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Research Article

Restricted Coherent Risk Measures and Actuarial Solvency

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We prove a general dual representation form for restricted coherent risk measures, and we apply it to a minimization problem of the required solvency capital for an insurance company.

1. Overview of the Paper

The aim of this paper is to propose a min-max scheme for the specification of the *solvency capital* for an insurance company. We actually suppose that the initial surplus of the company is equal to u. According to the evolution of the premiums and the investment payoffs of the company both with the evolution of the cumulative claim obligations till a certain time-period T, the final surplus U_T of the company is formulated by the corresponding stochastic process $\{U_t, t \in [0,T]\}$. In order to anticipate the probable risk which arises from the variability of U_T at the different states of the world, the insurance company needs a capital being expressed either in units of money or in an amount of some shares of a numeraire asset. In order to measure this risk, we use a risk measure, which is in general one of the form $\rho : \mathcal{U} \to \mathbb{R} \cup \{\infty\}$, it is more rational to define the risk measure on the set $\mathcal{U}$, being the set of financial positions which represent the final surplus variables U_T for the company, than defining the risk measure on the entire space L (which is in general an ordered normed linear space), where these financial positions lie in. Also, the ∞ value is in general excluded from the duality representations we deduce, hence we refer to risk measures $\rho : \mathcal{U} \to \mathbb{R}$. Classical properties of coherent and convex risk measures, like *translation invariance*, are not transferred in the exact form they have in articles of unrestricted risk measures. Relative definitions are given in the following sections. In the present paper, we deduce a dual representation result for such restricted coherent risk measures and we also propose a min-max method of solution to the above solvency capital optimization problem, which is rightly stated if we suppose that the capital constraint $\rho(U_T) \leq u$ holds, where u represents the initial capital of the insurance company. We use the dual representation of the corresponding risk measure which provides inf-compactness in this optimization problem. This is the main point of the paper. The proof of a dual representation theorem for restricted coherent risk measures which are defined on wedges of reflexive spaces is given. The domain of the risk measure we desire to minimize is a subspace $\mathcal{U}$ of random variables, as we prove in the following. The specification of the surplus position U_T which minimizes a coherent risk measure $\rho : \mathcal{U} \to \mathbb{R}$ of this kind, under the constraint $\rho(U) \leq u$, $U \in \mathcal{U}$ implies a threshold of solvency capital needed by this way for any feasible (in the sense that $\rho(U) \leq u$ holds) surplus position. Namely, the optimal value of the problem is the part of the initial capital u that the insurance company expends in order to anticipate the risk, no matter what will happen (and which will be the final surplus $U_T = U$ of the company). The existence of a minimum point is also deduced. The calculation of the solvency capital—or else of this threshold—indicates a decision criterion for the insurance company, because its optimal value is going to be spent by the company. The sole calculation of the risk for any surplus variable $U \in \mathcal{U}$ with respect to $\rho : \mathcal{U} \to \mathbb{R}$ does not involve any decision criterion. Hence a decision criterion for the solvency of the insurance company described by the model below may be indicated by the solution of the minimization of the risk subject to the constraint $\rho(U) \leq u, U \in \mathcal{U}$.

2. Previous Results and Essential Notions

First, we remind two general definitions in which A is a wedge of the space L.

Definition 2.1. A real-valued function $\rho : L \to \mathbb{R} \cup \{+\infty\}$ which satisfies the properties

(i) $\rho(x + ae) = \rho(x) - a$ (Translation Invariance),

(ii) $\rho(\lambda x + (1 - \lambda)x) \leq \lambda\rho(x) + (1 - \lambda)\rho(y)$ for any $\lambda \in [0, 1]$ (Convexity) and,

(iii) $y \geq_A x$ implies $\rho(y) \leq \rho(x)$ (A-Monotonicity),

where $x, y \in L$ is called (A, e)-convex risk measure.

Definition 2.2. A real-valued function $\rho : \mathrm{L} \to \mathbb{R} \cup \{+\infty\}$ is a (A, e)-coherent risk measure if it is an (A, e)-convex risk measure and it satisfies the following property: $\rho(\lambda x) = \lambda\rho(x)$ for any $x \in E$ and any $\lambda \in \mathbb{R}_+$ (Positive Homogeneity).

In [1, Theorem 2.12], the coherent or convex (see the Definition 5.8) risk measures which are defined on a domain which is a proper subset of L^p, $1 \leq p \leq \infty$ are called *restricted*. We have to remind that in [1], the restricted convex risk measures are studied on *cash-invariant* subsets of L^p-spaces, where $1 \leq p \leq \infty$. A dual representation theorem is proved there [1, Theorem 2.12] for $(L^p_+, \mathbf{1})$-convex risk measures. A primal reference for the study of restricted convex risk measures is [2], in which such risk measures are studied in L^∞ and mainly by means of theory of conjugate convex functions defined on normed linear spaces. A similarity between the frame of [2] and the frame we propose is that authors consider a convex cone $\mathcal{P}$ which implies a partial ordering for a locally convex topological vector space L and a numeraire $\Pi \in L \setminus \{0\}$. The examples of [2] focus on normed linear spaces and moreover on L^p spaces ordered by their usual (componentwise) partial ordering. In [2, Lemmas 3.5 and 3.6], the conditions are directly related to the boundedness of the base that the numeraire asset, (see also [3, Theorem 4.4.4]). It is valid only in the case where $p = \infty$, because $\mathbf{1}$ is an interior point of L^∞_+. In the present paper, we are going to define *restricted* *risk measures* on wedges of ordered normed linear spaces containing the numeraire asset E, which may be proper subspaces of reflexive spaces-like the spaces $L^p, 1 < p < \infty$. We define the restricted measures on such spaces, because this arises from our financial model (Itô processes correspond to L^2 spaces according to Itô Isometry). As it is well known the $L^p(\Omega, \mathcal{F}, \mu)$-spaces are associated to the existence of the moments $\mathbb{E}_\mu(X^p)$ for $1 \leq p < \infty$, where $X : \Omega \to \mathbb{R}$ is a $\mathcal{F}$-measurable random variable. If $X \in L^\infty$ all these moments are real numbers. Of course, the model of risk measurement would be set in a different mode in which the insurance company examines the effects of its decisions continuously, and it calculates the equivalent solvency capital according to the evolution of its investment portfolio, by using dynamic risk measures (see, e.g., [4–6]). But the intention of the present paper is to provide some results on static, *restricted coherent* risk measures and emphasizing in the application we discussed above. However, as it also quoted in [7] the time moments 0 and T cannot be very far, because the control of the decision effects of an insurer, even if it is set in a static place, cannot refers to a time interval which is great enough. Moreover in practice, the insurance companies do not pay their clients as far as the claims arrive, but after a certain time in which they can anticipate the total risk that they adopted by those claims partially by their investment portfolio and partially by the solvency capital they shaped. However, $\rho(U_T)$ is indicated as the *solvency capital* under ρ is indicated also for example in [8, page 3]. For more details about an actuarial definition of Solvency Capital Requirement functionals, we refer to [9]. If X represents the time-T liabilities of an insurance company and $K(X)$ is the *economic capital* associated with these liabilities, while $P(X)$ is the *value* of them calculated either by a quantile method, or by an additional margin method, or by a replicating portfolio method, then if the risk measure used is ρ, the *solvency capital* for X is equal to $\rho(X)$. The functionals K, P are connected to ρ by the identity $\rho(X) = K(X) + P(X)$ for any X which is a liability variable. This notion of the solvency capital partially excuses a static frame than a dynamic one, but a dynamic one is however accurate, too. We may say that the frame of the present paper is exactly the one of [7] expressed in the case of restricted coherent risk measures. However, the presentation of the results is autonomous since it is related to the solution of the problem (7.1). This problem is related to the determination of the solvency capital for a set of financial positions (which are liability variables of an insurance company) which are not consisted of a sole element. The minimization of a coherent risk measure over a set of positions as a problem is however studied for example in [10, 11], mainly for CVaR. The Duality Representation Theorem for restricted coherent risk measures on wedges (see Theorem 6.6) also relies both on the strong separation theorem for convex sets in locally convex spaces and on the Hahn-Banach extension theorem for linear functionals, by which we reach the primal [12, Theorem 2.3], which refers to the case of coherent risk measures defined on $L^\infty(\Omega, \mathcal{F}, \mu)$. In this case, L^∞ is ordered by its usual partial ordering and the numeraire asset is $\mathbf{1}$, which is an interior point of L^∞_+. The important is that without the assumption of the existence of interior points in the positive cone L_+ which was provided from the idea of the application of the Krein-Rutman Theorem [3, Corollary 1.6.2] and the extension result [3, Proposition 3.1.8]. This is a point of similarity to the equivalent result in [7]. This is the cause of presence of the classes of cones in Examples 3.1 and 3.2 in this paper.

3. A Financial Model for the Surplus Evolution of an Insurance Company

The model we consider is described as follows. We assume a closed interval $[0, T]$ for some constant $T > 0$ representing the time-horizon in our model and an infinite set of states of the world Ω. We suppose that all the transaction decisions take place at time 0. The state $\omega \in \Omega$ faced by the investors is contained in some event $A \in \mathcal{F}$, where $\mathcal{F}$ represents some σ-algebra of subsets of Ω which provides information about the states being available at time T. We denoted by μ the objective probability measure on the measurable space $(\Omega, \mathcal{F})$ and we consider the probability space $(\Omega, \mathcal{F}, \mu)$. The information evolution along time-horizon $[0, T]$ is represented by a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$ of $(\Omega, \mathcal{F}, \mu)$.

We consider a diffusion model for the capital of an insurance company. We assume an insurance company, whose $\mathbb{F}$-adapted capital process $(C_t)_{t\in[0,T]}$ evolves according to the following stochastic differential equation;

$$dC_t = (m(t, C_t) - \phi_t r_t B_t)dt + \theta_t s(t, C_t)dW_t^{(1)}, \quad \mu\text{—}a.e., \tag{3.1}$$

where $C_0 = c$ is some constant which denotes the initial capital of it and $W^{(1)}$ is one-dimensional Brownian motion with respect to $\mathbb{F}$. The form of this stochastic differential equation is explained as follows: during an infinitesimal time, the company receives premiums equal to $m(t, C_t)$. The company borrows money from a "bank account asset," in order to invest to a risky asset, whose volatility process is $s(t, C_t)$, $t \in [0, T]$. The process $p_t = \int_0^t m(s, C_s)ds$ may be interpreted as the *total premium process*, or else it indicates the total premium payments that the insurance company receives till time-period t. The process $\{m(t, C_t),\ t \in [0, T]\}$ is the *premium payment intensity* and we may suppose that the pure diffusion term $\int_0^t \theta_u s(u, C_u)dW_u^{(1)}$ corresponds to the payoff of the investment of these received premia to some risky assets whose volatility-term is $s(t, C_t)$, $t \in [0, T]$ with respect to a one-dimensional Brownian motion $W^{(1)}$. The premia are invested to this asset as soon as they are received by the company. $B_t, t \in [0, T]$ denotes the stochastic process which represents the evolution of the "bank account" asset. Hence, we have $dB_t = r_t B_t dt$ and $B_0 = 1$. We may suppose that the interest-rate process $(r_t)_{t\in[0,T]}$ of the asset which represents a bank account is uniformly bounded, or else that $r_t(\omega) \le M$ for any $(t, \omega) \in [0, T] \times \Omega$. It is usual in the recent literature, the capital process $(C_t)_{t\in[0,T]}$ to have a diffusion form (see, e.g., [13, 14]). The set of admissible trading strategies for the investment of the capital of the insurance company to the risky asset is denoted by $\Theta = \{(\theta_t)_{t\in[0,T]}\}$. These processes are adapted to the filtration $\mathbb{F}$ of $(\Omega, \mathcal{F}, \mu)$. Θ includes those processes for which $\theta_t(\omega) = a_t$, $a_t \in \mathbb{R}$, $t \in [0, T]$ for any $\omega \in \Omega$. Φ is a set of admissible trading strategies for the bank account asset, while Θ and Φ have the same properties. The integrability properties that the investment strategies $\phi \in \Phi$, $\theta \in \Theta$ have to satisfy are the following:

$$\begin{gathered}\int_0^T |\theta_t r_t| dt < \infty, \quad \mu\text{—a.e.,} \\ \int_0^T |\phi_t r_t| dt < \infty, \quad \mu\text{—a.e.,} \\ \int_0^T \theta_t^2 s^2(t, C_t)dt < \infty, \quad \mu\text{—a.e.}\end{gathered} \tag{3.2}$$

These integrability conditions are implied from [15, Definition 2.1, page 7]. We also may suppose that for any $t \in [0, T]$ the random variables θ_t, ϕ_t are not necessarily stochastically independent, since the insurance company may borrow an amount of money during an infinitesimal time and this money can be invested for buying an amount of shares of the risky asset. The model presented here is an extension of the model described in [14]. Moreover, the claim payment process may be represented by an Itô process

$$dZ_t = q(t, Z_t)dt + \sigma(t, Z_t)dW_t^{(2)}, \quad \mu\text{—a.e.}, \tag{3.3}$$

where $Z_0 = z$ is the initial total claim and $W^{(2)}$ is a one-dimensional Brownian motion with respect to the same filtration $\mathbb{F}$, being independent from $W^{(1)}$. Hence, the stochastic process $\{U_t, t \in [0, T]\}$ of the surplus of the insurance company is an Itô process (by consideration of the Itô Lemma) and its dynamics are demonstrated in the form of a stochastic differential equation:

$$dU_t = \left(m(t, U_t + Z_t) - \phi_t B_t r_t - q(t, Z_t)\right)dt + [\theta_t s(t, U_t + Z_t), -\sigma(t, Z_t)] \cdot \begin{bmatrix} dW_t^{(1)} \\ dW_t^{(2)} \end{bmatrix}, \quad \mu\text{—a.e.}, \tag{3.4}$$

where $U_0 = u = c - z$. With notation of $dW_t = \begin{bmatrix} dW_t^{(1)} \\ dW_t^{(2)} \end{bmatrix}$ for the stochastic differential of the 2-dimensional Brownian motion W ($W^{(1)}, W^{(2)}$ are independent) and $l(t, U_t, Z_t, \phi_t) = m(t, U_t + Z_t) + \phi_t B_t r_t - q(t, Z_t)$, $t \in [0, T]$, while $v(t, U_t, Z_t, \theta_t) = [\theta_t s(t, U_t + Z_t), -\sigma(t, Z_t)]$, $t \in [0, T]$, the stochastic differential equation for $\{U_t, \ t \in [0, T]\}$ is obtained in the form

$$dU_t = l(t, U_t, Z_t, \phi_t)dt + v(t, U_t, Z_t, \theta_t)dW_t, \quad \mu\text{—a.e.}, \tag{3.5}$$

where $U_0 = u$. For the risk model we propose, we may refer to [16] in which both the capital of the insurance company and the total liabilities of the company are modelled as simple diffusion processes. The stochastic process $W = (W^{(1)}, W^{(2)})$ is a 2-dimensional Brownian motion with respect to the filtration $\mathbb{F}^W$ generated by it, that means $\mathbb{F}^W \subseteq \mathbb{F}$. Hence, the surplus process can be considered as a financial asset whose drift is $l(t, U_t, Z_t, \phi_t)$ and whose volatility process with respect to the 2-dimensional Brownian motion W is $v(t, U_t, Z_t, \theta_t)$, $t \in [0, T]$, with realizations as vectors of $\mathbb{R}^2$. It is not restrictive to require

$$v_t(U_t(\omega), Z_t(\omega), \theta_t(\omega)) \neq 0, \quad \lambda_{[0,T]} \otimes \mu\text{—a.e.}, \tag{3.6}$$

where $\lambda_{[0,T]}$ denotes the Lebesgue measure on $[0, T]$, for each portfolio strategy $(\theta_t)_{t \in [0,T]}$. It implies that

$$\operatorname{rank} v_t(U_t(\omega), Z_t(\omega), \theta_t(\omega)) = 1, \quad \lambda_{[0,T]} \otimes \mu\text{—a.e.} \tag{3.7}$$

This holds when for example

$$\sigma(t, Z_t(\omega)) \neq 0, \quad \lambda_{[0,T]} \otimes \mu\text{—a.e.} \tag{3.8}$$

If we consider the financial market consisted by the surplus asset and the "bank account" asset whose dynamics are driven by the stochastic differential equation

$$dB_t = r_t B_t dt, \qquad B_0 = 1, \quad \mu\text{—a.e.}, \tag{3.9}$$

where r denotes the short-term rate process, to conclude (see [17, Theorems 12.1.8, and 12.2.5]) that the market is incomplete. This criterion of incompleteness was established in [15, Theorem 6.6, page 24].

The attainable final surplus variables are in this case a subspace denoted by $\mathcal{U}$. This set is actually

$$\mathcal{U} = \left\{ U \in L^2(\Omega, \mathcal{F}, \mu) \mid U = u + \int_0^T l(t, U_t, Z_t, \phi_t)dt + \int_0^T v(t, U_t, Z_t, \theta_t)dW_t \right\}. \tag{3.10}$$

The fact that $\mathcal{U}$ is a subspace under the strategies is deduced as follows: if

$$U_1 = u + \int_0^T l\left(t, U_t, Z_t, \phi_t^1\right)dt + \int_0^T v\left(t, U_t, Z_t, \theta_t^1\right)dW_t, \tag{3.11}$$

for some corresponding strategies ϕ^1, θ^1 and also

$$U_2 = u + \int_0^T l\left(t, U_t, Z_t, \phi_t^2\right)dt + \int_0^T v\left(t, U_t, Z_t, \theta_t^2\right)dW_t, \tag{3.12}$$

for some corresponding strategies ϕ^2, θ^2, then

$$\begin{aligned}
\lambda_1 \cdot U_1 + \lambda_2 \cdot U_2 &= u + \int_0^T \left(m(t, U_t + Z_t) - \left(\lambda_1 \cdot \phi_t^1 + \lambda_2 \phi_t^2\right) B_t r_t - q(t, Z_t)\right)dt \\
&\quad + \int_0^T \left[\left(\lambda_1 \cdot \theta_t^1 + \lambda_2 \theta_t^2\right) s(t, U_t + Z_t), -\sigma(t, Z_t)\right] \cdot \begin{bmatrix} dW_t^{(1)} \\ dW_t^{(2)} \end{bmatrix} \\
&= u + \int_0^T l\left(t, U_t, Z_t, \lambda_1 \phi_t^1 + \lambda_2 \phi_t^2\right)dt \\
&\quad + \int_0^T v\left(t, U_t, Z_t, \left(\lambda_1 \theta_t^1 + \lambda_2 \theta_t^2\right)\right)dW_t \in \mathcal{U},
\end{aligned} \tag{3.13}$$

where $\lambda_1, \lambda_2 \in \mathbb{R}$.

The set of the attainable surplus variables $\mathcal{U}$ of the specific market is not equal to $L = L^2(\Omega, \mathcal{F}_T^W, Q)$ for any Equivalent Martingale Measure Q of it. Hence we consider as the space in which the positions $U = U_T$ lie in, any of the spaces $L^2(\Omega, \mathcal{F}_T^W, Q)$ for any Equivalent Martingale Measure Q of the market consisted by the surplus asset and the bank account asset.

Every such $Q_{\theta,\phi}$ is defined as follows: The market prices of risk for the surplus-bond market are given by the following process equation for u:

$$[\theta_t s(t, U_t + Z_t), -\sigma(t, Z_t)] \cdot u_t = m(t, U_t + Z_t) + \phi_t B_t r_t - q(t, Z_t) - r_t. \tag{3.14}$$

The equivalent Radon-Nikodym derivative of Q_u with respect to μ is

$$\frac{dQ_u}{d\mu} = \exp\left\{ -\int_0^T u_t dW_t - \frac{1}{2}\int_0^T \|u_t\|_2^2 dt \right\}. \tag{3.15}$$

The spaces $L^2(\Omega, \mathcal{F}_T^W, Q_u)$ are reflexive; therefore, we can refer to the same frame of ordered normed linear spaces which appears in [7], where the main results are proved under the assumptions that L is an ordered linear space with nonempty cone interior. As in [7], the order structure of L is given by a closed wedge L_+ which is not necessarily the positive cone L_+^p of the pointwise partial ordering if $L = L^p$. It is in general a wedge whose elements may be financial positions whose outcomes are negative at some of the states of the world. The geometry of the positive wedge reflects the joint beliefs of the investors about the payoffs of these positions. The beliefs of the investors determine the set of the positions which are "jointly considered to be the nice investments." Actually, we mean that the positive cone L_+ is itself an acceptance set in the sense of the definition contained in [18]. Moreover, we mention two families of cones with nonempty interior, since the results of the present paper join this frame. An element of these families may represent the positive cone L_+ of L.

Example 3.1. A family of cones in normed linear spaces having nonempty cone-interior are the Bishop-Phelps cones (see in [7]). The family of these cones in a normed linear space L is the following:

$$K(f, a) = \{x \in L \mid \langle x, f\rangle \geq a\|x\|\}, \quad f \in L^*, \; \|f\| = 1, \; a \in (0, 1). \tag{3.16}$$

An explanation for the existence of interior points in these cones is found in [3, page 127].

Example 3.2. Another family of cones with nonempty interior is the family of Henig Dilating cones. These cones are defined as follows: consider a closed, well-based cone C in the normed linear space L, which has a base B, such that $0 \notin \overline{B + \delta B(0,1)}$. Let $\delta \in (0, 1)$ be such that

$$2\delta B(0,1) \cap B = \emptyset, \tag{3.17}$$

where $B(0,1)$ denotes the closed unit ball in L. If

$$K_n = \overline{\text{cone}}\left(B + \frac{\delta}{j} B(0,1)\right), \quad j \in \mathbb{N}, \tag{3.18}$$

then $C \subseteq K_{j+1} \subseteq K_j$, $j \in \mathbb{N}$, K_j is a cone for any $j \geq 2$, $C \setminus \{0\} \subseteq \text{int}(K_j)$, $j \geq 1$. About these cones, see for example [19, Lemma 2.1]. For example, a Bishop-Phelps cone $C = K(f, a)$ in a reflexive space which is a well-based cone as the construction of the K_n, $n \in \mathbb{N}$ requires, provides a set of interior points $C \setminus \{0\}$ of the cone $K_j,\ j \geq 1$. If we consider the base $B_f = \{x \in C \mid \langle x, f\rangle = 1\}$ defined by f, this base is a closed set where $0 \notin B_f$. Hence there is a $g \neq 0,\ g \in L^*$ such that $g(y) \geq \delta' > 0$ for any $y \in B_f$. g can be selected to be such that $\|g\| = 1$, hence $\|y\| \geq g(y) \geq \delta' > 0$. By setting $\delta' = 2\delta$, we may construct a sequence of approximating cones $K_n, n \in \mathbb{N}$, since we can set $g = f, \delta' = a \in (0,1)$. We remind that if D is a convex set, then the set $\text{cone}(D) = \{x \in L \mid x = \lambda d,\ d \in D,\ \lambda \in \mathbb{R}_+\}$ is a wedge and by $\overline{\text{cone}}(D)$, we denote its norm (or weak) closure.

We remind that in general, the *acceptance set* $\mathcal{A}_\rho$ of a risk measure $\rho : L \to \mathbb{R} \cup \{+\infty\}$ (where the infinity value may be omitted) is the set $\mathcal{A}_\rho = \{X \in L \mid \rho(X) \leq 0\}$. If the risk measure is restricted, namely it is of the form $\rho_1 : \mathcal{U} \to \mathbb{R} \cup \{+\infty\}$, where $\mathcal{U} \subseteq L$, the acceptance set of ρ_1 is $\mathcal{A}_{\rho_1} = \{X \in \mathcal{U} \mid \rho_1(X) \leq 0\}$.

4. Some Quotes on the Content of the Paper

The insurance company determines a numeraire asset E which corresponds to an insurable claim and belongs to an acceptance set $\mathcal{A}$. This set $\mathcal{A}$, namely demonstrates the exposure limits of risk for this insurance company. If $U \in \mathcal{A}$, then the company can anticipate the risk arising from the variability of U among the states of Ω. $\mathcal{A}$ is a wedge of L, because it should satisfy the main properties of the acceptance sets of the coherent risk measures introduced in [18]:

(i) $\mathcal{A} + \mathcal{A} \subseteq \mathcal{A}$.

(ii) $\lambda\mathcal{A} \subseteq \mathcal{A},\ \lambda \in \mathbb{R}_+$.

We remind that a set $C \subseteq L$ satisfying $C + C \subseteq C$ and $\lambda C \subseteq C$ for any $\lambda \in \mathbb{R}_+$ is called *wedge*. A wedge for which $C \cap (-C) = \{0\}$ is called *cone*. The coherence of the acceptance set $\mathcal{A}$ is related to the property of Monotonicity. In [18, Ax. 2.1], a coherent acceptance set is supposed to contain the cone L_+ of the space in which the financial positions lie in. In the case where the risk measure is defined in the whole subspace $\mathcal{U}$, this property can be expressed in the following way:

$$\mathcal{U} \cap L_+ \subseteq \mathcal{A}, \tag{4.1}$$

which indicates that the cone of the induced partial ordering of the subspace $\mathcal{U}$ of L is contained in the acceptance set. The acceptance set $\mathcal{A}$ is not necessarily a subset of $\mathcal{U}$ because the risk exposure limits are rather independent from the specific surplus process $(U_t,\ t \in [0,T])$. Also, if we talk about convex risk measures which are not coherent $\mathcal{A}$ is an unbounded convex subset of L containing L_+.

If D is a subspace of L, this subspace becomes a partially ordered linear space itself by using the order structure of $(L, \geq)$. In this case, we call D an *ordered subspace* of $(L, \geq)$ whose *positive wedge* is $D \cap L_+$. For the rest notions related to the partially ordered linear spaces, the reader may append to the Appendix of [7] or to the brief Appendix at the end of this article.

If $\rho : \mathcal{U} \to \mathbb{R}$, then the value $\rho(U),\ U \in \mathcal{U}$ can be understood as the *solvency capital* for the financial position U under ρ or else as the minimum amount of shares of E that allows U jointly with these shares of E to belong to the acceptance set of ρ,

$$\mathcal{A}_\rho = \{U \in \mathcal{U} \mid \rho(U) \leq 0\}. \tag{4.2}$$

On the other hand, the *solvency position* that the insurance company needs in order to be insured in this case is $\rho(U)E$. Since the numeraire asset E may be normalized according to a price-functional $\pi \in L^*$, namely a strictly positive functional of L_+ such that $\langle E, \pi \rangle = 1$, then the solvency capital of E with respect to ρ is equal to -1. Also note that if the space L is reflexive, then the numeraire asset E defines a *base* on the cone L_+^0, because $L_+^{00} = (L_+^0)^0 = L_+$. This base is weakly compact due to the fact that E is an interior point of L_+ and also due to [3, Theorem 3.8.6(i)] which indicates that *if a wedge P of a locally convex space L has an interior point, then P^0 has a $\sigma(L^*, L)$-compact base*, which is actually the base defined by this interior point. This is due to [3, Theorem 3.8.12] which indicates that If L is an ordered normed linear space with positive wedge P, then a linear functional f of L is uniformly monotonic if and only if it is an interior point of P^0 with respect to the norm topology of L^*. We may come to the same conclusion by using [20, Theorem 4]. About the notation—even it is explained in the Appendix—we have that $L_+^0 = \{\pi \in L^* \mid \langle X, \pi \rangle \geq 0, \ \forall X \in L_+\}$. This is the reason of considering families of cones with nonempty interior. They actually provide the conditions for the solution of the optimization problem (7.1), which is mentioned below.

5. Restricted Coherent and Restricted Convex Risk Measures

Restrictions in cash invariance or actually in numeraire invariance may appear in the attempt of the insurance company to draw the limits of its risk exposure. The sorts of invariance are described in the following definitions.

Definition 5.1. The subset $N \subseteq L$ is *E- translation invariant (cash translation invariant)* if $E \in N$ ($\mathbf{1} \in N$) and for any $U \in N$ and any $\lambda \in \mathbb{R}$, $U + \lambda E \in N$ ($U + \lambda \mathbf{1} \in N$).

Definition 5.2. The subset $N \subseteq L$ is *weak E- translation invariant (weak cash translation invariant)* if there is some $t \in \mathbb{R}$ such that for any $U \in N$, $U + \lambda E \in N$ ($U + \lambda \mathbf{1} \in N$), where $\lambda \geq t$.

Lemma 5.3. *A wedge N of L is weak E-translation invariant with respect to any element E of it.*

Proof. If $U \in N$, then for any $t \geq 0$, $U + tE \in N$. Hence N is weak E-translation invariant. □

The definition of translation invariance for a risk measure defined on a wedge N takes a more general form.

Definition 5.4. Let N be a wedge of L and ρ be a function $\rho : N \to \mathbb{R}$. ρ is E-translation invariant if

$$\rho(U + aE) = \rho(U) - a, \tag{5.1}$$

for any $U \in N$ and any $a \in \mathbb{R}$ such that $U + aE \in N$.

Definition 5.5. Let N be a wedge of L, ρ be a function $\rho : N \to \mathbb{R}$, and L_+ be a cone of L. Then ρ is L_+-monotone (or equivalently it satisfies the L_+-monotonicity), if the inclusion

$$N \cap L_+ \subseteq \mathcal{A}_\rho, \tag{5.2}$$

holds.

Definition 5.6. Let N be a wedge of L and the function $\rho : N \to \mathbb{R}$ satisfy the properties of (L_+, E)-Translation Invariance, Subadditivity, Positive Homogeneity and L_+-Monotonicity. Then we call it (L_+, E)-*restricted coherent*.

Definition 5.7. A risk measure $\rho : N \to \mathbb{R}$ satisfies the Convexity property if it is a convex function

$$\rho(\lambda U_1 + (1-\lambda)U_2) \leq \lambda\rho(U_1) + (1-\lambda)\rho(U_2), \tag{5.3}$$

for any $\lambda \in [0,1]$ and $U_1,\ U_2 \in N$.

Definition 5.8. If N is a wedge of the ordered linear space L and a function $\rho : N \to \mathbb{R}$ satisfies the properties of (L_+, E)-translation invariance, convexity, and L_+-monotonicity, then it is called (L_+, E)-*restricted convex*.

The rest of results concerning the primal insurance and the reinsurance companies' risk measures are simple implications of the following results.

Remark 5.9. The main condition under which the following results hold, is that N is a (closed) wedge of an ordered normed space L (which is reflexive if it is required and equal to some appropriately defined L^2-space, too) such that $N \cap L_+$ is a nonempty wedge of L containing the numeraire E.

6. Dual Representation of Restricted Coherent Risk Measures

The following Theorem relies on Hahn-Banach extension theorem for linear functionals defined on subspaces of linear spaces.

Theorem 6.1 ((Krein-Rutman) [3, Corollary 1.6.2]). *Let L be an ordered linear space and J be a linear subspace of it containing an order unit of L. Then a positive linear functional defined on J has a positive extension to L.*

The following Proposition assures the continuity of this extension.

Proposition 6.2 ([3, Pr. 3.1.8]). *Let L be an ordered locally convex space with positive wedge P and suppose that $E \in \operatorname{int} P$. Let J be a linear subspace containing E and f a positive linear functional defined on J. Then f has a continuous, positive extension to L.*

The above theorem and proposition are crucial for the similarity between the proof of the Theorem of dual representation for coherent risk measures in case where the domain is the whole reflexive space L and the cone interior $\operatorname{int} L_+$ is nonempty (like the ones studied in [7]). The results in [7] and other related articles cannot be proved in the restricted case, because the Separation Theorem for convex sets cannot be applied in this case, in the sense that we desire the separating functionals to lie in L^*. Thus, we assure that every positive linear functional has a continuous extension all over L, by the previous theorem and proposition.

For the application of the above theorem and the above proposition we need the following.

Lemma 6.3. *If D is a wedge of L then $D - D$ is a subspace of L.*

Proof. It suffices to verify that the set $D - D$ is closed under the addition of vectors and the scalar multiplication. If $y_1,\ y_2 \in D - D$, then $y_1 = x_1 - x_2,\ y_2 = x_3 - x_4$, where $x_1,\ x_2,\ x_3,\ x_4 \in D$. Then

$$y_1 + y_2 = (x_1 + x_3) - (x_2 + x_4), \tag{6.1}$$

and by the properties of D as a wedge, $x_1 + x_3 \in D,\ x_2 + x_4 \in D$. Also, if $\lambda \geq 0$, $\lambda y_1 = \lambda x_1 - \lambda x_2$ and by the properties of D as a wedge, $\lambda x_1 \in D,\ \lambda x_2 \in D$. Finally if $\lambda < 0, \lambda y_1 = (-\lambda)x_2 - (-\lambda)x_1$ and by the properties of D as a wedge, $(-\lambda)x_1 \in D, (-\lambda)x_2 \in D$. □

Lemma 6.4. *The set $D = N \cap L_+$ is a cone of L.*

Proof. Obviously, $D = N \cap L_+$ is a wedge. In order to prove that it is a cone, it suffices to prove that

$$D \cap (-D) = \{0\}. \tag{6.2}$$

Suppose that $n \in D \cap (-D)$. Then $n \in L_+,\ n \in (-L_+)$. Hence, $n \in L_+ \cap (-L_+) = \{0\}$. □

Corollary 6.5. *The set $N \cap L_+ - N \cap L_+$ is a subspace of L.*

We remark—as it is also mentioned in the Appendix—that $\hat{}\ : L \to L^{**}$ denotes the natural embedding map from L to the second dual space L^{**} of L.

Theorem 6.6. *If N is a closed wedge of a reflexive ordered normed space L with nonempty cone interior containing the L_+-interior point E, $\rho : N \to \mathbb{R}$ is a (L_+, E)-restricted coherent risk measure with $\sigma(L,\ L^*)$-closed acceptance set $\mathcal{A}_\rho$, then*

$$\rho(U) = \sup\{\langle -U, \pi\rangle \mid \pi \in B_E\}, \tag{6.3}$$

for any $U \in N$, where $B_E = \{y^* \in L_+^0 \ \mid\ \widehat{E}(y^*) = 1\}$. On the other hand, every ρ defined through (6.3)*, is a $(L_+,\ E)$-restricted coherent risk measure.*

Proof. If we consider a (L_+, E)-restricted coherent risk measure ρ, then ρ admits a representation like the one indicated in (6.3). To see this, we remark that for any $U \in N$ and for any $\epsilon > 0$ we have that

$$U + (\rho(U) - \epsilon)E \notin \mathcal{A}_\rho. \tag{6.4}$$

The singleton $\{U + (\rho(U) - \epsilon)E\}$ is a convex, weakly compact set and $\mathcal{A}_\rho$ is by assumption a weakly closed set of L which is also convex (since it is a wedge), since ρ is an $(L_+,\ E)$-restricted coherent risk measure. Since these two sets are disjoint, from the Strong

Separation Theorem for convex sets in locally convex spaces there is some $\ell \in L^*$, $\ell \neq 0$, an $\alpha \in \mathbb{R}$ and a $\delta > 0$ such that

$$\langle U + (\rho(U) - \epsilon)E, \ell \rangle \geq \alpha + \delta > \alpha \geq \langle V, \ell \rangle \tag{6.5}$$

for any $V \in \mathcal{A}_\rho$. Hence we take that

$$\langle U + (\rho(U) - \epsilon)E, \ell \rangle > \sup\{\langle V, \ell \rangle \mid V \in \mathcal{A}_\rho\}. \tag{6.6}$$

We have to notice that every ℓ depends on U. The functional ℓ takes negative values on $N \cap L_+$ since if there is some $Z_0 \in N \cap L_+ \setminus \{0\}$ such that $\langle Z_0, \ell \rangle > 0$, then for any $\lambda \in \mathbb{R}_+$, we take $\lambda Z_0 \in N \cap L_+$. Then if $\lambda \to +\infty$

$$\langle \lambda \cdot Z_0, \ell \rangle > \langle U + (\rho(U) - \epsilon)E, \ell \rangle, \tag{6.7}$$

being a contradiction according to the previous separation argument. Then since we have that $-\ell$ is a positive linear functional on the subspace $J = N \cap L_+ - N \cap L_+$. But from Theorem [3, Corollary 1.6.2] and Proposition [3, Proposition 3.1.8] since E is an order unit of L which belongs to $N \cap L_+$, $-\ell$ has a continuous, linear extension all over L. For the sake of brevity, we denote again this functional by $-\ell$ and we have that $-\ell \in L_+^0$. If we suppose that $\langle E, -\ell \rangle = 0$, then we would take $\langle U, -\ell \rangle = 0$ for any $U \in L$. This is true since $U \in [-n_0(U)E, n_0 E]$, for appropriate $n_0(U) \in \mathbb{N}$, for any $U \in L$, because E is an order-unit of L. Hence $n_0(U)\langle E, \ell \rangle \leq -\langle U, \ell \rangle \leq -n_0(U)\langle E, \ell \rangle$. But since $-\langle E, \ell \rangle = 0$ by assumption, this implies $\langle U, -\ell \rangle = 0$ for any $U \in L$. Hence $-\ell = 0$ because $L = \cup_{n=1}^{\infty}[-nE, nE]$. This implies that the initial assumption that $\langle E, -\ell \rangle = 0$ is not true and $\langle E, -\ell \rangle > 0$ holds. Hence we may suppose that

$$\langle E, -\ell \rangle = 1 \tag{6.8}$$

holds, or else $-\ell \in B_E$.

Since $-\ell$ lies on B_E, we have by the above separation inequalities and the fact that $0 \in \mathcal{A}_\rho$ that

$$\langle U, \ell \rangle - \rho(U) + \epsilon > 0. \tag{6.9}$$

Hence if we denote $-\ell$ by π we have that $\rho(U) < \langle -U, \pi \rangle + \epsilon \leq \sup\{\langle -U, \pi \rangle \mid \pi \in B_E\} + \epsilon$ for any $\epsilon > 0$. On the other hand, we have that for any $U \in N$, $U + \rho(U)E \in \mathcal{A}_\rho$. This implies $\langle U, \pi \rangle + \rho(U) \geq 0$ for any $\pi \in B_E$, since $B_E \subseteq L_+^0$ and since $L_+ \subseteq \mathcal{A}_\rho$ and this implies $\mathcal{A}_\rho^0 \subseteq L_+^0$. $\pi(U) + \rho(U) \geq 0$ implies $\rho(U) \geq \langle -U, \pi \rangle$ for any $\pi \in B_E$. This implies $\rho(U) \geq \sup\{\langle -U, \pi \rangle \mid \pi \in B_E\}$. Finally we get that for any $U \in N$ and any $\epsilon > 0$,

$$\sup\{\langle -U, \pi \rangle \mid \pi \in B_E\} \leq \rho(U) < \sup\{\langle -U, \pi \rangle \mid \pi \in B_E\} + \epsilon. \tag{6.10}$$

This implies $\rho(U) = \sup\{\langle -U, \pi \rangle \mid \pi \in B_E\}$, for any $U \in N$.

For the opposite direction, it suffices to show that any $\rho : N \to \mathbb{R}$, defined through (6.3), is an (L_+, E)-restricted coherent risk measure. For this, we have to verify that ρ satisfies the properties of a (L_+, E)-restricted coherent risk measure as follows.

(i) (E-translation invariance):

$$\begin{aligned}\rho(U+aE) &= \sup\{\langle -U-aE, \pi\rangle \mid \pi \in B_E\}\\ &= \sup\{\langle -U, \pi\rangle - a \mid \pi \in B_E\} = \sup\{\langle -U, \pi\rangle \mid \pi \in B_E\} - a = \rho(U) - a,\end{aligned} \tag{6.11}$$

for any $U \in N$ and for these $a \in \mathbb{R}$ for which $U + aE \in N$.

(ii) (Subadditivity):

$$\begin{aligned}\rho(U_1+U_2) &= \sup\{\pi(-U_1-U_2) \mid \pi \in B_E\}\\ &\leq \sup\{\langle -U_1, \pi\rangle \mid \pi \in B_E\}\\ &\quad + \sup\{\langle -U_2, \pi\rangle \mid \pi \in B_E\} = \rho(U_1) + \rho(U_2),\end{aligned} \tag{6.12}$$

where $U_1, U_2 \in N$ and $U_1 + U_2 \in N$ because N is a wedge.

(iii) (Positive homogeneity):

$$\begin{aligned}\rho(\lambda U) &= \sup\{\langle -\lambda \cdot U, \pi\rangle \mid \pi \in B_E\}\\ &= \sup\{\lambda \cdot \langle -U, pi\rangle \mid \pi \in B_E\}\\ &= \lambda \cdot \sup\{\langle -U, \pi\rangle \mid \pi \in B_E\}\\ &= \lambda \cdot \rho(U),\end{aligned} \tag{6.13}$$

where $\lambda \in \mathbb{R}_+$ and $\lambda \cdot U \in N$ for any $U \in N$, since N is a wedge.

(iv) (L_+-Monotonicity): If $U' \geq U$ where $U,\ U' \in N$ in terms of the partial ordering of L induced by L_+, then $-U \geq -U'$, hence

$$\langle -U, \pi\rangle \geq \langle -U', \pi\rangle \tag{6.14}$$

for any $\pi \in B_E$ since $B_E \subseteq L_+^0$. Then by taking suprema over B_E we have $\rho(U) \geq \rho(U')$. □

Remark 6.7. A difference between the representation Theorem [1, Theorem 2.12] and the previous dual characterization theorem for restricted coherent risk measures is that our result holds for closed wedges and for coherent risk measures defined on reflexive spaces, while [1, Theorem 2.12] holds for any convex, closed, and cash-invariant subset of some L^p space with $1 \leq p \leq \infty$. Also, in [1, Theorem 2.12] an initial lower semicontinuous, unrestricted measure is mentioned, so that the equivalent restricted one is weakly (weak-star in the case of L^∞) continuous. Such conditions are absent in our Theorem and the only condition is that the acceptance set of the restricted coherent measure ρ is weakly closed.

7. The Optimization Problem

The optimization problem that an insurance company faces at time-period 0 is the following:

$$\text{Minimize } \rho(U) \quad \text{subject to } U \in \mathcal{U},\ \rho(U) \leq u, \tag{7.1}$$

where u denotes the surplus of the insurance company at time-period 0. The problem (7.1) expresses the attempt of the insurance company to minimize the solvency capital given an initial economic capital constraint. The set $\mathcal{U}$ is the subspace of the attainable surplus variables. A direct method of determination of optimal points of ρ on $N_1 = \{U \in \mathcal{U}, \rho(U) \leq u\}$, is via the min-max theorem reminded by Delbaen in [12, page 10]. Specifically, the statement of the previously mentioned min-max theorem is the following: *Let K be a compact, convex subset of a locally convex space Y. Let V be a convex subset of an arbitrary vector space M. Suppose that F is a bilinear function $F : M \times Y \to \mathbb{R}$. For each $l \in V$, we suppose that the partial (linear) function $F(l, \cdot)$ is continuous on Y. Then we have that*

$$\inf_{l\in V} \sup_{k\in K} F(l,k) = \sup_{k\in K} \inf_{l\in V} F(l,k). \tag{7.2}$$

The case of the spaces mentioned are mainly the case where $L = L^2(\Omega, \mathcal{F}, Q)$ and Q is any EMM of the financial market shaped by the surplus asset of the insurance company and the bank-account asset whose interest rate process is constant and equal to r. Under the above assumptions, we have the following.

Theorem 7.1. *The problem* (7.1) *has a solution.*

Proof. Suppose that M is the space $L = L^2(\Omega, \mathcal{F}, Q)$ endowed with the weak topology. V represents the convex set $N_1 \subseteq L^2(\Omega, \mathcal{F}, Q)$ of the constraints. The compact set K is the weakly compact base B_E defined by the numeraire asset on the cone L^0_+. The bilinear function we need in order to apply the above min-max Theorem is defined as follows: $F : M \times K \to \mathbb{R}$ ($F : L^2 \times L^2 \to \mathbb{R}$), where

$$F(x, \pi) = \langle -x, \pi \rangle. \tag{7.3}$$

The partial function $F_x : L^2 \to \mathbb{R}$ is weakly continuous, since for each $x \in L^2$ and each weakly convergent net $(\pi_a)_{a\in A} \subseteq L^2$, we have that $\langle x, \pi_a \rangle \to \langle x, \pi \rangle$, where π is the net's weak limit. Hence we have that $\langle -x, \pi_a \rangle \to \langle -x, \pi \rangle$, for any $x \in L^2 = M$. This implies that weak continuity holds, since the images of $(\pi_a)_{a\in A}$ through F_x for any x converge to the image of π through F_x, too. Hence, by Representation Theorem 6.6 and by weak compactness of the base $K = B_E$ we have that

$$\inf_{U\in N_1} \rho(U) = \inf_{U\in N_1} \sup_{\pi\in B_E} F(U,\pi) = \sup_{\pi\in B_E} \inf_{U\in N_1} F(U,\pi) = F(U^*,\pi^*) = \rho(U^*), \tag{7.4}$$

because the following result holds. □

Proposition 7.2 (see [21, Propsition 3.1]). *A function satisfying the min-max equality if and only if it has a saddle-point.*

The existence of the last saddle-point (U^*, π^*) implies the existence of a minimization point U^* of ρ over N_1, hence a solution to (7.1).

Hence, the minimum solvency capital is $\rho(U^*)$ in this case or else the part of the capital u that will be surely subtracted, so that the risk which arises from the cumulative obligations of the company to be eliminated (with respect to ρ), if we suppose that the constraint $\rho(U) \leq u$ is active.

8. A Note for the Specification of the Optimal Solution

We proved the existence of solution through saddle-point theory and inf-compactness. If we would like to specify optimal strategies $(\theta^*, \phi^*) \in \Theta \times \Phi$ which lead to an optimal solution $U_T^*(\theta^*, \phi^*)$, we actually have to restate the optimization problem (7.1) as follows:

$$\text{Minimize } \rho\big(U_T(\theta, \phi)\big), \quad \text{subject to } U \in \mathcal{U}, \rho(U) \leq u, \tag{8.1}$$

while the evolution of $U_t, t \in [0, T]$ is given through

$$dU_t = l\big(t, U_t, Z_t, \phi_t\big)dt + v\big(t, U_t, Z_t, \theta_t\big)dW_t, \tag{8.2}$$

where $W_t = (W_t^{(1)}, W_t^{(2)}),\ t \in [0, T]$ is a 2-dimensional Brownian motion and the claims' SDE is

$$dZ_t = q(t, Z_t)dt + \sigma(t, Z_t)dW_t^{(2)}, \tag{8.3}$$

where the Brownian motion $W_t^{(2)},\ t \in [0, T]$ is the second component of the above 2-dimensional Brownian motion, which is a 1-dimensional Brownian motion itself. Finally,

$$\theta \in \Theta, \quad \phi \in \Phi. \tag{8.4}$$

Equations (8.1)–(8.4) describe a *stochastic control problem*.

The interesting point is that if we completely restate the problem (8.1)–(8.4) according to the dual Representation Theorem 6.6, we take:

$$\text{Minimize } \sup_{\pi \in B_E} \big\langle -U_T(\theta, \phi), \pi \big\rangle, \quad \text{subject to } U \in \mathcal{U},\ \rho(U) \leq u, \tag{8.5}$$

while the evolution of $U_t,\ t \in [0, T]$ is given through

$$dU_t = l\big(t, U_t, Z_t, \phi_t\big)dt + v\big(t, U_t, Z_t, \theta_t\big)dW_t, \tag{8.6}$$

where $W_t = (W_t^{(1)},\ W_t^{(2)}),\ t \in [0, T]$ is a 2-dimensional Brownian motion and the claims' SDE is

$$dZ_t = q(t, Z_t)dt + \sigma(t, Z_t)dW_t^{(2)}, \tag{8.7}$$

where the Brownian motion $W_t^{(2)}$, $t \in [0, T]$ is the second component of the above 2-dimensional Brownian motion, which is a 1-dimensional Brownian motion itself. Finally,

$$\theta \in \Theta, \quad \phi \in \Phi. \tag{8.8}$$

In the statement indicated by (8.5)–(8.8), π are normalized functionals and not probability measures, like in classical approaches such as [15] in the Chapter which refer to utility optimization; hence, there are crucial differences related to the model presented here.

Appendix

In this paragraph, we give some essential notions and results from the theory of partially ordered linear spaces which are used in the previous sections of this paper.

Let L be a (normed) linear space. A set $C \subseteq L$ satisfying $C + C \subseteq C$ and $\lambda C \subseteq C$ for any $\lambda \in \mathbb{R}_+$ is called *wedge*. A wedge for which $C \cap (-C) = \{0\}$ is called *cone*. A pair $(L, \geq)$ where L is a linear space and $\geq$ is a binary relation on L satisfying the following properties:

(i) $x \geq x$ for any $x \in L$ (reflexive),

(ii) If $x \geq y$ and $y \geq z$ then $x \geq z$, where $x, y, z \in L$ (transitive),

(iii) If $x \geq y$ then $\lambda x \geq \lambda y$ for any $\lambda \in \mathbb{R}_+$ and $x + z \geq y + z$ for any $z \in L$, where $x, y \in L$ (compatible with the linear structure of L),

is called *partially ordered linear space*.

The binary relation $\geq$ in this case is a *partial ordering* on L. The set $P = \{x \in L \mid x \geq 0\}$ is called *(positive) wedge* of the partial ordering $\geq$ of L. Given a wedge C in L, the equivalent partial ordering relation $\geq_C$ defined as follows:

$$x \geq_C y \Longleftrightarrow x - y \in C \tag{A.1}$$

is a partial ordering on L, called *partial ordering induced by* C *on* L. If the partial ordering $\geq$ of the space L is *antisymmetric*, namely if $x \geq y$ and $y \geq x$ implies $x = y$, where $x, y \in L$, then P is a cone. If L is partially ordered by C, then any set of the form $[x, y] = \{r \in L \mid y \geq_C r \geq_C x\}$ where $x, y \in C$ is called *order-interval of* L. e is an *order unit* of L, if $L = \cup_{n=1}^{\infty}[-ne, ne]$, where $[-ne, ne]$ is the order-interval of L and we suppose that L is ordered by L_+, while $n \in \mathbb{N}$. If e is an interior point, then e is an order unit of L. If e is an order unit and L is a Banach space, then e is an interior point of L. L' denotes the linear space of all linear functionals of L, while L^* is the norm dual of L^*, in case where L is a normed linear space.

Suppose that C is a wedge of L. A functional $f \in L'$ is called *positive functional* of C if ≥ 0 for any $x \in C$. $f \in L'$ is a *strictly positive functional* of C if $\langle x, f \rangle > 0$ for any $x \in C \setminus C \cap (-C)$. A linear functional $f \in L'$ where L is a normed linear space, is called *uniformly monotonic functional* of C if there is some real number $a > 0$ such that $\langle x, f \rangle \geq a\|x\|$ for any $x \in C$. In case where a uniformly monotonic functional of C exists, C is a cone. $C^0 = \{f \in L^* \mid \langle x, f \rangle \geq 0$ for any $x \in C\}$ is the *dual wedge of* C *in* L^*. Also, by C^{00} we denote the subset $(C^0)^0$ of L^{**}. It can be easily proved that if C is a closed wedge of a reflexive space, then $C^{00} = C$. If C is a wedge of L^*, then the set $C_0 = \{x \in L \mid \widehat{x}(f) \geq 0$ for any $f \in C\}$ is the *dual wedge of* C *in* L, where $\widehat{\ }: L \to L^{**}$ denotes the natural embedding map from L to the second dual space L^{**}

of L. Note that if for two wedges K, C of $LK \subseteq C$ holds, then $C^0 \subseteq K^0$. If C is a cone, then a set $B \subseteq C$ is called *base* of C if for any $x \in C \setminus \{0\}$ there exists a unique $\lambda_x > 0$ such that $\lambda_x x \in B$. The set $B_f = \{x \in C \mid \langle x, f \rangle = 1\}$ where f is a strictly positive functional of C is the *base of C defined by f*. B_f is bounded if and only if f is uniformly monotonic. If B is a bounded base of C such that $0 \notin \overline{B}$ then C is called *well based*. If C is well based, then a bounded base of C defined by a $g \in L^*$ exists.

The partially ordered vector space L whose positive cone is P is a *vector lattice* if for any $x, y \in L$, the supremum and the infimum of $\{x, y\}$ with respect to the partial ordering defined by the cone P exist in L. In this case $\sup\{x, y\}$ and $\inf\{x, y\}$ are denoted by $x \vee y$ and $x \wedge y$, respectively. If so, $|x| = \sup\{x, -x\}$ is the *absolute value* of x and if L is also a normed space such that $\||x|\| = \|x\|$ for any $x \in L$, then E is called *normed lattice*.

Finally, we remind that the *usual partial ordering* of an $L^p(\Omega, \mathcal{F}, \mu)$ space, where $(\Omega, \mathcal{F}, \mu)$ is a probability space is the following: $x \geq y$ if and only if the set $\{\omega \in \Omega : x(\omega) \geq y(\omega)\}$ is a set lying in $\mathcal{F}$ of μ-probability 1.

All the previously mentioned notions and related propositions concerning partially ordered linear spaces are contained in [3].

A subset F of a convex set C in L is called *extreme set* or else *face* of C, if whenever $x = az + (1-a)y \in F$, where $0 < a < 1$ and $y, z \in C$ implies $y, z \in F$. If F is a singleton, F is called *extreme point* of C.

A subset of some Euclidean space $\mathbb{R}^n$ of the form $\{x \in \mathbb{R}^n \mid a_1x_1 + a_2x_2 + \cdots + a_nx_n \leq b_1\}$, where $a_i \in \mathbb{R}$, $i = 1, 2, \ldots, n, b_1 \in \mathbb{R}$ is called *half-space* and more specifically *closed half-space* of $\mathbb{R}^n$. The intersection of finitely many closed half-spaces is called *polyhedral set*.

References

[1] M. Kaina and L. Rüschendorf, "On convex risk measures on L^p-spaces," *Mathematical Methods of Operations Research*, vol. 69, no. 3, pp. 475–495, 2009.

[2] D. Filipović and M. Kupper, "Monotone and cash-invariant convex functions and hulls," *Insurance: Mathematics & Economics*, vol. 41, no. 1, pp. 1–16, 2007.

[3] G. Jameson, *Ordered Linear Spaces*, vol. 141 of *Lecture Notes in Mathematics*, Springer, Berlin, Germany, 1970.

[4] K. Detlefsen and G. Scandolo, "Conditional and dynamic convex risk measures," *Finance and Stochastics*, vol. 9, no. 4, pp. 539–561, 2005.

[5] B. Roorda, J. M. Schumacher, and J. Engwerda, "Coherent acceptability measures in multiperiod models," *Mathematical Finance*, vol. 15, no. 4, pp. 589–612, 2005.

[6] F. Riedel, "Dynamic coherent risk measures," *Stochastic Processes and Their Applications*, vol. 112, no. 2, pp. 185–200, 2004.

[7] D. G. Konstantinides and C. E. Kountzakis, "Risk measures in ordered normed linear spaces with non-empty cone-interior," *Insurance: Mathematics & Economics*, vol. 48, no. 1, pp. 111–122, 2011.

[8] H. H. Panjer, "Measurement of risk, solvency requirements and allocation of capital within financial conglomerates," Working Paper, Department of Statistics and Actuarial Science, University of Waterloo, Waterloo, Canada.

[9] G. Dhaene, M. J. Goovaerts, R. Kaas, Q. Tang, S. Vanduffel, and D. Vyncke:, "Solvency capital, risk measures and comonotonicity: a review," Research Report OR0416, Department of Applied Economics, Catholic University of Leuven, 2003.

[10] R. T. Rockafellar and S. Uryasev, "Optimization of conditional value-at-risk," *Journal of Risk* , vol. 2, pp. 22–41, 2000.

[11] G. Ch. Pflug, "Some remarks on the value-at-risk and the conditional value-at-risk," in *Probabilistic Constrained Optimization*, S. Uryasev, Ed., vol. 49, pp. 272–281, Kluwer Academic Publishers, Dordrecht, The Netherlands, 2000.

[12] F. Delbaen, "Coherent risk measures on general probability spaces," in *Advances in Finance and Stochastics*, pp. 1–38, Springer, Berlin, Germany, 2002.

[13] P. Grandits, F. Hubalek, W. Schachermayer, and M. Žigo, "Optimal expected exponential utility of dividend payments in a Brownian risk model," *Scandinavian Actuarial Journal*, vol. 2007, no. 2, pp. 73–107, 2007.

[14] C. Hipp and H. Schmidli, "Asymptotics of ruin probabilities for controlled risk processes in the small claims case," *Scandinavian Actuarial Journal*, vol. 2004, no. 5, pp. 321–335, 2004.

[15] I. Karatzas and S. E. Shreve, *Methods of Mathematical Finance*, vol. 39 of *Applications of Mathematics-Stochastic Modelling and Applied Probability*, Springer, New York, NY, USA, 1998.

[16] R. Norberg, "Ruin problems with assets and liabilities of diffusion type," *Stochastic Processes and their Applications*, vol. 81, no. 2, pp. 255–269, 1999.

[17] B. Øksendal, *Stochastic Differential Equations: An Introduction with Aplications*, Springer, Berlin, Germany, 2000.

[18] P. Artzner, F. Delbaen, J.-M. Eber, and D. Heath, "Coherent measures of risk," *Mathematical Finance*, vol. 9, no. 3, pp. 203–228, 1999.

[19] X. H. Gong, "Connectedness of efficient solution sets for set-valued maps in normed spaces," *Journal of Optimization Theory and Applications*, vol. 83, no. 1, pp. 83–96, 1994.

[20] I. A. Polyrakis, "Demand functions and reflexivity," *Journal of Mathematical Analysis and Applications*, vol. 338, no. 1, pp. 695–704, 2008.

[21] V. Barbu and T. Precupanu, *Convexity and optimization in Banach spaces*, vol. 10, D. Reidel Publishing, Kluwer Academic Publishers, Dordrecht, The Netherland, 1986.