

$$W_T := \prod_{t=0}^{T-1} \left(1 + X_f + \mathbf{w}_t'(\mathbf{X}_{t+1} - X_f\mathbf{e})\right). \tag{2.1}$$

Suppose that a utility function $U : x \mapsto U(x)$ is differentiable, concave, and strictly increasing for each $x \in \mathbb{R}$. Consider an investor's problem

$$\max_{\{\mathbf{w}_t\}_{t=0}^{T-1}} E[U(W_T)]. \tag{2.2}$$

Following a formulation by the dynamic programming (e.g., Bellman [6]), it is convenient to express the expected terminal wealth in terms of a value function V_t:

$$\begin{aligned} V_t &\equiv \max_{\{\mathbf{w}_s\}_{s=t}^{T-1}} E[U(W_T) \mid \mathcal{F}_t] \\ &= \max_{\mathbf{w}_t} E\left[\max_{\{\mathbf{w}_s\}_{s=t}^{T-1}} E[U(W_T) \mid \mathcal{F}_{t+1}] \mid \mathcal{F}_t \right] \\ &= \max_{\mathbf{w}_t} E[V_{t+1} \mid \mathcal{F}_t], \end{aligned} \tag{2.3}$$

subject to the terminal condition $V_T = U(W_T)$. The recursive equation (2.3) is the so-called Bellman equation and is the basis for any recursive solution of the dynamic portfolio choice problem. The first-order conditions (FOCs) (here $(\partial/\partial \mathbf{w}_t\ )E[V_{t+1}|\mathcal{F}_t] = E[(\partial/\partial \mathbf{w}_t)V_{t+1}|\mathcal{F}_t]$.is assumed). in order to obtain an optimal solution at each time t are

$$\frac{\partial V_t}{\partial \mathbf{w}_t} = E\left[\partial_1 U(W_T)\left(\mathbf{X}_{t+1} - X_f \mathbf{e}\right) \mid \mathcal{F}_t\right] = \mathbf{0}, \tag{2.4}$$

where $\partial_1 U(x_0) = (\partial/\partial x)U(x)|_{x=x_0}$. These FOCs make up a system of nonlinear equations involving integrals that can in general be solved for $\mathbf{w}_t$ only numerically.

According to the literature (e.g., [5]), we can simplify this problem in case of a constant relative risk aversion (CRRA) utility function, that is,

$$U(W) = \frac{W^{1-\gamma}}{1-\gamma}, \quad \gamma \neq 1, \tag{2.5}$$

where γ denotes the coefficient of relative risk aversion. In this case, the Bellman equation simplifies to

$$\begin{aligned}
V_t &= \max_{\mathbf{w}_t} E\left[\max_{\{\mathbf{w}_s\}_{s=t+1}^{T-1}} E\left[\frac{1}{1-\gamma}(W_T)^{1-\gamma} \mid \mathcal{F}_{t+1}\right] \mid \mathcal{F}_t\right] \\
&= \max_{\mathbf{w}_t} E\left[\max_{\{\mathbf{w}_s\}_{s=t+1}^{T-1}} E\left[\frac{1}{1-\gamma}\left(\prod_{s=0}^{T-1}(1+X_f+\mathbf{w}_s'(\mathbf{X}_{s+1}-X_f\mathbf{e})\right)^{1-\gamma} \mid \mathcal{F}_{t+1}\right] \mid \mathcal{F}_t\right] \\
&= \max_{\mathbf{w}_t} E\left[\frac{1}{1-\gamma}\left(\prod_{s=0}^{t}(1+X_f+\mathbf{w}_s'(\mathbf{X}_{s+1}-X_f\mathbf{e}))\right)^{1-\gamma}\right. \\
&\quad \left.\times \max_{\{\mathbf{w}_s\}_{s=t+1}^{T-1}} E\left[\left(\prod_{s=t+1}^{T-1}(1+X_f+\mathbf{w}_s'(\mathbf{X}_{s+1}-X_f\mathbf{e}))\right)^{1-\gamma} \mid \mathcal{F}_{t+1}\right] \mid \mathcal{F}_t\right] \\
&= \max_{\mathbf{w}_t} E\left[\frac{1}{1-\gamma}(W_{t+1})^{1-\gamma} \max_{\{\mathbf{w}_s\}_{s=t+1}^{T-1}} E\left[\left(W_{t+1}^T\right)^{1-\gamma} \mid \mathcal{F}_{t+1}\right] \mid \mathcal{F}_t\right] \\
&= \max_{\mathbf{w}_t} E[U(W_{t+1})\Psi_{t+1} \mid \mathcal{F}_t],
\end{aligned} \tag{2.6}$$

where $W_{t+1}^T = \prod_{s=t+1}^{T-1}(1+X_f+\mathbf{w}_s'(\mathbf{X}_{s+1}-X_f\mathbf{e}))$ and $\Psi_{t+1} = \max_{\{\mathbf{w}_s\}_{s=t+1}^{T-1}} E[(W_{t+1}^T)^{1-\gamma}|\mathcal{F}_{t+1}]$.

From this, the value function V_t can be expressed as

$$V_t = U(W_t)\Psi_t, \tag{2.7}$$

and Ψ_t also satisfies a Bellman equation

$$\Psi_t = \max_{\mathbf{w}_t} E\left[(1+X_f+\mathbf{w}_t'(\mathbf{X}_{t+1}-X_f\mathbf{e}))^{1-\gamma}\Psi_{t+1} \mid \mathcal{F}_t\right], \tag{2.8}$$

subject to the terminal condition $\Psi_T = 1$.

The corresponding FOCs (in terms of Ψ_t) are

$$E\left[(1+X_f+\mathbf{w}_t'(\mathbf{X}_{t+1}-X_f\mathbf{e}))^{-\gamma}\Psi_{t+1}(\mathbf{X}_{t+1}-X_f\mathbf{e}) \mid \mathcal{F}_t\right] = \mathbf{0}. \tag{2.9}$$

3. Estimation

Suppose that $\{\mathbf{X}_t = (X_1(t),\ldots,X_m(t))';\ t \in \mathbb{Z}\}$ is an m-vector AR(1) process defined by

$$\mathbf{X}_t = \boldsymbol{\mu} + A(\mathbf{X}_{t-1} - \boldsymbol{\mu}) + \boldsymbol{\epsilon}_t, \tag{3.1}$$

where $\boldsymbol{\mu} = (\mu_1,\ldots,\mu_m)'$ is a constant m-dimensional vector, $\boldsymbol{\epsilon}_t = (\epsilon_1(t),\ldots,\epsilon_m(t))'$ are independent and identically distributed (i.i.d.) random m-dimensional vectors with $E[\boldsymbol{\epsilon}_t] = \mathbf{0}$

and $E[\boldsymbol{\epsilon}_t \boldsymbol{\epsilon}_t'] = \Gamma$ (Γ is a nonsingular m by m matrix), and A is a nonsingular m by m matrix. We make the following assumption.

Assumption 3.1. $\det\{I_m - Az\} \neq 0$ on $\{z \in \mathbb{C}; |z| \leq 1\}$.

Given $\{\mathbf{X}_{-n+1}, \ldots, \mathbf{X}_0, \mathbf{X}_1, \ldots, \mathbf{X}_t\}$, the least-squares estimator $\widehat{A}^{(t)}$ of A is obtained by solving

$$\widehat{\Gamma}^{(t)} \widehat{A}^{(t)} = \sum_{s=-n+2}^{t} \widehat{\mathbf{Y}}_{s-1}^{(t)} \left(\widehat{\mathbf{Y}}_{s}^{(t)}\right)', \tag{3.2}$$

where $\widehat{\mathbf{Y}}_s^{(t)} = \mathbf{X}_s - \widehat{\boldsymbol{\mu}}^{(t)}$, $\widehat{\Gamma}^{(t)} = \sum_{s=-n+1}^{t} \widehat{\mathbf{Y}}_s^{(t)} (\widehat{\mathbf{Y}}_s^{(t)})'$ and $\widehat{\boldsymbol{\mu}}^{(t)} = (1/(n+t)) \sum_{s=-n+1}^{t} \mathbf{X}_s$. Then, the error $\widehat{\boldsymbol{\epsilon}}_s^{(t)} = (\widehat{\epsilon}_1^{(t)}(s), \ldots, \widehat{\epsilon}_m^{(t)}(s))'$ is "recovered" by

$$\widehat{\boldsymbol{\epsilon}}_s^{(t)} := \widehat{\mathbf{Y}}_s^{(t)} - \widehat{A}^{(t)} \widehat{\mathbf{Y}}_{s-1}^{(t)}, \qquad s = -n+2, \ldots, t. \tag{3.3}$$

Let $F_n^{(t)}(\cdot)$ denote the distribution which puts mass $1/(n+t)$ at $\widehat{\boldsymbol{\epsilon}}_s^{(t)}$. Let $\{\boldsymbol{\epsilon}_s^{(b,t)*}\}_{s=t+1}^{T}$ (for $b = 1, \ldots, B (\in \mathbb{N})$) be i.i.d. bootstrapped observations from $F_n^{(t)}$.

Given $\{\boldsymbol{\epsilon}_s^{(b,t)*}\}$, define $\mathbf{Y}_s^{(b,t)*}$ and $\mathbf{X}_s^{(b_1,b_2,t)*}$ by

$$\begin{gathered} \mathbf{Y}_s^{(b,t)*} = \left(\widehat{A}^{(t)}\right)^{s-t} \left(\mathbf{X}_t - \widehat{\boldsymbol{\mu}}^{(t)}\right) + \sum_{k=t+1}^{s} \left(\widehat{A}^{(t)}\right)^{s-k} \epsilon_k^{(b,t)*}, \\ \mathbf{X}_s^{(b_1,b_2,t)*} = \widehat{\boldsymbol{\mu}}^{(t)} + \widehat{A}^{(t)} \mathbf{Y}_{s-1}^{(b_1,t)*} + \boldsymbol{\epsilon}_s^{(b_2,t)*}, \end{gathered} \tag{3.4}$$

for $s = t+1, \ldots, T$.

Based on the above $\{\mathbf{X}_s^{(b_1,b_2,t)*}\}_{b_1,b_2=1,\ldots,B; s=t+1,\ldots,T}$ for each $t = 0, \ldots, T-1$, we construct an estimator of the optimal portfolio weight $\widetilde{\mathbf{w}}_t$ as follows.

Step 1. First, we fix the current time t which implies that the observed stretch $n+t$ is fixed. Then, we can generate $\{\mathbf{X}_s^{(b_1,b_2,t)*}\}$ by (3.4).

Step 2. Next, for each $b_0 = 1, \ldots, B$, we obtain $\widehat{\mathbf{w}}_{T-1}^{(b_0,t)}$ as the maximizer of

$$E_{T-1}^{*}\left[\left(1 + X_f + \mathbf{w}'\left(\mathbf{X}_T^{(b_0,b,t)*} - X_f \mathbf{e}\right)\right)^{1-\gamma}\right] = \frac{1}{B} \sum_{b=1}^{B} \left(1 + X_f + \mathbf{w}'\left(\mathbf{X}_T^{(b_0,b,t)*} - X_f \mathbf{e}\right)\right)^{1-\gamma}, \tag{3.5}$$

or the solution of

$$\begin{aligned} &E_{T-1}^{*}\left[\left(1 + X_f + \mathbf{w}'\left(\mathbf{X}_T^{(b_0,b,t)*} - X_f \mathbf{e}\right)\right)^{1-\gamma} \left(\mathbf{X}_T^{(b_0,b,t)*} - X_f \mathbf{e}\right)\right] \\ &\quad = \frac{1}{B} \sum_{b=1}^{B} \left(1 + X_f + \mathbf{w}'\left(\mathbf{X}_T^{(b_0,b,t)*} - X_f \mathbf{e}\right)\right)^{1-\gamma} \left(\mathbf{X}_T^{(b_0,b,t)*} - X_f \mathbf{e}\right) \\ &\quad = \mathbf{0}, \end{aligned} \tag{3.6}$$

with respect to $\mathbf{w}$. Here we introduce a notation "$E_s^*[\cdot]$" as an estimator of conditional expectation $E[\cdot \mid \mathcal{F}_s]$, which is defined by $E_s^*[h(\mathbf{X}_{s+1}^{(b_0,b,t)*})] = (1/B)\sum_{b=1}^{B} h(\mathbf{X}_{s+1}^{(b_0,b,t)*})$ for any function h of $\mathbf{X}_{s+1}^{(b_0,b,t)*}$. This $\widehat{\mathbf{w}}_{T-1}^{(b_0,t)}$ corresponds to the estimator of myopic (single period) optimal portfolio weight.

Step 3. Next, we construct estimators of Ψ_{T-1}. Since it is difficult to express the explicit form of Ψ_{T-1}, we parameterize it as linear functions of $\mathbf{X}_{T-1}$ as follows;

$$\Psi^{(1)}(\mathbf{X}_{T-1}, \boldsymbol{\theta}_{T-1}) := \left[1, \mathbf{X}'_{T-1}\right]\boldsymbol{\theta}_{T-1}, \tag{3.7}$$

$$\Psi^{(2)}(\mathbf{X}_{T-1}, \boldsymbol{\theta}_{T-1}) := \left[1, \mathbf{X}'_{T-1},\ \mathrm{vech}\left(\mathbf{X}_{T-1}\mathbf{X}'_{T-1}\right)'\right]\boldsymbol{\theta}_{T-1}. \tag{3.8}$$

Note that the dimensions of $\boldsymbol{\theta}_{T-1}$ in $\Psi^{(1)}$ and $\Psi^{(2)}$ are $m+1$ and $m(m+1)/2+m+1$, respectively. The idea of $\Psi^{(1)}$ and $\Psi^{(2)}$ is inspired by the parameterization of the conditional expectations in [5].

In order to construct the estimators of $\Psi^{(i)}$ $(i = 1, 2)$, we introduce the conditional least squares estimators of the parameter $\boldsymbol{\theta}_{T-1}^{(i)}$, that is,

$$\widehat{\boldsymbol{\theta}}_{T-1}^{(i)} = \arg\min_{\boldsymbol{\theta}} Q_{T-1}^{(i)}(\boldsymbol{\theta}), \tag{3.9}$$

where

$$\begin{aligned} Q_{T-1}^{(i)}(\boldsymbol{\theta}) &= \frac{1}{B}\sum_{b_0=1}^{B} E_{T-1}^*\left[\left(\Psi_{T-1} - \Psi^{(i)}\right)^2\right] \\ &= \frac{1}{B}\sum_{b_0=1}^{B}\left[\frac{1}{B}\sum_{b=1}^{B}\left\{\Psi_{T-1}\left(\mathbf{X}_T^{(b_0,b,t)*}\right) - \Psi_{T-1}^{(i)}\left(\mathbf{X}_{T-1}^{(b_0,b_0,t)*}, \boldsymbol{\theta}\right)\right\}^2\right], \end{aligned} \tag{3.10}$$

$$\Psi_{T-1}\left(\mathbf{X}_T^{(b_0,b,t)*}\right) = \left(1 + X_f + \left(\widehat{\mathbf{w}}_{T-1}^{(b_0,t)}\right)'\left(\mathbf{X}_T^{(b_0,b,t)*} - X_f\mathbf{e}\right)\right)^{1-\gamma}.$$

Then, by using $\widehat{\boldsymbol{\theta}}_{T-1}^{(i)}$, we can compute $\Psi^{(i)}(\mathbf{X}_{T-1}^{(b_0,b,t)^*}, \widehat{\boldsymbol{\theta}}_{T-1}^{(i)})$.

Step 4. Based on the above $\Psi^{(i)}$, we obtain $\widehat{\mathbf{w}}_{T-2}^{(b_0,t)}$ as the maximizer of

$$\begin{aligned} &E_{T-2}^*\left[\left(1 + X_f + \mathbf{w}'\left(\mathbf{X}_{T-1}^{(b_0,b,t)*} - X_f\mathbf{e}\right)\right)^{1-\gamma}\Psi^{(i)}\left(\mathbf{X}_{T-1}^{(b_0,b,t)*}, \widehat{\boldsymbol{\theta}}_{T-1}^{(i)}\right)\right] \\ &\quad = \frac{1}{B}\sum_{b=1}^{B}\left(1 + X_f + \mathbf{w}'\left(\mathbf{X}_{T-1}^{(b_0,b,t)*} - X_f\mathrm{e}\right)\right)^{1-\gamma}\Psi^{(i)}\left(\mathbf{X}_{T-1}^{(b_0,b,t)*}, \widehat{\boldsymbol{\theta}}_{T-1}^{(i)}\right), \end{aligned} \tag{3.11}$$

or the solution of

$$E_{T-2}^{*}\left[\left(1+X_f+\mathbf{w}'\left(\mathbf{X}_{T-1}^{(b_0,b,t)*}-X_f\mathbf{e}\right)\right)^{1-\gamma}\left(\mathbf{X}_{T-1}^{(b_0,b,t)*}-X_f\mathbf{e}\right)\Psi^{(i)}\left(\mathbf{Y}_{T-1}^{(b_0,b,t)*},\widehat{\boldsymbol{\theta}}_{T-1}^{(i)}\right)\right]$$
$$=\frac{1}{B}\sum_{b=1}^{B}\left(1+X_f+\mathbf{w}'\left(\mathbf{X}_{T-1}^{(b_0,b,t)*}-X_f\mathbf{e}\right)\right)^{1-\gamma}\left(\mathbf{X}_{T-1}^{(b_0,b,t)*}-X_f\mathbf{e}\right)\Psi^{(i)}\left(\mathbf{X}_{T-1}^{(b_0,b,t)*},\widehat{\boldsymbol{\theta}}_{T-1}^{(i)}\right)$$
$$=\mathbf{0}. \tag{3.12}$$

with respect to $\mathbf{w}$. This $\widehat{\mathbf{w}}_{T-2}^{(b_0,t)}$ does not correspond to the estimator of myopic (single period) optimal portfolio weight due to the effect of $\Psi^{(i)}$.

Step 5. In the same manner of Steps 3–4, we can obtain $\widehat{\boldsymbol{\theta}}_s^{(i)}$ and $\widehat{\mathbf{w}}_s^{(b_0,t)}$, recursively, for $s = T-2, T-1, \ldots, t+1$.

Step 6. Then, we define an optimal portfolio weight estimator at time t as $\widehat{\mathbf{w}}_t^{(t)} := \widehat{\mathbf{w}}_t^{(b_0,t)}$ by Step 4. Note that $\widehat{\mathbf{w}}_t^{(t)}$ is obtained as only one solution because $\mathbf{X}_{t+1}^{(b_0,b,t)*}(=\widehat{\boldsymbol{\mu}}^{(t)}+\widehat{A}^{(t)}(\mathbf{X}_t-\widehat{\boldsymbol{\mu}}^{(t)})+\boldsymbol{\epsilon}_{t+1}^{(b,t)*})$ is independent of b_0.

Step 7. For each time $t = 0, 1, \ldots, T-1$, we obtain $\widehat{\mathbf{w}}_t^{(t)}$ by Steps 1–6. Finally, we can construct an optimal investment strategy as $\{\widehat{\mathbf{w}}_t^{(t)}\}_{t=0}^{T-1}$.

4. Examples

In this section we examine our approach numerically. Suppose that there exists a risky asset with the excess return X_t at time t and a risk-free asset with the excess return $X_f = 0.01$. We assume that X_t is defined by the following univariate AR(1) model:

$$X_t = \mu + A(X_{t-1}-\mu) + \epsilon_t, \qquad \epsilon_t \sim N(0,\Gamma). \tag{4.1}$$

Let w_t be a portfolio weight for the risky asset at the beginning of time $t+1$. Suppose that an investor is interested in the investment strategy from time 0 to time T. Then the terminal wealth is written as (2.1). Applying our method, the estimator $\widehat{W}_T$ can be obtained by

$$\widehat{W}_T = \prod_{t=0}^{T-1}\left(1+X_f+\widehat{w}_t\left(X_{t+1}-X_f\right)\right), \tag{4.2}$$

where $\widehat{w}_t$ is the estimator of optimal portfolio under the CRRA utility function defined by (2.5). In what follows, we examine the effect of $\widehat{W}_T$ for a variety of n (initial sample size), B (resampling size), A (AR parameter), Γ (variance of ϵ_t), γ (relative risk aversion parameter), and Ψ (defined by (3.7) or (3.8)).

Example 4.1 (myopic (single period) versus dynamic (multiPeriod)). Let $\mu = 0.02$, $A = 0.1$, $\Gamma = 0.05$, $n = 100$, $T = 10$, and $B = 100$. We generate the excess return process $\{X_t\}_{t=-n+1,\ldots,T}$

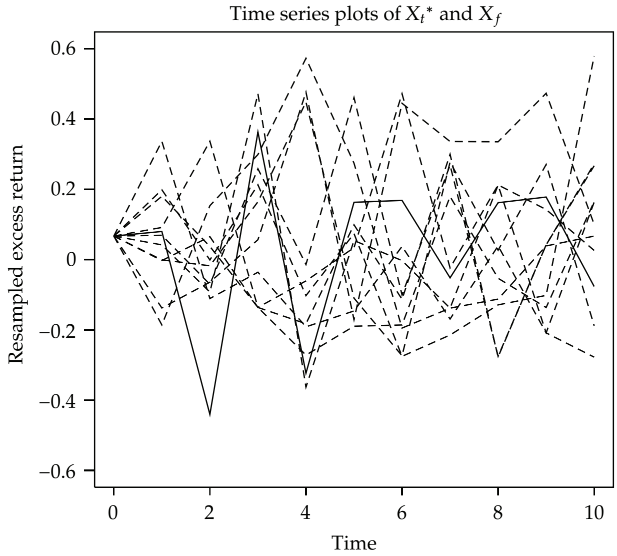

Figure 1: Resampled excess return.

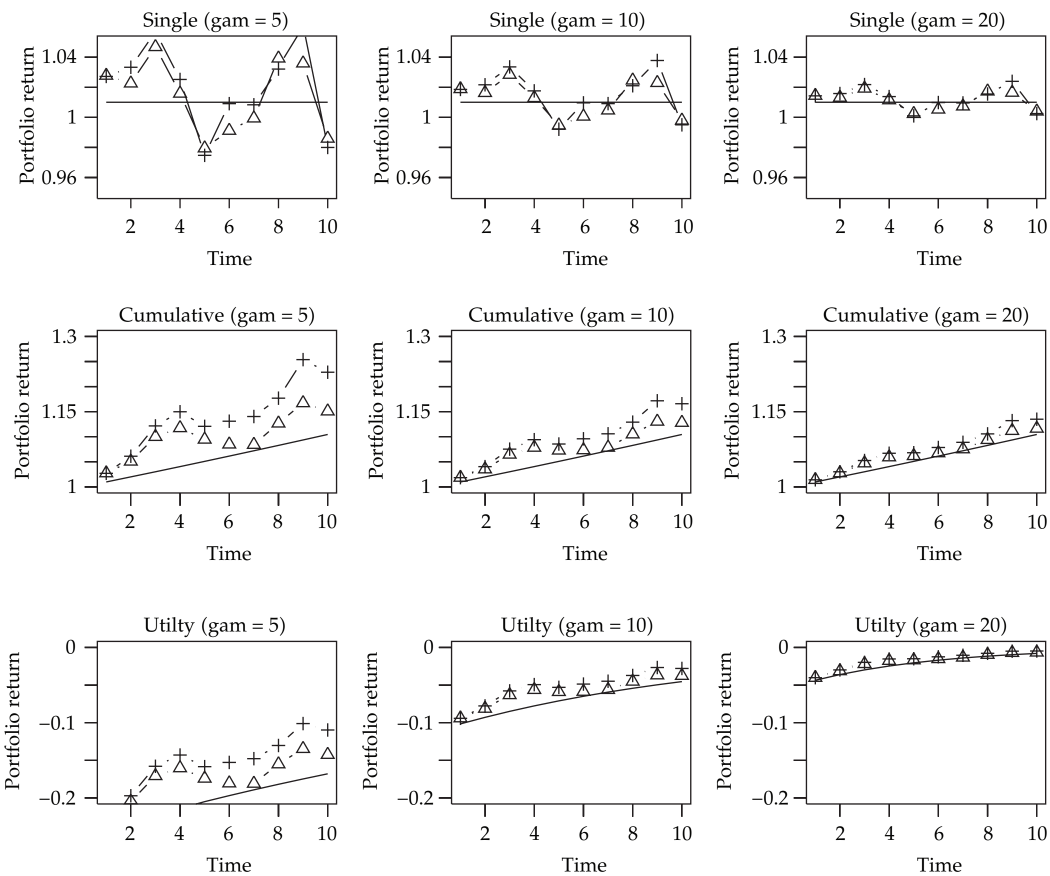

Figure 2: Myopic and dynamic portfolio return.

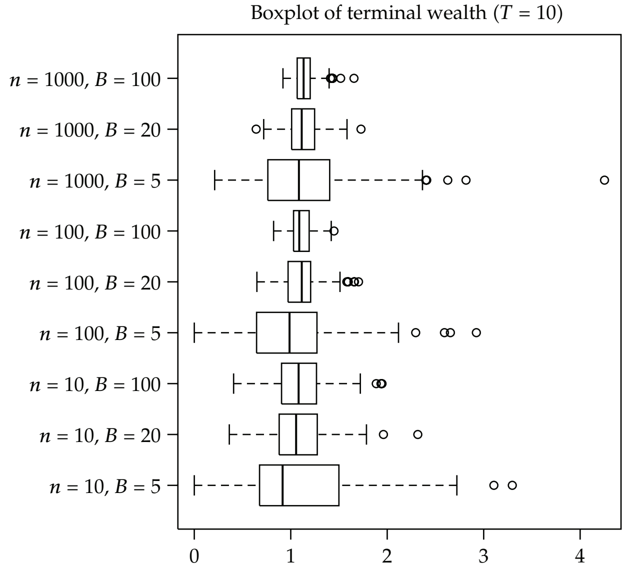

Figure 3: Boxplot1.

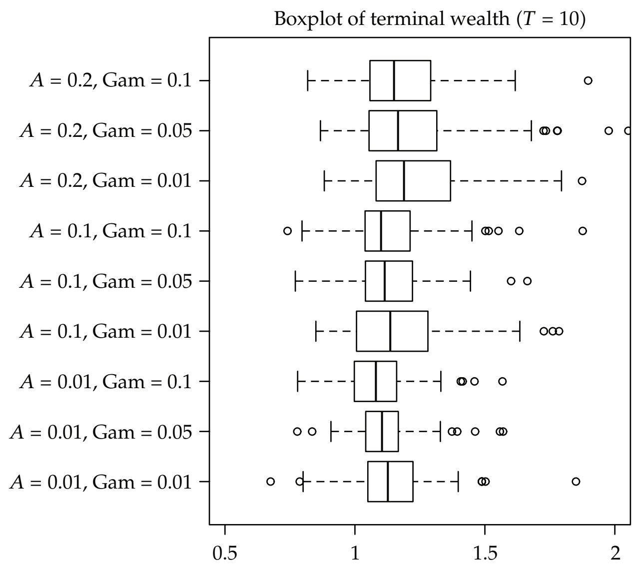

Figure 4: Boxplot2.

by (4.1). First, for each $t = 0, \dots, T-1$ we generate $\{X_s^{(b_1,b_2,t)*}\}_{b_1,b_2=1,\dots,B;s=t+1,\dots,T}$ by (3.4) based on $\{X_s\}_{s=-n+1}^{t}$ (as Step 1). We plot $\{X_t\}_{t=1,\dots,T}$ and $\{X_s^{(b_1,b_2,t)*}\}_{b=1,\dots,10;\ s=1,\dots,T}$ in Figure 1.

It can be seen that $X_s^{(b_1,b_2,t)*}$ show similar behavior with X_t.

Table 1: Dynamic portfolio returns for $\gamma = 5$.

	Myopic		Dynamic ($\Psi^{(1)}$)		Dynamic ($\Psi^{(2)}$)	
T	Mean	$(q_{0.25}, q_{0.5}, q_{0.75})$	Mean	$(q_{0.25}, q_{0.5}, q_{0.75})$	Mean	$(q_{0.25}, q_{0.5}, q_{0.75})$
A: Terminal wealth						
1	1.013564	(0.9924, 1.0091, 1.0192)	1.013667	(0.9920, 1.0096, 1.0176)	1.013814	(0.9920, 1.0096, 1.0176)
2	1.024329	(0.9917, 1.0192, 1.0445)	1.024396	(0.9923, 1.0177, 1.0436)	1.024667	(0.9924, 1.0183, 1.0437)
5	1.065896	(1.0021, 1.0504, 1.1125)	1.065988	(1.0000, 1.0509, 1.1115)	1.066355	(0.9999, 1.0505, 1.1106)
10	1.137727	(1.0273, 1.1062, 1.2024)	1.137707	(1.0264, 1.1041, 1.2005)	1.138207	(1.0265, 1.1043, 1.2002)
B: Utility of terminal wealth						
1	−0.24158	(−0.257, −0.241, −0.231)	−0.24139	(−0.258, −0.240, −0.233)	−0.24130	(−0.258, −0.240, −0.233)
2	−0.23609	(−0.258, −0.231, −0.210)	−0.23595	(−0.257, −0.233, −0.210)	−0.23578	(−0.257, −0.232, −0.210)
5	−0.21761	(−0.247, −0.205, −0.163)	−0.21761	(−0.249, −0.204, −0.163)	−0.21703	(−0.250, −0.205, −0.164)
10	−0.18349	(−0.224, −0.166, −0.119)	−0.18339	(−0.225, −0.168, −0.120)	−0.18287	(−0.225, −0.168, −0.120)

Table 2: Dynamic portfolio returns for $\gamma = 10$.

	Myopic		Dynamic ($\Psi^{(1)}$)		Dynamic ($\Psi^{(2)}$)	
T	Mean	$(q_{0.25}, q_{0.5}, q_{0.75})$	Mean	$(q_{0.25}, q_{0.5}, q_{0.75})$	Mean	$(q_{0.25}, q_{0.5}, q_{0.75})$
A: Terminal wealth						
1	1.011802	(1.0011, 1.0095, 1.0146)	1.011859	(1.0010, 1.0098, 1.0138)	1.011944	(1.0010, 1.0098, 1.0138)
2	1.022249	(1.0059, 1.0196, 1.0323)	1.022286	(1.0065, 1.0190, 1.0319)	1.022439	(1.0065, 1.0192, 1.0319)
5	1.058344	(1.0276, 1.0512, 1.0825)	1.058373	(1.0254, 1.0509, 1.0823)	1.058584	(1.0253, 1.0507, 1.0818)
10	1.120369	(1.0658, 1.1070, 1.1544)	1.120323	(1.0687, 1.1068, 1.1533)	1.120595	(1.0666, 1.1060, 1.1532)
B: Utility of terminal wealth						
1	−0.10224	(−0.109, −0.101, −0.097)	−0.10215	(−0.110, −0.101, −0.098)	−0.10210	(−0.110, −0.101, −0.098)
2	−0.09530	(−0.105, −0.093, −0.083)	−0.09523	(−0.104, −0.093, −0.083)	−0.09515	(−0.104, −0.093, −0.083)
5	−0.07581	(−0.086, −0.070, −0.054)	−0.07582	(−0.088, −0.071, −0.054)	−0.07557	(−0.088, −0.071, −0.054)
10	−0.05007	(−0.062, −0.044, −0.030)	−0.05003	(−0.061, −0.044, −0.030)	−0.04986	(−0.062, −0.044, −0.030)

Table 3: Dynamic portfolio returns for $\gamma = 20$.

	Myopic		Dynamic ($\Psi^{(1)}$)		Dynamic ($\Psi^{(2)}$)	
T	Mean	$(q_{0.25}, q_{0.5}, q_{0.75})$	Mean	$(q_{0.25}, q_{0.5}, q_{0.75})$	Mean	$(q_{0.25}, q_{0.5}, q_{0.75})$
A: Terminal wealth						
1	1.010905	(1.0055, 1.0097, 1.0123)	1.010934	(1.0055, 1.0099, 1.0119)	1.010979	(1.0054, 1.0099, 1.0119)
2	1.021181	(1.0131, 1.0198, 1.0262)	1.021200	(1.0133, 1.0195, 1.0260)	1.021281	(1.0133, 1.0196, 1.0260)
5	1.054646	(1.0396, 1.0512, 1.0668)	1.054655	(1.0381, 1.0509, 1.0668)	1.054767	(1.0381, 1.0508, 1.0667)
10	1.112289	(1.0853, 1.1062, 1.1297)	1.112256	(1.0876, 1.1060, 1.1291)	1.112396	(1.0865, 1.1056, 1.1290)
B: Utility of terminal wealth						
1	−0.04386	(−0.047, −0.043, −0.041)	−0.04382	(−0.047, −0.043, −0.042)	−0.04379	(−0.047, −0.043, −0.042)
2	−0.03705	(−0.041, −0.036, −0.032)	−0.03702	(−0.040, −0.036, −0.032)	−0.03699	(−0.040, −0.036, −0.032)
5	−0.02189	(−0.025, −0.020, −0.015)	−0.02189	(−0.025, −0.020, −0.015)	−0.02181	(−0.025, −0.020, −0.015)
10	−0.00881	(−0.011, −0.007, −0.005)	−0.00880	(−0.010, −0.007, −0.005)	−0.00876	(−0.010, −0.007, −0.005)

Next, we construct the optimal portfolio estimator $\widehat{w}_t^{(t)}$ along the lines with Steps 2–7. Here we apply the approximated solution for (3.5) or (3.11) following [5], that is,

$$\widehat{w}_s^{(b_0,t)} = \frac{1}{2E_s^*\left[D_{3,s+1}^{(b_0,b,t)*}\right]}\left\{E_s^*\left[D_{2,s+1}^{(b_0,b,t)*}\right] + 3\left(\widehat{\overline{w}}_s^{(b_0,t)}\right)^2 E_s^*\left[D_{4,s+1}^{(b_0,b,t)*}\right] + 4\left(\widehat{\overline{w}}_s^{(b_0,t)}\right)^3 E_s^*\left[D_{5,s+1}^{(b_0,b,t)*}\right]\right\}, \tag{4.3}$$

where

$$\begin{aligned}
D_{2,s+1}^{(b_0,b,t)*} &= \left(1+X_f\right)^{-\gamma}\left(X_{s+1}^{(b_1,b_2,t)*} - X_f\right)\Psi^{(i)}\left(X_{s+1}^{(b_1,b_2,t)*}, \widehat{\boldsymbol{\theta}}_{s+1}^{(i)}\right),\\
D_{3,s+1}^{(b_0,b,t)*} &= \frac{-\gamma}{2}\left(1+X_f\right)^{-1-\gamma}\left(X_{s+1}^{(b_1,b_2,t)*} - X_f\right)^2\Psi^{(i)}\left(X_{s+1}^{(b_1,b_2,t)*}, \widehat{\boldsymbol{\theta}}_{s+1}^{(i)}\right),\\
D_{4,s+1}^{(b_0,b,t)*} &= \frac{(-\gamma)(-1-\gamma)}{6}\left(1+X_f\right)^{-2-\gamma}\left(X_{s+1}^{(b_1,b_2,t)*} - X_f\right)^3\Psi^{(i)}\left(X_{s+1}^{(b_1,b_2,t)*}, \widehat{\boldsymbol{\theta}}_{s+1}^{(i)}\right),\\
D_{5,s+1}^{(b_0,b,t)*} &= \frac{(-\gamma)(-1-\gamma)(-2-\gamma)}{24}\left(1+X_f\right)^{-3-\gamma}\left(X_{s+1}^{(b_1,b_2,t)*} - X_f\right)^4\Psi^{(i)}\left(X_{s+1}^{(b_1,b_2,t)*}, \widehat{\boldsymbol{\theta}}_{s+1}^{(i)}\right),\\
\widehat{\overline{w}}_s^{(b_0,t)} &= -\frac{E_s^*\left[D_{2,s+1}^{(b_0,b,t)*}\right]}{2E_s^*\left[D_{3,s+1}^{(b_0,b,t)*}\right]}.
\end{aligned} \tag{4.4}$$

This approximate solution describes a fourth-order expansion of the value function around $1 + X_f$ ($\widehat{\widehat{w}}_s$ describes a second-order expansion). According to [5], a second-order expansion of the value function is sometimes not sufficiently accurate, but a fourth-order expansion includes adjustments for the skewness and kurtosis of returns and their effects on the utility of the investor.

Figure 2 shows time series plots for single portfolio return ($=1+X_f+\widehat{w}_t(X_{t+1}-X_f)$, Line 1), cumulative portfolio return ($=\widehat{W}_T$, Line 2), and value of utility function ($=1/(1-g)\widehat{W}_T^{1-g}$, Line 3) for $\gamma = 5, 10$ and 20. The solid line shows the investment only for risk-free asset (i.e., $\widehat{w}_t = 0$), the dotted line with $\triangle$ shows myopic (single period) portfolio (i.e., $\Psi^{(i)} = 1$) and the dotted line with + shows dynamic (multiperiod) portfolio by using $\Psi^{(1)}$.

Regarding the single-portfolio return, we can not argue the best investment strategy among the risk-free, the myopic portfolio and the dynamic portfolio investment. However, to look at the cumulative portfolio return or the value of utility function, it is obviously that the dynamic portfolio investment is the best one. The difference between the myopic and dynamic portfolio is due to Ψ and is called "hedging demands" because by deviating from the single period portfolio choice, the investor tries to hedge against changes in the investment opportunities. In view of the effect of γ, we can see that the magnitude of the hedging demands decreases with increased amount of γ.

Next, we repeat the above algorithm 100 times using the different generated data. Tables 1, 2, and 3 show means, 25 percentiles ($q_{0.25}$), medians ($q_{0.5}$), and 75 percentiles ($q_{0.75}$) of terminal wealth ($\widehat{W}_T$) and the values of utility function ($1/(1-g)\widehat{W}_T^{1-g}$) for $T = 1, 2, 5, 10$, and $\gamma = 5, 10, 20$.

We can see that for all T, the means of terminal wealth $\widehat{W}_T$ are larger than that of risk-free investment (i.e., $(1+X_f)^T$). In view of the distribution of $\widehat{W}_T$, the means are larger than the medians ($q_{0.5}$) which shows the asymmetry of the distribution. Among the myopic, dynamic portfolio using $\Psi^{(1)}$ and $\Psi^{(2)}$, dynamic portfolio using $\Psi^{(2)}$ is the best investment strategy in view of the means of $\widehat{W}_T$ or $1/(1-g)\widehat{W}_T^{1-g}$. There are some cases that the means of $\widehat{W}_T$ for dynamic portfolio using $\Psi^{(1)}$ are smaller than those for myopic portfolio. This phenomenon would show the inaccuracy of the approximation of Ψ. In addition, in view of the dispersion of $\widehat{W}_T$, the dynamic portfolio's one is relatively smaller than the myopic portfolio's one.

Example 4.2 ((sample size (n) and resampling size (B)). In this example, we examine effect of the initial sample size (n) and the resample size (B). Let $\mu = 0.02$, $A = 0.1$, $\Gamma = 0.05$, $T = 10$, and $\gamma = 5$. In the same manner as Example 4.1, we consider the effect of $\widehat{W}_T$ for $n = 10, 100, 1000$ and $B = 5, 20, 100$. Figure 3 shows the box plots of the terminal wealth $\widehat{W}_T$ for each n and B.

It can be seen that the medians tend to increase with increased amount of n and B. In addition, the wideness of the box plots decreases with increased amount of n and B. This phenomenon shows the accuracy of the approximation of X_t^*.

Example 4.3 (AR Parameter (A) and variance of $\in_t$ (Γ)). In this example, we examine effect of the AR parameter (A) and the variance of ϵ_t (Γ). Let $\mu = 0.02$, $n = 100$, $B = 100$, $T = 10$, and $\gamma = 5$. In the same manner as Example 4.1, we consider the effect of $\widehat{W}_T$ for $A = 0.01, 0.1, 0.2$, and $\Gamma = 0.01, 0.05, 0.10$. Figure 4 shows the box plots of the terminal wealth $\widehat{W}_T$ for each A and Γ.

Obviously, the medians increase with decreased amount of Γ which shows that the investment result is preferred when the amount of ϵ_t is small. On the other hand, the wideness of the box plots increases with increased amount of A which shows that the difference of the investment result is wide when the amount of A is large.

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