Hindawi Publishing Corporation
Advances in Decision Sciences
Volume 2012, Article ID 671942, 7 pages
doi:10.1155/2012/671942

Research Article

Almost Sure Central Limit Theorem of Sample Quantiles

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Received 19 April 2012; Accepted 12 August 2012

Academic Editor: Saralees Nadarajah

We obtain the almost sure central limit theorem (ASCLT) of sample quantiles. Furthermore, based on the method, the ASCLT of order statistics is also proved.

1. Introduction

To describe the results of the paper, suppose that we have an independent and identically distributed sample of size n from a distribution function $F(x)$ with a continuous probability density function $f(x)$. Let $F_n(x)$ denote the sample distribution function, that is,

$$F_n(x) = \frac{1}{n}\sum_{i=1}^{n} 1_{\{X_i \le x\}}, \quad -\infty < x < \infty. \tag{1.1}$$

Let us define the pth quantile of F by

$$\xi_p = \inf\{x : F(x) \ge p\}, \quad p \in (0,1), \tag{1.2}$$

and the sample quantile $\hat{\xi}_{np}$ by

$$\hat{\xi}_{np} = \inf\{x : F_n(\mathrm{x}) \ge p\}, \quad p \in (0,1). \tag{1.3}$$

It is well known that $\widehat{\xi}_{np}$ is a natural estimator of ξ_p. Since the quantile can be used for describing some properties of random variables, and there are not the restrictions of moment conditions, it is being widely employed in diverse problems in finance, such as quantile-hedging, optimal portfolio allocation, and risk management.

In practice, the large sample theory which can give the asymptotic properties of sample estimator is an important method to analyze statistical problems. There are numerous literatures to study the sample quantiles. Let $p \in (0,1)$, if ξ_p is the unique solution x of $F(x-) \le p \le F(x)$, then $\widehat{\xi}_{np} \xrightarrow{\text{a.e.}} \xi_p$ (see [1]). In addition, if $F(x)$ possesses a continuous density function $f(x)$ in a neighborhood of ξ_p and $f(\xi_p) > 0$, then

$$\frac{n^{1/2} f(\xi_p)\left(\widehat{\xi}_{np} - \xi_p\right)}{\left[p(1-p)\right]^{1/2}} \longrightarrow N(0,1), \quad \text{as } n \longrightarrow \infty, \tag{1.4}$$

where $N(0,1)$ denotes the standard normal variable (see [1, 2]). Suppose that $F(x)$ is twice differentiable at ξ_p, with $F'(\xi_p) = f(\xi_p) > 0$, then Bahadur [3] proved

$$\widehat{\xi}_{np} = \xi_p + \frac{p - F_n(\xi_p)}{f(\xi_p)} + \widetilde{R}_n, \quad \text{a.e.}, \tag{1.5}$$

where $\widetilde{R}_n = O(n^{-3/4}(\log n)^{3/4})$, a.e, as $n \to \infty$. Very recently, Xu and Miao [4] obtained the moderate deviation, large deviation and Bahadur asymptotic efficiency of the sample quantiles $\widehat{\xi}_{np}$. Xu et al. [5] studied the Bahadur representation of sample quantiles for negatively associated sequences under some mild conditions.

Based on the above works, in the paper, we are interested in the almost sure central limit theorem (ASCLT) of sample quantiles $\widehat{\xi}_{np}$. The theory of ASCLT has been first introduced independently by Brosamler [6] and Schatte [7]. The classical ASCLT states that when $\mathbb{E}X = 0$, $\operatorname{Var}(X) = \sigma^2$,

$$\lim_{n\to\infty} \frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k} 1_{\{S_k \le \sqrt{k}\sigma x\}} = \Phi(x), \quad \text{a.s.} \tag{1.6}$$

for any $x \in \mathbb{R}$, where S_k denotes the partial sums $S_k = X_1 + \cdots + X_k$. Moreover, from the method to prove the ASCLT of sample quantiles, in Section 3, we obtain the ASCLT of order statistics.

2. Main Results

Theorem 2.1. *Let $X_1, X_2, \ldots, X_n$ be a sequence of independent identically distributed random variables from a cumulative distribution function F. Let $p \in (0,1)$ and suppose that $f(\xi_p) := F'(\xi_p)$ exists and is positive. Then one has*

$$\lim_{n\to\infty} \frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k} 1_{\{\sqrt{k}(\widehat{\xi}_{kp} - \xi_p) \le \sigma x\}} = \Phi(x), \quad \text{a.s.} \tag{2.1}$$

for any $x \in \mathbb{R}$, where $\sigma^2 = p(1-p)/f^2(\xi_p)$.

Proof. Firstly, it is not difficult to check

$$\left\{\sqrt{k}\left(\widehat{\xi}_{kp}-\xi_p\right)\leq \sigma x\right\}=\left\{F_k\left(\xi_p+\frac{\sigma x}{\sqrt{k}}\right)\geq p\right\}=:\left\{\frac{1}{\sqrt{k}}\sum_{i=1}^{k}Y_{i,k}\leq \omega_k\right\}, \tag{2.2}$$

where

$$\begin{gathered} Y_{i,k}=\mathbb{E}1_{\{X_i\leq \xi_p+\sigma x/\sqrt{k}\}}-1_{\{X_i\leq \xi_p+\sigma x/\sqrt{k}\}}, \\ \omega_k=\sqrt{k}\left(\mathbb{E}1_{\{X_i\leq \xi_p+\sigma x/\sqrt{k}\}}-p\right). \end{gathered} \tag{2.3}$$

From the Taylor's formula, it follows

$$\begin{aligned} \mathbb{E}1_{\{X_i\leq \xi_p+\sigma x/\sqrt{k}\}} &= F\left(\xi_p+\frac{\sigma x}{\sqrt{k}}\right) \\ &= F(\xi_p)+F'(\xi_p)\frac{\sigma x}{\sqrt{k}}+o\left(\frac{1}{\sqrt{k}}\right) \\ &= p+f(\xi_p)\frac{\sigma x}{\sqrt{k}}+o\left(\frac{1}{\sqrt{k}}\right), \end{aligned} \tag{2.4}$$

which implies

$$\omega_k=f(\xi_p)\sigma x+o(1). \tag{2.5}$$

By the Lindeberg's central limit theorem, we can get

$$\frac{1}{f(\xi_p)\sigma\sqrt{k}}\sum_{i=1}^{k}Y_{i,k}\xrightarrow{d}N(0,1),\quad \text{as } k\longrightarrow\infty. \tag{2.6}$$

Hence, (2.1) is equivalent to

$$\lim_{n\to\infty}\frac{1}{\log n}\sum_{k=1}^{n}\frac{1}{k}1_{\{(1/f(\xi_p)\sigma\sqrt{k})\sum_{i=1}^{k}Y_{i,k}\leq x+o(1)\}}=\Phi(x),\quad \text{a.s.} \tag{2.7}$$

Throughout the following proof, C denotes a positive constant, which may take different values whenever it appears in different expressions.

Put that

$$Z_{i,k}:=\frac{1}{f(\xi_p)\sigma}Y_{i,k}. \tag{2.8}$$

Let g be a bounded Lipschitz function bounded by C, then from (2.6), we have

$$\mathbb{E}g\left(\frac{1}{\sqrt{k}}\sum_{i=1}^{k}Z_{i,k}\right)\longrightarrow \mathbb{E}g(N),\quad \text{as } k\longrightarrow\infty, \tag{2.9}$$

where N denotes the standard normal random variable. Next, we should notice that (2.7) is equivalent to

$$\lim_{n\to\infty}\frac{1}{\log n}\sum_{k=1}^{n}\frac{1}{k}g\left(\frac{1}{\sqrt{k}}\sum_{i=1}^{k}Z_{i,k}\right)=\mathbb{E}g(N)\quad \text{a.s.} \tag{2.10}$$

from Section 2 of Peligrad and Shao [8] and Theorem 7.1 of Billingsley [9]. Hence, to prove (2.7), it suffices to show that as $n\to\infty$,

$$\begin{aligned} R_n &= \frac{1}{\log n}\sum_{k=1}^{n}\frac{1}{k}\left[g\left(\frac{1}{\sqrt{k}}\sum_{i=1}^{k}Z_{i,k}\right)-\mathbb{E}g\left(\frac{1}{\sqrt{k}}\sum_{i=1}^{k}Z_{i,k}\right)\right] \\ &=: \frac{1}{\log n}\sum_{k=1}^{n}\frac{1}{k}T_k \longrightarrow 0,\quad \text{a.s.} \end{aligned} \tag{2.11}$$

It is obvious that

$$\mathbb{E}R_n^2=\frac{1}{\log^2 n}\left[\sum_{k=1}^{n}\frac{1}{k^2}\mathbb{E}T_k^2+2\sum_{k=1}^{n-1}\sum_{j=k+1}^{n}\frac{1}{kj}\mathbb{E}T_kT_j\right]. \tag{2.12}$$

Since g is bounded, we have

$$\frac{1}{\log^2 n}\sum_{k=1}^{n}\frac{1}{k^2}\mathbb{E}T_k^2\le\frac{C}{\log n}. \tag{2.13}$$

Furthermore, for $1\le k<j\le n$, we have

$$\begin{aligned} \left|\mathbb{E}T_kT_j\right| &= \left|\operatorname{Cov}\left(g\left(\frac{1}{\sqrt{k}}\sum_{i=1}^{k}Z_{i,k}\right),g\left(\frac{1}{\sqrt{j}}\sum_{i=1}^{j}Z_{i,j}\right)\right)\right| \\ &= \left|\operatorname{Cov}\left(g\left(\frac{\sum_{i=1}^{k}Z_{i,k}}{\sqrt{k}}\right),g\left(\frac{\sum_{i=1}^{j}Z_{i,j}}{\sqrt{j}}\right)-g\left(\frac{\sum_{i=k+1}^{j}Z_{i,j}}{\sqrt{j}}\right)\right)\right| \\ &\le \frac{C}{\sqrt{j}}\mathbb{E}\left|\sum_{i=1}^{k}Z_{i,j}\right|\le\frac{C\sqrt{k}}{\sqrt{j}}\left(\mathbb{E}Z_{1,j}^2\right)^{1/2}, \end{aligned} \tag{2.14}$$

where

$$\mathbb{E}Z_{1,j}^2=1+O\left(\frac{1}{\sqrt{j}}\right). \tag{2.15}$$

Therefore, we have

$$\begin{aligned}\frac{1}{\log^2 n}\sum_{k=1}^{n-1}\sum_{j=k+1}^{n}\frac{1}{kj}\left|\mathbb{E}T_kT_j\right| &\le \frac{C}{\log^2 n}\sum_{k=1}^{n-1}\sum_{j=k+1}^{n}\frac{1}{k^{1/2}j^{3/2}}\left(\mathbb{E}Z_{1,j}^2\right)^{1/2} \\ &= \frac{C}{\log^2 n}\sum_{j=2}^{n}\sum_{k=1}^{j-1}\frac{1}{k^{1/2}j^{3/2}}\left(\mathbb{E}Z_{1,j}^2\right)^{1/2} \le \frac{C}{\log n}.\end{aligned} \tag{2.16}$$

From the above discussions, it follows that

$$\mathbb{E}R_n^2 \le \frac{C}{\log n}. \tag{2.17}$$

Take $n_k = e^{k^\tau}$, where $\tau > 1$. Then by Borel-Cantelli lemma, we have

$$R_{n_k} \longrightarrow 0, \quad \text{a.s. as} \quad k \longrightarrow \infty. \tag{2.18}$$

Since g is bounded function, then for $n_k < n \le n_{k+1}$, we obtain

$$\begin{aligned}|R_n| &\le \frac{1}{\log n_k}\left|\sum_{l=1}^{n_k}\frac{1}{l}\left[g\left(\frac{1}{\sqrt{l}}\sum_{i=1}^{l}Z_{i,l}\right) - \mathbb{E}g\left(\frac{1}{\sqrt{l}}\sum_{i=1}^{l}Z_{i,l}\right)\right]\right| \\ &\quad + \frac{1}{\log n_k}\sum_{l=n_k+1}^{n_{k+1}}\frac{1}{l}\left|g\left(\frac{1}{\sqrt{l}}\sum_{i=1}^{l}Z_{i,l}\right) - \mathbb{E}g\left(\frac{1}{\sqrt{l}}\sum_{i=1}^{l}Z_{i,l}\right)\right| \\ &\le |R_{n_k}| + \frac{C}{\log n_k}\sum_{l=n_k+1}^{n_{k+1}}\frac{1}{l} \longrightarrow 0, \quad \text{a.s., as } n \longrightarrow \infty,\end{aligned} \tag{2.19}$$

where we used the fact

$$\frac{\log n_{k+1}}{\log n_k} = \frac{(k+1)^\tau}{k^\tau} \longrightarrow 1, \quad \text{as } k \longrightarrow \infty. \tag{2.20}$$

So, the proof of the theorem is completed. □

3. Further Results

Another method to estimate the quantile is to use the order statistics. Based on the sample $\{X_1, \ldots, X_n\}$ of observations on $F(x)$, the ordered sample values:

$$X_{(1)} \le X_{(2)} \le \cdots \le X_{(n)} \tag{3.1}$$

are called the order statistics. For more details about order statistics, one can refer to Serfling [1] or David and Nagaraja [10]. Suppose that F is twice differentiable at ξ_p with

$F'(\xi_p) = f(\xi_p) > 0$, then the Bahadur representation for order statistics was first established by Bahadur [3], as $n \to \infty$

$$X_{(k_n)} = \xi_p + \frac{(k_n/n) - F_n(\xi_p)}{f(\xi_p)} + O\left(n^{-3/4}(\log n)^{(1/2)(\delta+1)}\right) \quad \text{a.e.,} \tag{3.2}$$

where

$$k_n = np + o\left(\sqrt{n}(\log n)^{\delta}\right), \quad n \longrightarrow \infty, \text{ for some } \delta \geq \frac{1}{2}. \tag{3.3}$$

From the idea of the Bahadur representation for order statistics, many important properties of order statistics can be easily proved. For example, Miao et al. [11] proved asymptotic properties of the deviation between order statistics and pth quantile, which included large and moderate deviation, Bahadur asymptotic efficiency.

Though there are some papers to study the ASCLT for the order statistics (e.g., Peng and Qi [12], Hörmann [13], Tong et al. [14], etc.), based on the method to deal with the sample quantile, we can also obtain the ASCLT of the order statistics.

Theorem 3.1. *Let $X_1, X_2, \ldots, X_n$ be a sequence of independent identically distributed random variables from a cumulative distribution function F. Let $p \in (0,1)$ and suppose that $f(\xi_p) := F'(\xi_p)$ exists and is positive. Let $k_n = np + o(\sqrt{n})$, then one has*

$$\lim_{n\to\infty} \frac{1}{\log n} \sum_{j=1}^{n} \frac{1}{j} 1_{\{\sqrt{j}(X_{(k_j)} - \xi_p) \leq \sigma x\}} = \Phi(x), \quad \text{a.s.} \tag{3.4}$$

for any $x \in \mathbb{R}$, where $\sigma^2 = p(1-p)/f^2(\xi_p)$.

Proof. Firstly, it is easy to see that the following two events are equivalent:

$$\begin{aligned}\left\{\sqrt{j}\left(X_{(k_j)} - \xi_p\right) \leq \sigma x\right\} &= \left\{\sum_{i=1}^{j} 1_{\{X_i \leq \xi_p + \sigma x/\sqrt{j}\}} \geq k_j\right\} \\ &=: \left\{\frac{\sum_{i=1}^{j} \overline{Y}_{i,j}}{\sqrt{j}} \leq \overline{\omega}_j\right\},\end{aligned} \tag{3.5}$$

where

$$\begin{aligned}\overline{Y}_{i,j} &= \mathbb{E}1_{\{X_i \leq \xi_p + \sigma x/\sqrt{j}\}} - 1_{\{X_i \leq \xi_p + \sigma x/\sqrt{j}\}}, \\ \overline{\omega}_j &= \frac{1}{\sqrt{j}}\left(jF\left(\xi_p + \frac{\sigma x}{\sqrt{j}}\right) - k_j\right) = f(\xi_p)\sigma x + o(1).\end{aligned} \tag{3.6}$$

Hence, by the same proof of Theorem 2.1, we can obtain the desired result. □

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