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Research Article

On Ideals of Implication Groupoids

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Ideals of implication groupoids are considered. Given a subset of a distributive implication groupoid, the smallest ideal containing it is constructed. A characterization of ideals in distributive implication groupoid using upper sets is given.

1. Introduction

In 50-ties L-Henkin and T-Skolem introduced the notion of Hilbert algebra as an algebraic counterpart of intuitionistic logic. A Hilbert algebra [1] is an algebra $\mathcal{H} = (H, *, 1)$ of type $(2, 0)$ satisfying the axioms:

$(H1)$ $x * (y * x) = 1$,

$(H2)$ $(x * (y * z)) * ((x * y) * (x * z)) = 1$,

$(H3)$ $x * y = 1$ and $y * x = 1$ imply $x = y$.

One can easily show that $(H2)$ can be replaced by two rather simpler axioms:

(LD) $x * (y * z) = (x * y) * (x * z)$ (left distributivity),

(E) $x * (y * z) = y * (x * z)$ (exchange).

Chajda and Halaš [2] introduced the concept of distributive implication groupoid and studied deductive systems, ideals, and congruence relations in distributive implication groupoid. In this paper we consider ideals in distributive implication groupoid. Given a subset of a distributive implication groupoid, we make the smallest ideal containing it. We provide an equivalent condition of the ideals using the notion of upper sets.

2. Preliminaries

Definition 2.1 (see [2]). An algebra $(A, *, 1)$ of type $(2, 0)$ is called an implication groupoid if it satisfies the identities:

(1) $x * x = 1$,

(2) $1 * x = x$ for all $x, y \in A$.

Example 2.2. Let $A = \{1, a, b\}$ in which $*$ is defined by

$*$	1	a	b
1	1	a	b
a	a	1	b
b	a	b	1

(2.1)

Then $(A, *, 1)$ is an implication groupoid.

Example 2.3. Let $A = \{1, a, b, c\}$ in which $*$ is defined by

$*$	1	a	b	c
1	1	a	b	c
a	1	1	b	b
b	1	a	1	a
c	1	a	b	1

(2.2)

Then $(A, *, 1)$ is an implication groupoid.

Definition 2.4 (see [2]). An implication groupoid $(A, *, 1)$ of type $(2, 0)$ is called a distributive implication groupoid if it satisfies the following identity:

$$(LD) \quad x * (y * z) = (x * y) * (x * z) \quad \text{(left distributivity)} \tag{2.3}$$

for all $x, y, z \in A$.

Example 2.5. Let $A = \{1, a, b, c, d\}$ in which $*$ is defined by

$*$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	b	b	1
b	1	a	1	1	d
c	1	a	1	1	d
d	1	1	c	c	1

(2.4)

Then $(A, *, 1)$ is a distributive implication groupoid.

In every implication groupoid, one can introduce the so-called induced relation $\leq$ by the setting

$$x \leq y \quad \text{iff } x * y = 1. \tag{2.5}$$

Lemma 2.6 (see [2]). *Let $(A, *, 1)$ be a distributive implication groupoid. Then A satisfies the identities*

$$x * 1 = 1, \qquad x * (y * x) = 1. \tag{2.6}$$

Moreover, the induced relation $\leq$ is a quasiorder on A, and the following relationships are satisfied:

(i) $x \leq 1$,

(ii) $x \leq y * x$,

(iii) $x * ((x * y) * y) = 1$,

(iv) $1 \leq x$ *implies* $x = 1$,

(v) $y * z \leq (x * y) * (x * z)$,

(vi) $x \leq y$ *implies* $y * z \leq x * z$,

(vii) $x * (y * z) \leq y * (x * z)$,

(viii) $x * y \leq (y * z) * (x * z)$.

3. On Ideals of Implication Groupoids

In this section, we study some properties of ideals in a distributive implication groupoid and give the smallest ideal containing a subset of a distributive implication groupoid. We characterize ideals in terms of upper sets.

Definition 3.1 (see [2]). Let $\mathcal{A} = (A, *, 1)$ be an implication groupoid. A subset $I \subseteq A$ is called an ideal of $\mathcal{A}$ if

$(I1)$ $1 \in I$,

$(I2)$ $x \in A$, $y \in I$ imply $x * y \in I$,

$(I3)$ $x \in A$, $y_1, y_2 \in I$ imply $(y_2 * (y_1 * x)) * x \in I$.

Remark 3.2. If I is an ideal of an implication groupoid $\mathcal{A} = (A, *, 1)$ and $a \in I$, $x \in A$, then $(a * x) * x \in I$.

Definition 3.3 (see [2]). Let $\mathcal{A} = (A, *, 1)$ be an implication groupoid. A subset $D \subseteq A$ is called a deductive system of $\mathcal{A}$ if

$(D1)$ $1 \in D$,

$(D2)$ $x \in D$ and $x * y \in D$ imply $y \in D$.

Lemma 3.4 (see [2]). *Let $\mathcal{A}$ be an implication groupoid. Then every ideal of $\mathcal{A}$ is a deductive system of $\mathcal{A}$.*

Converse of the above lemma does not hold in general.

Example 3.5. From Example 2.2, we can see that $\{1, a\}$ is its deductive system which is not an ideal since $b * a = b \notin \{1, a\}$.

Theorem 3.6 (see [2]). *A nonempty subset I of a distributive implication groupoid $\mathcal{A}$ is an ideal if and only if it is a deductive system of $\mathcal{A}$.*

For any $x_1, x_2, \ldots, x_n,\ a \in A$, we define

$$\prod_{i=1}^{n} x_1 * a = x_n * (\cdots * (x_1 * a) \cdots). \tag{3.1}$$

Lemma 3.7. *Let A be a distributive implication groupoid and $x, y, z \in A$ such that $x \leq y$. Then $z * x \leq z * y$.*

Proof. Let $x, y, z \in A$ and $x \leq y$. Then $x * y = 1$ and hence $(z * x) * (z * y) = z * (x * y) = z * 1 = 1$. Therefore $z * x \leq z * y$. □

Lemma 3.8. *Let A be a distributive implication groupoid and $x, y \in A$ such that $x * y = 1$. Then for all $a_1, a_2, \ldots, a_n \in A$, $\prod_{i=1}^{n} a_i * x = 1$ implies $\prod_{i=1}^{n} a_i * y = 1$.*

Proof. We have $x * y = 1$; that is, $x \leq y$, and from Lemma 3.7, we can see that

$$1 = \prod_{i=1}^{n} a_i * x \leq \prod_{i=1}^{n} a_i * y. \tag{3.2}$$

Therefore, from Lemma 2.6(iv), $\prod_{i=1}^{n} a_i * y = 1$. □

We denote the set of all ideals of A by $\mathcal{I}(A)$. It is obvious that $\{1\}, A \in \mathcal{I}(A)$.

Example 3.9. From Example 2.2, we can see that $\mathcal{I}(A) = \{\{1\}, A\}$.

Example 3.10. From Example 2.5, we can see that $\mathcal{I}(A) = \{\{1\}, \{1, a, d\}, \{1, b, c\}, A\}$.

Example 3.11. Let $A = \{1, a, b, c, d\}$ in which $*$ is defined by

$*$	1	a	b	c	d
1	1	a	b	c	d
a	a	1	c	d	d
b	a	a	1	c	c
c	a	a	a	1	c
d	a	a	a	a	1

(3.3)

Then $(A, *, 1)$ is an implication groupoid. We can see that $\mathcal{I}(A) = \{\{1\}, \{1, a\}, \{1, a, c, d\}, A\}$.

The following theorem is straightforward.

Theorem 3.12. *If I_i $(i \in \Delta)$ are ideals of an implication groupoid A, then $\bigcap_{i\in\Delta} I_i$ is an ideal of A.*

Note 1. In an implication groupoid, union of two ideals need not be an ideal. From Example 2.3, we can see that $I = \{1, a\}$ and $J = \{1, b\}$ are ideals of A but $I \cup J = \{1, a, b\}$ is not an ideal of A.

The following is a characterization of ideals

Theorem 3.13. *Let I be a subset of a distributive implication groupoid A containing 1. Then $I \in \mathcal{I}(A)$ if and only if for any $a, b \in I$ and $x \in A$, $a * (b * x) = 1$ implies $x \in I$.*

Proof. Let $I \in \mathcal{I}(A)$. Assume $a, b \in I$ and $x \in A$ such that $a * (b * x) = 1$. Since I is an ideal of A, we have $a * (b * x) \in I$. Since every ideal of A is deductive system, by applying $(D2)$ twice, we conclude that $x \in I$. Conversely, assume that the condition holds. Since ideals and deductive systems coincide in distributive implication groupoid, it is enough to show that I satisfies $(D1)$ and $(D2)$. Since $1 \in I$, the condition $(D1)$ holds. Suppose $x \in I$ and $x * a \in I$. Then $x * ((x * a) * a) = (x * (x * a)) * (x * a) = ((x * x) * (x * a)) * (x * a) = (1 * (x * a)) * (x * a) = (x * a) * (x * a) = 1$. Therefore $x * ((x * a) * a) \in I$ and hence $a \in I$. Thus $I \in \mathcal{I}(A)$. □

Corollary 3.14. *Let I be a subset of a distributive implication groupoid A containing 1. Then $I \in \mathcal{I}(A)$ if and only if for any $a_1, a_2, \ldots, a_n \in I$ and $x \in A$, $\prod_{i=1}^{n} a_i * x = 1$ implies $x \in I$.*

Definition 3.15. For every subset $\mathrm{X} \subseteq A$, the smallest ideal of A which contains X, that is, the intersection of all ideals $I \supseteq \mathrm{X}$, is said to be the ideal generated by X, and will be denoted by $(\mathrm{X}]$. Obviously, $(\emptyset] = \{1\}$.

Lemma 3.16. *Let A be a distributive implication groupoid and $x, y, z \in A$. Then $x * (y * z) = 1$ if and only if $y * (x * z) = 1$.*

Proof. Let $x * (y * z) = 1$. Then $y * (x * (y * z)) = y * 1 = 1$ and hence $(y * x) * (y * (y * z)) = 1$. Therefore $(y * x) * (y * z) = 1$. Thus $y * (x * z) = 1$. Similarly, we can prove the converse. □

Theorem 3.17. *Let A be a distributive implication groupoid and $\mathrm{X}(\neq \emptyset) \subseteq A$. Then*

$$(\mathrm{X}] = \left\{ x \in A : x = 1 \text{ or } \prod_{i=1}^{n} a_i * x = 1 \text{ for some } \mathrm{a}_1, \mathrm{a}_2, \ldots, \mathrm{a}_\mathrm{n} \in \mathrm{X} \right\}. \tag{3.4}$$

Proof. Let $I = \{x \in A : x = 1 \text{ or } \prod_{i=1}^{n} a_i * x = 1 \text{ for some } a_1, a_2, \ldots a_n \in \mathrm{X}\}$. Since $a * a = 1$ for all $a \in \mathrm{X}$, we obtain $\mathrm{X} \subseteq I$. Obviously $1 \in I$. Let $x * y \in I$ and $x \in I$. To prove $y \in I$, we will consider three cases. Case 1: $x = 1$. Then $y = 1 * y \in I$. Case 2: $x * y = 1$ and $x \neq 1$. Since $x \in I$ and $x \neq 1$, we conclude that $\prod_{i=1}^{n} a_i * x = 1$ for some $a_1, a_2, \ldots, a_n \in \mathrm{X}$. From Lemma 3.8, $\prod_{i=1}^{n} a_i * y = 1$. Therefore $y \in I$. Case 3: $x * y \neq 1$ and $x \neq 1$. Then there are

$a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_m \in X$ such that $\prod_{i=1}^{n} a_i * (x * y) = 1$ and $\prod_{j=1}^{m} b_j * x = 1$. Applying Lemma 3.16, we deduce that $x \leq \prod_{i=1}^{n} a_i * y$ and by Lemma 3.7, we see that

$$1 = \prod_{j=1}^{m} b_j * x \leq \prod_{j=1}^{m} b_j * \left(\prod_{i=1}^{n} a_i * y \right). \tag{3.5}$$

By Lemma 2.6(iv), $\prod_{j=1}^{m} b_j * (\prod_{i=1}^{n} a_i * y) = 1$. Hence I is an ideal of A.

Suppose that U is any ideal of A containing X. Let $x \in I$. If $x = 1$, then obviously $x \in U$. Assume that $x \neq 1$. Then there are $a_1, a_2, \ldots, a_n \in X$ such that $\prod_{i=1}^{n} a_i * x = 1$. Since $X \subseteq U$, it follows that $a_1, a_2, \ldots, a_n \in U$. Therefore $x \in U$ by Corollary 3.14. Thus $I \subseteq U$ and hence $I = (X]$. □

Let $I_1, I_2 \in \mathcal{I}(A)$; we define the meet of I_1 and I_2 (denoted by $I_1 \wedge I_2$) by $I_1 \wedge I_2 = I_1 \cap I_2$ and the join of I_1 and I_2 (denoted by $I_1 \vee I_2$) by $I_1 \vee I_2 = (I_1 \cup I_2]$. We note that $(\mathcal{I}(A), \wedge, \vee)$ is a lattice.

Theorem 3.18. *$(\mathcal{I}(A), \wedge, \vee)$ is a complete lattice.*

Let A be a distributive implication groupoid. For any $x, y \in A$, consider a set

$$A(x) = \{z \in A \mid x * z = 1\}, \qquad A(x, y) = \{z \in A \mid x * (y * z) = 1\}. \tag{3.6}$$

The set $A(x)$ (resp., $A(x, y)$) is called an upper set of x (resp., of x and y). Obviously, $1, x \in A(x)$ and $1, x, y \in A(x, y)$. We know that $A(1) = \{1\}$ is always an ideal of A. But the sets $A(x)$ and $A(x, y)$ need not be ideals of A in an implication groupoid, since $A(a) = \{a\}$ and $A(a, 1) = \{a\}$ are not ideals of A in Example 2.2. The following lemma can be proved easily.

Lemma 3.19. *If A is an implication groupoid, then $A(u) = A(u, 1)$.*

Theorem 3.20. *If A is a distributive implication groupoid, then, for any $x, y \in A$, the set $A(x, y)$ is an ideal of A.*

Proof. Let A be a distributive implication groupoid. Clearly $1 \in A(x, y)$. Let $r \in A(x, y)$ and $r * s \in A(x, y)$. Then $x * (y * r) = 1$ and $x * (y * (r * s)) = 1$. Now $x * (y * (r * s)) = 1$ implies that $(x * (y * r)) * (x * (y * s)) = 1$ which gives $x * (y * s) = 1$. Therefore $s \in A(x, y)$. Hence $A(x, y)$ is an ideal of A. □

Corollary 3.21. *Let A be a distributive implication groupoid. Then for any $x \in A$, the set $A(x)$ is an ideal of A.*

Lemma 3.22. *If A is a distributive implication groupoid, then $A(x) \subseteq A(x, y)$ for any $x, y \in A$.*

Theorem 3.23. *Let A be a distributive implication groupoid and $a \in A$. Then the following are equivalent:*

(i) *$a \leq x$ for any $x \in A$,*

(ii) *$A = A(a)$,*

(iii) *$A = A(a, x) = A(x, a)$ for any $x \in A$.*

Proof. (i) $\Leftrightarrow$ (ii): straightforward.

(ii) $\Rightarrow$ (iii): by Lemma 3.22, $A = A(a) \subseteq A(a,x) \subseteq A$.

(iii) $\Rightarrow$ (ii): $A = A(a,1) = A(a)$. □

Theorem 3.24. *Let A be a distributive implication groupoid and $a \in A$. Then $A(a) = \bigcap_{b\in A} A(a,b)$.*

Proof. By Lemma 3.22, $A(a) \subseteq A(a,b)$ for any $a,b \in A$. Therefore $A(a) \subseteq \bigcap_{b\in A} A(a,b)$. If $c \in \bigcap_{b\in A} A(a,b)$, then $c \in A(a,b)$ for all $b \in A$ and so $c \in A(a,1)$. Hence $1 = a * (1 * c) = a * c$, which proves $c \in A(a)$. This means that $\bigcap_{b\in A} A(a,b) \subseteq A(a)$. □

Corollary 3.25. *Let A be a distributive implication groupoid. Then for any $a \in A$, $A(a) = A(a,1) = \bigcap_{b\in A} A(a,b)$.*

Theorem 3.26. *Let A be a distributive implication groupoid. Then $A(a,b) = A(b,a)$ for any $a,b \in A$.*

Proof. It follows from Lemma 3.16. □

The following is a characterization of ideals.

Theorem 3.27. *Let I be a nonempty subset of a distributive implication groupoid A. Then I is an ideal of A if and only if $A(a,b) \subseteq I$ for all $a,b \in I$.*

Proof. Let I be an ideal of A and $a,b \in I$. If $c \in A(a,b)$, then $a * (b * c) \in I$ and so $z \in I$. Hence $A(a,b) \subseteq I$. Conversely, assume that $A(a,b) \subseteq I$ for all $a,b \in I$. Note that $1 \in A(a,b) \subseteq I$. Let $x \in I$ and $x * y \in I$. Since $(x * y) * (x * y) = 1$, we have $y \in A(x * y, x) \subseteq I$. We conclude that I is an ideal of A. □

Corollary 3.28. *Let A be a distributive implication groupoid. If I is an ideal of A, then $A(a) \subseteq I$ for any $a \in I$.*

The converse of the above corollary need not be true in general. Consider the following example.

Example 3.29. Let $A = \{1,a,b,c,d,e,f,g\}$ in which $*$ is defined by

$*$	a	b	c	d	e	f	g	1
a	1	1	1	1	1	1	1	1
b	c	1	c	g	1	1	g	1
c	f	f	1	f	1	f	1	1
d	c	e	c	1	e	1	1	1
e	a	f	f	d	1	f	g	1
f	c	e	c	g	e	1	g	1
g	a	b	c	f	e	f	1	1
1	a	b	c	d	e	f	g	1

(3.7)

Then $(A, *, 1)$ is a distributive implication groupoid. Here $I = \{1, b, e, f, g\}$ contains $A(1), A(b), A(e), A(f), A(g)$ but I is not an ideal of A.

Theorem 3.30. *Let A be a distributive implication groupoid and $x, y \in A$. Then $y \in A(x)$ if and only if $A(x) = A(x, y)$.*

Proof. Assume that $y \in A(x)$. Then $x * y = 1$. We know that $A(x) \subseteq A(x, y)$. For any $z \in A(x, y)$, we have $1 = x*(y*z) = (x*y)*(x*z) = x*z$ and so $z \in A(x)$. Hence $A(x) = A(x, y)$. Conversely, if $A(x) = A(x, y)$, then $y \in A(x, y) = A(x)$. $\square$

Theorem 3.31. *Let A be a distributive implication groupoid and $x, y \in A$. Then $x \le y$ if and only if $A(y) \subseteq A(x)$.*

Proof. Let $x \le y$. Then $x * y = 1$. For any $z \in A(y)$, we have $y * z = 1$. Also $x * z = 1 * (x * z) = (x * y) * (x * z) = x * (y * z) = x * 1 = 1$ and so $z \in A(x)$. Hence $A(y) \subseteq A(x)$. Conversely, if $A(y) \subseteq A(x)$, then $y \in A(x)$ and hence $x \le y$. $\square$

Corollary 3.32. *Let A be a distributive implication groupoid and $x, y \in A$. Then $x \le y$ and $y \le x$ if and only if $A(x) = A(y)$.*

Example 3.33. Let $A = \{1, a, b, c\}$ be a set with the following table:

$*$	1	a	b	c
1	1	a	b	c
a	1	1	b	1
b	1	c	1	c
c	1	1	b	1

(3.8)

Then $(A, *, 1)$ is a distributive implication groupoid. We can see that $a \le c$, $c \le a$ and $A(a) = A(c) = \{1, a, c\}$.

Theorem 3.34. *Let I be an ideal of A. Then $I = \bigcup_{x,y \in I} A(x, y)$.*

Proof. We know that $A(x, y) \subseteq I$ for all $x, y \in I$. Therefore $\bigcup_{x,y \in I} A(x, y) \subseteq I$. Let $z \in I$. Then $z \in A(z) = A(z, 1) \subseteq \bigcup_{x,y \in I} A(x, y)$. Then $I \subseteq \bigcup_{x,y \in I} A(x, y)$. $\square$

Corollary 3.35. *If I is an ideal of A, $I = \bigcup_{x \in I} A(x, 1)$.*

Finally we conclude this paper with the following theorem.

Theorem 3.36. *Let I be an ideal of A. Then $I = \bigcup_{x \in I} A(x)$.*

Proof. Since $A(x, 1) = A(x)$, we have, by Corollary 3.35, $I = \bigcup_{x \in I} A(x)$. $\square$

References

[1] W. A. Dudek, "On ideals in Hilbert algebras," *Mathematica*, vol. 38, pp. 31–34, 1999.

[2] I. Chajda and R. Halaš, “Distributive implication groupoids,” *Central European Journal of Mathematics*, vol. 5, no. 3, pp. 484–492, 2007.